Vector Calculus

30th December 2008

I promised myself I would not do this, but I am spending two hours writing on Vector Calculus for you, Chris, since you were saying you were a bit rusty on it. This is my idea of fun!

I am going to introduce the following topics:

- The naive idea of a set, and an ordered set.
- Different types of number.
- The axioms (rules) for a vector space.
- Dimension and duality.
- Scalar products.
- Vector (cross) products.
- Quaternions.
- Covariance and contravariance.
- Generalisation of vectors to tensors.
- Differential (Riemannian) geometry.
- Stokes' and the divergence theorem.

This is all going to be very brief, so I am going to present the ideas, and often bypass the definitions and the proofs.

I am also going to append "what you do *not* need to know" sections to the above items, which explore as ideas further topics, but do not go into detail.

(1) The Idea of a Set, and an Ordered Set.

A *set* may be thought of as a collection of objects. Thus we could have, say, a set of socks in a drawer. Each individual sock is a *member*, or *element*, of the set, which can be thought of as a container (the drawer) for the elements.

There are two principal axiom systems which describe sets – Zermelo-Fraenkel and von Neumann-Gödel-Bernays. We will not go into those here, but we mention some ideas.

The *notation* for *membership* is the $\in$ symbol. Thus if x is a sock, and X is the drawer for the socks, we write

$$x \in X$$

and say "x *belongs to* X".

The axioms for set theory are expressed in terms of what is known as *symbolic logic*. I will discuss two types of symbolic logic here: *propositional calculus* and its extension – *predicate calculus.* The latter is needed to express the rules of set theory.

For propositional calculus, we consider *sentences* which may have the value *true* or *false* (only). I will consider three *logical connectives,* although it is possible to concoct more.

Consider the sentences (the '≡' symbol means "is equivalent to") A ≡ "the moon is made of green cheese" and B ≡ "Obama will make a better president than Bush".

Then NOT A, sometimes written as ¬A, is the logical operation:

"if A is *true*, then ¬A is *false*"

"if A is *false*, then ¬A is *true*".

We can write the above as a *truth table*, with T for *true* and F for *false*:

¬A	**A**
F	T
T	F

Then A AND B, sometimes written as A&B, is the logical operation represented by the truth table:

A&B	**A**	**B**
T	T	T
F	T	F
F	F	T
F	F	F

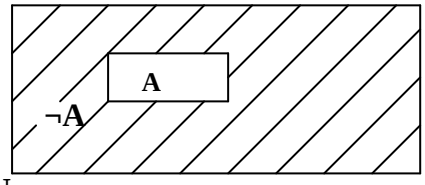
Likewise, A OR B, sometimes written as AVB, is the logical operation which means A OR B *OR BOTH*, represented by the truth table:

AVB	**A**	**B**
T	T	T
T	T	F
T	F	T
F	F	F

Now these truth tables can be represented in naive set theory, by *Venn diagrams.* All we have to do is use particular examples of sentences A and B. So now we use the sentences

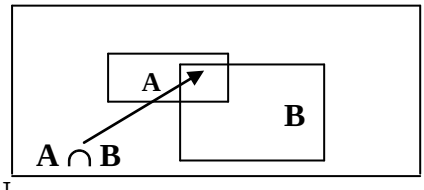
A ≡ "x ∈ *set* A" and **B** ≡ "x ∈ *set* B".

We can now represent **¬A** by the shaded part of the diagram

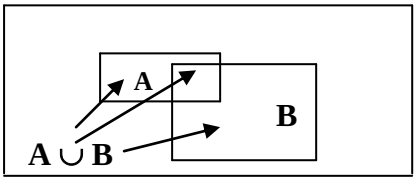


The whole container, **U**, is the *universe.* As you can see, here the mathematician is living in a pretty small universe! For sets, we generally do not use the symbolism **¬A**, but instead ∁A, for the *complement* of A.

Likewise for **A&B**, the set theoretic analogue is A ∩ B (A *intersection* B)



And for **AVB**, the set theoretic analogue is A ∪ B (A *union* B)



This completes what I want to say about the *propositional calculus*.

The *predicate calculus* includes two new symbols – *there exists*, written as ∃, and *for every*, written as ∀. These are applied to sentences. Thus a property of *integers* – positive or negative whole numbers, given by the set symbol **Z**, may be written in the form

"*for every* z *that belongs to* **Z**, *there exists an element* –z *that belongs to* **Z**, *such that*

z + -z = 0 *belongs to* **Z**",

or using our symbolism:

"∀ z ∈ **Z,** ∃ -z ∈ **Z**: **(**z + -z = 0) & 0 ∈ **Z**".

Strictly speaking, the introduction of the symbol ∀ is unnecessary, because this is equivalent to ¬∃¬.

We now go on to describe the idea of *order*. There are various definitions of this, from the weak to the more strongly restrictive, which we will not go into here.

If the elements of the set are denoted by a, b, c, … etc., then the set that lists these elements is often denoted by the elements written in curly brackets:

A ≡ {a, b, c, …}.

However we note that we can have ordered sets. The idea of an ordered set is just that the order of the elements matters. To put matters in an alternative notation for this, we use the *Cartesian product* symbol, X, and write

A ≡ a X b X c X ….

We will need this idea when we discuss vectors.

We note a particular axiom of set theory: *the axiom of choice*. This states that if we have an arbitrary mapping, say A to itself, then choosing this mapping is feasible, i.e. we can do it.

Starting from Cartesian products, we can also derive further Cartesian products. Consider a set of *pairs*, the first element of each pair being from A, and the second from B. Thus we can have the Cartesian product

A X B ≡ {{pair of a's from A and B respectively}, (pair of b's}, …}.

What you do not need to know. You have seen above that we have interrelated the ideas of logic and set theory, and also we are using what I would like to call *primitive concepts* – part of a syntax that cannot be interpreted in a more fundamental way, in particular with the symbols '∈' and '∃'.

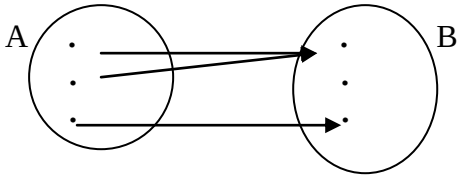
Well, all the above may be subsumed in a more general theory called *topos theory*, which extends the idea of a 'static' set which we have been dealing with, to that of a 'variable' set, and in which the ideas of variable set and logic are merged. Indeed the logic of the propositional calculus need no longer hold in these theories. We do not have to have

A or ¬**A** is *true*,

The above being called *the law of the excluded middle.*

In terms of the symbols '$\in$' and '$\exists$', these are replaced by another symbolism. This is a part of a programme of mathematics to replace 'everything' by a theory of transformations. The names for *transformations* come in various guises. You may have heard of *functions*, but other names are *maps, mappings, arrows* and *morphisms* – the latter being the favoured term in what is known as *category theory*, in terms of which this programme is generally expressed.

Thus morphisms are represented by the '$\rightarrow$' symbol, and we can have mappings between sets, or more particularly between elements of sets. Here is a diagram called a *cograph* mapping elements of set A to elements of set B:



In particular in these descriptions, the '$\in$' symbol is replaced by a '$\rightarrow$' symbol. Thus the *idea* of '*belonging*' is replaced by the idea of a *mapping*.

Also, the idea of *existence* in these topos theories is subsumed by the idea of *one set being embedded in another, elements being represented as 'subobjects'* – and these inclusion maps are represented by the general '$\in$' being replaced by an '$\rightarrow$' idea already mentioned.

We also note that the idea of ordered pairs (Cartesian products) can also be represented by mappings in category theory.

(2) **Different Types of Number.**

Having introduced the idea of a set, we now describe various ideas of *number*.

Set theory generally specifies that there exists at least one type of set – that given by the positive whole numbers – otherwise known as *natural numbers,* denoted by the symbol **N**. The axiom stating **N** exists is known as the *axiom of infinity*, because it is a property of these numbers that if you choose any finite natural number, then there is always a number that is one more than it, and also, and more to the point, if you put the number of elements of each *subset* of natural numbers in a one-to-one correspondence mapping with a natural number equal to its number of elements, then the whole set **N** does not map to any number in **N**.

In what follows we will assume that **N** does not include zero, although there is some dispute about this definition. In other words, we start from 1, and all elements in **N** are generated by adding 1 to each successive element so generated. This way of defining **N** is known as the *Peano Axioms*.

The next step up is to define the *integers*. We have already given the notation for these. They are denoted by **Z**. This, I believe, comes from the German word *Zahl*, for number. Integers are positive, zero or negative whole numbers. They are generally described by their properties, so this is an opportunity to present these properties, in a form which is known as an *abelian*, or *commutative group*.

The idea of a group developed in mathematics from the study of *permutations*, which it was found necessary to describe formally, because of their interest in solving what are known as *polynomial equations* – equations, say, like the *quadratic equation*

$$x^2 + ax + b = 0,$$

or more generally *equations of the nth degree*, like

$$x^n + ax^{n-1} + \ldots + bx + c = 0.$$

It was found that discussing *permutation of the roots* – the solutions of the above equation – resulted in the conclusion that there were no solutions involving general roots, additions, multiplications, subtractions and divisions of these roots, when n was greater or equal to 5. This theory is known as *Galois Theory*.

Like permutations, **Z** under the operation of + is also a *group*, but a special sort of group – it is commutative. Here are the axioms:

(a) **Z** is a set, and $1 \in \mathbf{Z}$ (this latter property is not necessary for a group).
(b) **Z** has an operation +, so that if x and $y \in \mathbf{Z}$, then $x + y \in \mathbf{Z}$.
(c) The operation + is *associative*: if w, x and $y \in \mathbf{Z}$, then $w + (x + y) = (w + x) + y \in \mathbf{Z}$.
(d) There is a unique element $0 \in \mathbf{Z}$, such that $x + 0 = x = 0 + x$.
(e) **Z** is an *abelian* or *commutative* group: $x + y = y + x \in \mathbf{Z}$.
(f) Each element x of **Z** has a unique *inverse*, $-x \in \mathbf{Z}$, satisfying $x + -x = 0$.

So, $1 + 5 \in \mathbf{Z}$, $(1 + 5) + 7 = 1 + (5 + 7) \in \mathbf{Z}$, $5 + 0 = 5 = 0 + 5 \in \mathbf{Z}$, $5 + 7 = 7 + 5 \in \mathbf{Z}$ and -7 exists in **Z** so that $7 + (-7) = 0$.

The next step is to introduce more complicated ideas of numbers. The natural numbers **N**, without zero can form multiplicatively the positive fractions. Just take a number, $n \in \mathbf{N}$, and multiply it by 1/n to form the number 1. Since $1 \in \mathbf{N}$, the set of fractions generated multiplicatively – and this generated set includes numbers n/m, where n and $m \in \mathbf{N}$, *also acts like a group*. The difference, in this case, is we do not have an additive group, we have a group under *multiplication*, and instead of having the property of having a zero, we have the number 1, which behaves very like the number zero for the group **Z**, +. We call this group the *positive rational numbers*.

We can similarly define a wider group than this: the *set of non-zero positive or negative rational numbers, under multiplication*. Call this group **Q**, **.**. So this contains the numbers of the form $\pm n/m$, where n, $m \in \mathbf{N}$. Let us go through the group axioms again in this case.

(a) **Q** is a set, and $0 \notin Q$ (this notation means zero does not belong to **Q**).
(b) **Q** has an operation **.**, so that if x and $y \in \mathbf{Q}$, then $x \cdot y \in \mathbf{Q}$.
(c) The operation **.** is *associative*: if w, x and $y \in \mathbf{Q}$, then $w \cdot (x \cdot y) = (w \cdot x) \cdot y \in \mathbf{Q}$.
(d) There is a unique element $1 \in \mathbf{Q}$, such that $x \cdot 1 = x = 1 \cdot x$.
(e) **Q** is an *abelian* or *commutative* group: $x \cdot y = y \cdot x \in \mathbf{Q}$.
(f) Each element x of **Q** has a unique *inverse*, $1/x \in \mathbf{Q}$, satisfying $x \cdot 1/x = 1$.

So, $1 \cdot (-3/5) \in \mathbf{Q}$, $(1/4 \cdot 3/5) \cdot -7/2 = 1/4 \cdot (3/5 \cdot -7/2) \in \mathbf{Q}$, $3/5 \cdot 1 = 3/5 = 1 \cdot 3/5 \in \mathbf{Q}$, $3/5 \cdot 7/2 = 7/2 \cdot 3/5 \in \mathbf{Q}$ and -17/49 exists in **Q** so that $-49/17 \cdot (-17/49) = 1$.

We can also form *algebraic numbers* – numbers like $[(2/3) - \sqrt{5}]/[7 + \sqrt{2}]$, and *real numbers* – numbers like $\sqrt{2}$ or $\pi = 3.14159\ldots$, described by non-terminating decimals (numbers, like π, which are not representable by a finite sum of algebraic numbers are called *transcendental*). Transcendental numbers exist independently of their representation.

Real numbers form an additive abelian group, denoted by **R**, +, and a multiplicative abelian group, without zero, denoted by **R**, **.**.

Combining the structures of the additive abelian group **R**, +, and the multiplicative abelian group, **R**, **.**, we form what is known as a *field*, in this case denoted by **R**, +, **.**. Note that for a field, 0 belongs to the field, but it is not possible to divide by zero, so this has to be included in the axioms.

A field has extra rules defining how multiplication relates to addition, called *distributive laws*. These are of the type

(D1) if w, x and $y \in \mathbf{R}$, then $w \cdot (x + y) = (w \cdot x) + (w \cdot y)$.

Multiplicative and additive commutativity implies other such rules.

We now define *complex numbers*. One method of doing this (not the only one, we can also use geometric rules, and rules derived from matrix multiplication) is to consider complex numbers as *sets of pairs* of numbers, provided with a special multiplication. So a complex number, which can be represented by

$a + ib$

where $i = \sqrt{-1}$, can also be represented by the pair

$[a, b]$.

This method was first developed by William Rowan Hamilton.

The only unusual thing is the *multiplicative inverse*. This is defined to be

$[a, -b/\sqrt{(a^2 + b^2)}]$,

so

$[a, b] \cdot [a, -b/\sqrt{(a^2 + b^2)}] = 1$.

Complex numbers form a *field*, called, not surprisingly, the field of complex numbers, and denoted by **C**, +, **.**.

(3) **The Axioms (Rules) for a Vector Space.**

The rules for vector spaces are more lax than those for a field, because to begin with we do not have rules for multiplying vectors with other vectors – we will introduce such rules later.

Looked at geometrically, vectors are represented by *arrows*. Arrows in this sense are different from the sense we used them before, where they were another word for a transformation. We use arrows, because they define a *direction*, from base to tip of the arrow. The vectors also have a *magnitude*.

We can represent the space in which the vector is embedded as being defined by an ordered set of numbers. If $v_1, v_2, \ldots v_n$ are numbers (say real numbers), then the vector can be represented by the ordered set

$\{v_1, v_2, \ldots v_n\}$.

We introduced the Cartesian product before to express this ordering. So another way of saying this is

$\{v_1, v_2, \ldots v_n\} \in \mathbf{R} \times \mathbf{R} \times \ldots \times \mathbf{R}$.

We now need to distinguish two types of vector spaces. In the first type, a *vector space with base point*, the base of the vector is fixed somewhere, and all vectors are fixed to this base point. This base point is usually given by the symbol *.

In the second type, which we will just call a *vector space*, there is no base point, and all vectors are allowed to float around in the space. As a further intuition on this point, we say that all vectors of the same magnitude and direction, but in different places, are *equivalent* in this definition.

Equivalence has its own definition. Using ≡ for equivalence as before, and **u**, **v**, and **w** in bold as vectors, an *equivalence class* for the vectors satisfies

u ≡ **u** (identity)
if **u** ≡ **v** then **v** ≡ **u** (symmetric)
if **u** ≡ **v** and **v** ≡ **w** then **u** ≡ **w** (transitive).

Since vectors have a magnitude, it is convenient to be able to *boost* or *contract* them, and this idea is inherent in the definition of a vector space. We do this by introducing the idea of a *scalar*. Often these scalars are thought of belonging to a field, say **R**, +, **.**, or we could have complex vectors boosted by complex scalars. Suppose r and s are scalars and **u**, **v**, and **w** are vectors as before. 0 is the zero scalar and **0** is the zero vector (no magnitude, no direction). Then the axioms for a *vector space* are

(a) $0\mathbf{u} = \mathbf{0}$
(b) $1\mathbf{u} = \mathbf{u}$
(c) $(s\ r)\mathbf{u} = s(r\mathbf{u})$
(d) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(e) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
(f) $(s + r)\mathbf{u} = s\mathbf{u} + r\mathbf{u}$
(g) $s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$.

These rules can be demonstrated in diagrams.

(4) **Dimension and Duality.**

It is possible to define the dimension of a vector space in a way that is quite formal mathematically and not (just) intuitive.

The axioms for a vector space which we have introduced mean that a general vector **u** can be represented as a sum of scalars a_1, a_2, … a_n with other *basis vectors* – given by an ordered n-tuple $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ as follows.

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n.$$

If the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n \neq \mathbf{0}$, where **0** is the zero vector, then the number n that is used to represent the vector **u** is called the *dimension* of the space. The dimension can be infinite.

Intuitively, two vectors, **u** and **v**, are *linearly dependent* if each *component* $a_1\mathbf{u}_1$, or respectively $b_1\mathbf{v}_1$, etc. is the same scalar multiple of the other.

Thus for a one dimensional subspace, two vectors are linearly dependent if they are along the same line. The idea can be extended to higher dimensions. In a plane, three vectors *have* to be linearly dependent, since the third must be some linear combination of the other two.

Sometimes mathematicians want to migrate from one set of basis vectors to another. This is part of the project already mentioned of describing mathematics in terms of transformations. The following definition is useful in such formulations.

Turning the above around, we define two vectors, **u** and **v**, to be *linearly **in**dependent* if

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n$$

and

$$\mathbf{v} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \ldots + b_n\mathbf{u}_n,$$

so that when we multiply each vector by a scalar to form the general *sum*

$$r\mathbf{u} + s\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n,$$

then this can be the zero vector only when each of the scalars c_1, c_2 etc. is zero.

Another formulation (which we could, but do not, prove) is that the dimension is the maximum number of linearly independent vectors in the space.

A *subspace*, **S**, of a vector space **V** is a *subset* of **V** which is also a *vector space*. We have seen already that the zero vector space, **0**, and the vector space **V** itself are such subspaces. Subspaces which are not the same as these two are called *proper* subspaces.

Every vector space includes the zero vector space, **0**.

It is possible to describe a vector subspace which is *perpendicular* to another subspace. Consider an n-dimensional vector space **V**, e.g. with base point, say with vectors hooked to the origin, 0. An m-dimensional subspace **S** of **V**, m < n, can be described in the example which follows, in terms of basis vectors with n components, the last n – m components being 0

$$\mathbf{u}_1 = \{1, 0, \ldots., 0\},$$

$\mathbf{u}_2 = \{0, 1, \ldots, 0\}$,

etc., i.e. the rth component of $\mathbf{u}_r = 1$, and the remainder are 0. Then it is possible to describe a vector space which is perpendicular to **S**, denoted by $\mathbf{S}^{\perp}$, the basis of which contains that part of the basis of **V** which does not belong to **S**. The zero vector space **0** is included in $\mathbf{S}^{\perp}$. Then we have

$(\mathbf{S}^{\perp})^{\perp} = \mathbf{S}$.

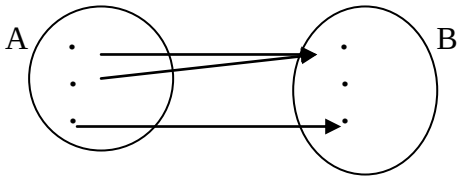
The formulation we have given here does not depend on the existence of a scalar or a vector product of vectors. That is why we are discussing it before the sections on scalar and vector products.

What you do not need to know (so much). We have mentioned the mathematical programme for transformations. Well, the linear transformations of vector spaces have been studied.

We now introduce *duality*, a property which relates certain types of *transformations* – called *linear transformations* – of vector spaces, with the vector spaces themselves. We will see an interesting property is that the dual of the dual of a vector space is equivalent to the original space.

Duality will crop up again when we study the deeper properties of covariance and contravariance.

We have studied morphisms, or arrows, represented by the '$\rightarrow$' symbol, for which we showed mappings between sets, or more particularly between elements of sets. Here again is the diagram of a cograph mapping elements of set A to elements of set B:



For a morphism, the set we start out with is called the *domain* (or *source*) and the set we map to is called the *codomain* (or *target*).

The *process of duality* is described by *reverse all arrows*, and *swap round the domain and codomain*.

We note a feature of some types of maps (called *functions*) is that every element of the domain is mapped to some element of the codomain. Well, that happens in the diagram above. But note that not every element of B is the target of a map. So our dual map contains empty maps from an element of B to no element of A, and so is not a function. Nevertheless, we can convert every statement about maps to their dual statement.

In particular, it follows from the definition, that *the dual map of a dual map is identical to the original map.*

(5) **Scalar Products.**

We now introduce ideas of multiplication. The first idea is to combine two vectors together to form a scalar. This is the *scalar product*. It is often stated that the scalar product of two vectors $\mathbf{u} = \{u_1, u_2, \ldots u_n\}$ and $\mathbf{v} = \{v_1, v_2, \ldots v_n\}$ is the scalar

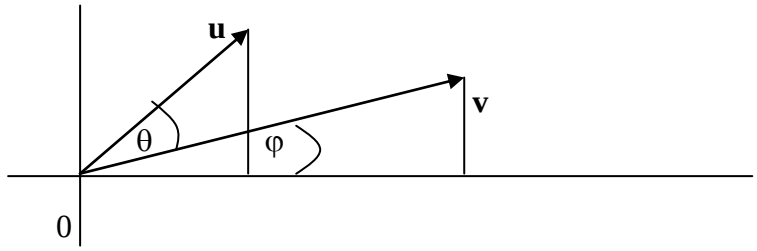
$$\mathbf{u.v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n.$$

We will go along with this for the moment, but it is not general enough. We first notice that the scalar product as defined above of a vector **u** with itself has the value

$$\mathbf{u.u} = u_1^2 + u_2^2 + \ldots + u_n^2,$$

that is, it is precisely the magnitude of the vector given by the n dimensional Pythagoras theorem.

If we represent vectors in a *Cartesian diagram* – the geometric representation of our algebraic description in terms of Cartesian products, then we have the following representation of a scalar product of vectors **u** and **v** (we are using here two dimensions, for demonstration purposes).



Using

$$\cos(A + B) = \cos A\cos B - \sin A\sin B,$$

(this can be worked out using the Euler relation

$$e^{iA} = \cos A + i\sin A,$$

together with

$$e^{i(A+B)} = e^{iA}e^{iB})$$

in which we put

$$A = (\theta + \varphi) \text{ and } B = -\varphi,$$

then the scalar product is, from the above definition,

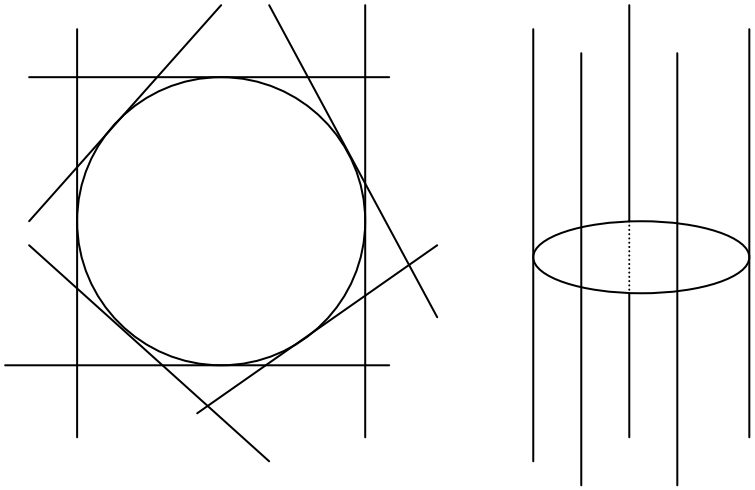
$$\sqrt{(\mathbf{u.u})}\sqrt{(\mathbf{v.v})}\cos\theta = \sqrt{(u_1^2 + u_2^2)}\cos(\theta + \varphi)\sqrt{(v_1^2 + v_2^2)}\cos(\varphi) + \sqrt{(u_1^2 + u_2^2)}\sin(\theta + \varphi)\sqrt{(v_1^2 + v_2^2)}\sin(\varphi).$$

Thus the scalar product is the magnitude of the two vectors times the cosine of the angle between them.

We have stated that this idea of scalar product is not general enough. We have indicated that the definition is consistent with the Pythagoras theorem. However, the Pythagoras theorem is not satisfied by the most general geometries.

We have, and still are, assuming here that the geometry of the space we are dealing with is *flat*. Even if the geometry of the space was one of, say a sphere, then the tangent plane to the sphere at any point, and therefore a local approximation in a neighbourhood of the point, *is* flat. So we will be considering here only spaces which are locally flat.

We mention, as an aside, that the (flat) tangent space to a curved n dimensional hypersurface is describable mathematically. We will take one particular example, the tangent space to a circle.



Shown on the left, is a series of tangents to the circle. Shown on the right is a representation of the tangent space, i.e. the *tangent vector bundle*. The end vectors of the vector bundle are aligned together *smoothly and in a non-overlapping manner* in this space. The tangent vector bundle has dimension two (a cylinder), compared with the dimension (one) of the circle. It is always the case that the tangent vector bundle is twice the dimension of the space to which it is tangent, but in higher dimensions this is often difficult to visualise.

However, even in the situation of a flat space, there are spaces which violate the n dimensional Pythagoras theorem. These general types of flat spaces are known as *Lie Algebras*, and they have been classified. The curved analogies to spheres or other curved surfaces for groups related to these spaces are known as *Lie Groups*. We do not go into details here, but note one special case.

For special relativity, one viewpoint (personally, I have a different angle on this) is that the 4 dimensional Pythagoras theorem does not hold. Instead of

$$s^2 = c^2t^2 + x^2 + y^2 + z^2,$$

where c is the speed of light, t is the time, and x, y and z are the space coordinates, we have

$$s^2 = c^2t^2 - x^2 - y^2 - z^2,$$

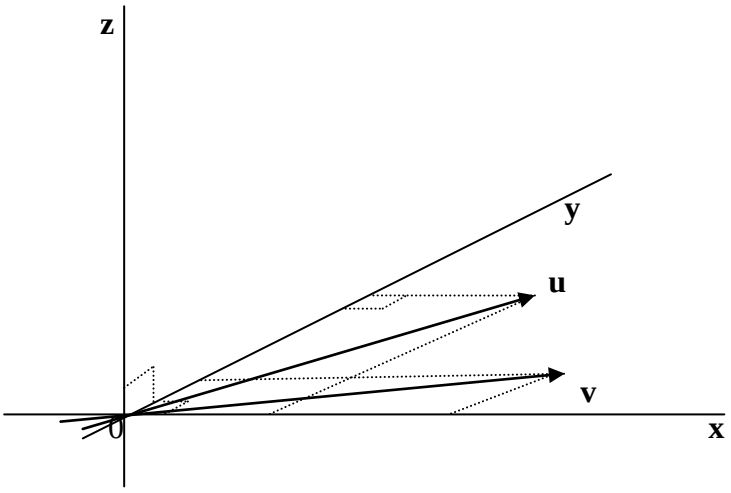
so that it is evident in this case that it might be interesting to consider the following different type of scalar product

$$\mathbf{u.v} = u_1v_1 - u_2v_2 - \ldots - u_nv_n.$$

(6) **Vector (Cross) Products.**

It is sometimes stated that vector products in general spaces do not (generally) exist. But in three dimensional Pythagorean space (we will from now on use the technical name of a *orthogonal* space – you will often also commonly hear, if you study

mathematics, the term *Euclidean* space) we certainly can define multiplying two vectors together to form a third. Let us do that right now.



Well, I am cheating a bit here, because I have the vectors **u** and **v** in the (x, y) plane.

I am just going to define the vector (cross) product of the vectors **u** and **v** to be the vector along the axis at right angles to **u** and **v** (which happens in this case to be along the **z** axis) – we have an arbitrary choice of the *handedness* – called the *chirality* – of this product vector (is it going in the up direction or the down direction?). We also have to define its *magnitude*. We are going to say that it is the magnitude of the vector **v** (denoted by $|\mathbf{v}| = \sqrt{[\mathbf{v}.\mathbf{v}]}$) times the magnitude of the component of the vector **u** which is at right angles to **v**. If we call the angle between **u** and **v**, θ, then the magnitude is just

$$|\mathbf{u}||\mathbf{v}|\sin\theta.$$

Our vector cross product is denoted by $\mathbf{u} \times \mathbf{v}$, and we have seen it involves a sine, rather than the cosine in the case of the scalar product. The '$\times$' here is not to be confused with the Cartesian product.

We note that if we change the order of **u** and **v**, then $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ (the handedness changes). This is our first example of a non-commutative operation.

(7) **Quaternions.**

Quaternions are interesting Victorian mathematics, invented by the Irish mathematician William Rowan Hamilton. They are now subsumed in matrix theory, in which they are representable.

Rather than deal with them in that way first, we return to their origins. Quaternions are used today embedded in the mathematical transformations which are the most efficient way of dealing with gyroscopes. They are also a natural extension of complex numbers to higher dimensions, with one important elaboration – they are *non-commutative*.

Instead of having one imaginary number, quaternions are excessive in having *three*. We will denote these by *j, k* and *l*. These satisfy

$j^2 = k^2 = l^2 = -1$,
$jk = -kj = l$, $lj = -jl = k$, $kl = -lk = j$
and consequently
$jkl = -1$.

Multiplication of complex numbers may be considered to happen in a *plane* – sometimes called an *Argand diagram*. A complex number may be represented by
$re^{iA} = r[\cos A + i\sin A]$,
so if r = 1, multiplication of two complex numbers is defined by the circular rotation
$e^{i(A+B)} = e^{iA}e^{iB}$,
where A and B are the *angles* (in *radians*) that the complex numbers make with the real axis.

Multiplication of quaternions happens by a similar process, except we are dealing with four dimensions, rather than the two of complex numbers. That is, a quaternion may be represented by the number
$re^{jA+kB+lC} = r[\cos A + j\sin A][\cos B + k\sin B][\cos C + l\sin C]$,
which can all be multiplied out using the above relations. Thus the general multiplication of quaternions with r = 1 is represented by rotations of a sphere.

We mentioned before that quaternions may be represented by matrices – e.g. 4 X 4 real matrices.

For multiplying two matrices, the element in the rth row and sth column of the result is obtained from multiplying successive elements of the rth row of the first matrix by successive elements of the sth column of the second matrix.

We first note that a complex number a + ib may be represented by the 2 X 2 matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

For a quaternion c + *j*d + *k*e +*l*f, the corresponding matrix is

$$\begin{pmatrix} c & d & -f & e \\ -d & c & -e & -f \\ f & -e & c & d \\ e & f & -d & c \end{pmatrix}.$$

What you do not need to know. We mentioned before the scalar product of vectors, and how this idea, commonly expressed, differs in special relativity from the usual orthogonal group type of scalar product.

Well, another way of looking at relativity, is to consider *time as a real variable*, and the *three space directions as* (*quaternionic*) *imaginary variables*. So, when measuring along, say, the x axis, we now assume we are measuring along the *j*x axis, when we are measuring y we are measuring along the *k*y axis, and likewise for z, being *l*z. Then the special relativistic *line element* – the equivalent of the Pythagorean theorem – is now expressed as the quaternionic (positive) *norm*
$s^2 = c^2t^2 + x^2 + y^2 + z^2$,
or expressed as a relativistic line element

$$s^2 = c^2t^2 - (jx)^2 - (ky)^2 - (lz)^2,$$

since we have $j^2 = k^2 = l^2 = -1$. We can then get out what are known as the *Lorentz transformations* for special relativity using this idea, i.e. expressions for ct and jx etc. in terms of the quaternionic velocity jv = d(jx)/dt. These are

$$t' = [t + (jv/c^2)jx]/\sqrt{[1 + (jv)^2/c^2]}$$
$$= [t - (v/c^2)x]/\sqrt{[1 - v^2/c^2]}$$

and

$$jx' = j[x - vt]/\sqrt{[1 + (jv)^2/c^2]}$$
$$x' = [x - vt]/\sqrt{[1 - v^2/c^2]},$$

the quaternionic and Lorentz equations being *identical*, not similar.

Another interesting comment on quaternions, is that their algebra is identical to that of four dimensional *spinors*. This is of interest in physics, because the relativistic electron in quantum mechanics is described by a Dirac spinor.

(8) **Covariance and Contravariance.**

Not many people go through life unable to distinguish between right and left, but there are many who will never learn the difference between covariance and contravariance. Yet these are important concepts. Indeed the laws of physics, according to Einstein, are expressed in ways that are generally covariant (the reason for this is these laws describe energy, in particular potential energy, and potential energy has characteristics of behaving covariantly, as we will see in the given related example). In order to understand how these laws are formulated, it is a significant step to understand what these concepts mean.

There are two cultures – that of physics, and that of mathematics, in expressing the ideas of covariance and contravariance. We deal with the ideas as usually expressed by physicists first, which is more specifically tied to differential geometry, then we deal with the mathematical aspect, and see how these two approaches are related. The mathematical aspect I have put in a 'what you do not need to know' section.

Covariance will be described with respect to the transformations you get when you change the coordinate system viewing a tank of water with static but variable temperatures within it. Contravariance is described in terms of the tank with a (*velocity*) *flow* of water within it, and the type of transformations you get, for example, when you change a coordinate system 'going with part of the flow'.

To get an intuitive idea of the difference between invariance, covariance, and contravariance, suppose we have an aquarium tank filled with water, and we define rectangular Cartesian coordinates (x, y, z) to identify each point in the tank. We could now express the temperature of the water at each point by the function T(x, y, z). This is just a single number associated with each point in the tank, representing the temperature (let's say in °C) at that point.

Now suppose we change our minds and decide to use polar coordinates (r, θ, φ) instead of rectangular coordinates (x, y, z). These new coordinates are known functions of the original coordinates r(x, y, z), θ(x, y, z) and φ(x, y, z).

Clearly the value of T is "invariant" with respect to changes in the coordinate system. Thus, the temperature $T(r, \theta, \varphi)$ in polar coordinates is related to the temperature $T(x, y, z)$ in rectangular coordinates by

$$T(r, \theta, \varphi) = T(x, y, z)$$

where r, θ and φ are each functions of x, y and z. This means the value of T at a given point is the same, regardless of the coordinate system we choose.

However, suppose we had determined the *gradient* $G(x, y, z)$ of the temperature at each point. This is a vector at each point (x, y, z) with the components

$$G\_1(x, y, z) = \partial T/\partial x$$
$$G\_2(x, y, z) = \partial T/\partial y$$
$$G\_3(x, y, z) = \partial T/\partial z,$$

where these are partial derivatives.

Now suppose we want to convert the gradient to polar coordinates. If the gradient was invariant with respect to coordinates changes we would expect the components to be unchanged at any given point. That is, we would expect the components of the gradient to be given by

$$G\_i(r, \theta, \varphi) = G\_i(x, y, z)$$

for i = 1, 2, 3. However, that's clearly not the case, because the components of the temperature gradient with respect to polar coordinates are

$$G\_1(r, \theta, \varphi) = \partial T/\partial r$$
$$G\_2(r, \theta, \varphi) = \partial T/\partial \theta$$
$$G\_3(r, \theta, \varphi) = \partial T/\partial \varphi,$$

and these represent the derivatives of temperature with respect to the polar coordinates, not with respect to the Cartesian coordinates. Thus, the components of the gradient at a given point clearly depend on the coordinate system we are using.

Fortunately, it is still possible to express the components of G_polar in terms of the components of G_xyz, but we need to take into account the relation between the polar coordinates and the Cartesian coordinates. For example, the conversion for G_1 looks like this

$$G\_1(r, \theta, \varphi) = \partial x/\partial r[G\_1(x, y, z)] + \partial y/\partial r[G\_2(x, y, z)] + \partial z/\partial r[G\_3(x, y, z)],$$

where r, θ and φ are each functions of x, y, z (the derivatives are partials).

Entities like the temperature gradient whose components transform according to this kind of rule are called *covariant.*

Finally, suppose the water in the tank is not perfectly motionless, but has some velocity at each point. Like the gradient, this is a vector at each point, and it has the components

$$V\_1(x, y, z) = \partial x/\partial t$$
$$V\_2(x, y, z) = \partial y/\partial t$$
$$V\_3(x, y, z) = \partial z/\partial t$$

where t stands for time.

It is worthwhile to compare these components with those of the temperature gradient considered previously. With the gradient we had $G\_1(x, y, z) = \partial T/\partial x$, whereas with the velocity we have $V\_1(x, y, z) = \partial x/\partial t$. So the gradient consists of partial derivatives of some "other" variable (T) with respect to the coordinates, whereas the

velocity consists of partial derivatives of the coordinates with respect to some "other" variable (t).

Like the gradient, the velocity vector is not invariant under coordinate transformations, and the conversion depends on the relation between the two sets of coordinates. However, the conversion has a different form. For example, the first component of the velocity vector in polar coordinates is given by

$$V\_1(r, \theta, \varphi) = \partial r/\partial x[V\_1(x, y, z)] + \partial r/\partial y[V\_2(x, y, z)] + \partial r/\partial z[V\_3(x, y, z)]$$

and similarly for the other two components. Entities that transform from one coordinate system to another according to this kind of rule are called *contravariant*.

This discussion has focused on scalars and vectors, but the same ideas apply to tensors (that is, vectors multiplied by one another) of any order. You can also have "mixed" tensors, which are covariant with respect to some of their indicies and contravariant with respect to others.

At this point people often wonder how we can talk about a vector being contravariant or covariant when the direction and magnitude of a vector (which are its defining properties) are actually invariant with respect to coordinate changes. This question points out a problem with the terminology. People commonly talk about contravariant and covariant vectors and tensors, when they really mean contravariant and covariant *components*. A given velocity vector (for example) has whatever direction and magnitude it has, independent of the coordinate system we use to express it. So it is true that the velocity vector itself doesn't change when we switch coordinate systems. However, the components of the vector change.

For example, suppose we have a velocity vector in the plane with components (1,1) relative to a particular x, y coordinate system. This vector has a magnitude of $\sqrt{2}$ and is pointing at 45 degrees up from the x axis. However, if we rotate the coordinate system about the origin so that the x-axis lines up with the vector, it now has coordinates ($\sqrt{2}$, 0). Notice that it's magnitude is still $\sqrt{2}$ because the magnitude is “invariant”, but the components of the vector are different. We didn't change the direction of the vector, we changed the orientation of the coordinate system, so the components of the vector had to change accordingly.

Now, if we accept that the components of a tensor have to change when we change coordinate systems, we might still wonder why they change differently depending on whether they are contravariant or covariant. The distinction between these two kinds of components is a bit subtle.

Essentially, contravariant components are directed PARALLEL to the coordinate AXES, whereas covariant components are directed NORMAL (perpendicular) to constant coordinate SURFACES. Of course, in the case of orthogonal Cartesian coordinates the axes are, by definition, normal to constant coordinate surfaces, so the distinction between contravariant and covariant components vanishes.

What you do not need to know. We now come to the mathematical description of covariance and contravariance. The main problem in conception is to show that the ideas that map onto descriptions used in physics are the same as those used by mathematicians.

I am going to start off with examples of covariance and contravariance – using matrices. Square matrices are arrays of n rows and n columns of numbers. If I denote a matrix by A, then in terms of its 'elements' – each item in a specific row or column – the matrix is written usually as a_{ij}. This means that element a_{ij} sits on the ith row and jth column. Two matrices are added by adding their elements – so we could write

$$A + B = a_{ij} + b_{ij}.$$

Matrices A and B are multiplied for the ith row and jth column by adding in sequence each element of the ith row multiplied by the corresponding element of the jth column, so if the first row of matrix A was (1, 2) and the first column of matrix B was [3, 4], then the first row and first column of matrix AB is 1x3 + 2x4 = 11. In general we will write

$$AB = a_{ik}b_{kj},$$

with the understanding that the repeated subscript k indicates we sum over k – this is known as the Einstein summation convention.

We can then form from matrix A, an inverse matrix A^{-1}. The inverse matrix multiplied by the original matrix becomes the unit diagonal matrix. So if $A = a_{ij}$, then $A^{-1}A = \delta_{ij}$, where $\delta_{ij} = 1$ when i = j, otherwise it is 0. Multiplication of matrices is covariant, because the mapping which here is an equality satisfies

$$AB = AB$$

and multiplication of inverse matrices is contravariant, because

$$(AB)^{-1} = B^{-1}A^{-1},$$

that is, contravariant multiplication of inverse matrices inverts the order of composition. This is the meaning of the essential difference between covariance and contravariance.

To deal with the mathematical language we need to go into the idea of a category in more detail. So I am going to give a set of axioms, or assumptions, which category theory follows. Category theory gives an overarching set of ideas that are supposed to hold for most mathematical objects. As we said before, category theory deals with transformations which it calls morphisms in its technical language.

So once again (a function is a type of morphism anyway), we are going to represent our morphisms in terms of the notation for functions. Let us look at the function

$$y = f(x) = x^2$$

as an example. There is a bit of subtle ambiguity here. We start off with a set of variables called x. This is the *domain* of the function or morphism f(x). We then multiply x by itself to get x^2, and that then belongs to a new set call the *codomain* of the function or morphism. We use the same symbol, f(x) for the codomain, and for the transformation itself, the morphism f(x) mapping the domain to the codomain.

We then state that we can *compose* two morphisms together, and that makes sense.

I now introduce the first disappointment for me in category theory: morphisms are *associative*. This means for morphisms h, then applying g, then applying f – yes, we are writing morphisms from right to left. I cannot understand this because I am dyslexic in that aspect, but this is a universal convention –

$$f(g(h(x))) = (fg)h(x).$$

This means that brackets don't matter. Well, there are some exceptions to that in mathematics (for example subtraction, division and exponentiation), as operations that aren't associative.

A fairly natural axiom is that there is a unit morphism, denoted by 1

$$1\ f = f, \text{ and } g\ 1 = g.$$

Well, that's it – the definition of a *category*. I hope it wasn't too painful, and we have been through these types of axioms before, so the pain should have been prior to this.

To put it another way, a category is a collection of elements in a domain and elements in a codomain, with morphisms between them satisfying the above rules.

Mathematicians like the idea that you can get rid of elements in the above description, so an element is said to be the same thing as a morphism (of an element to itself). I don't like that idea, because I assert that elements exist, and there is part of physics that adheres to a logical positivist philosophy that if you have described the transformations OK, that's all there is, and I just believe that is *wrong*, it gives the wrong physics (because there is a ground state where nothing is there, and these ground states exist, but the pure transformational approach doesn't know how to deal with them).

Before, I introduced the idea of *duality*. The idea of duality is just this: for each morphism, interchange the domain and codomain, and the direction of the morphisms, and then you have a *dual category*. Categories are often given the letter C to describe them. The dual, or *opposite* category, is often denoted C^{op}.

You might suspect if you have a theorem about a category, then there is a dual theory about the opposite category, and if this isn't due to non-categorical reasons, that is indeed the case.

Please note that for two morphisms f and g,

$$f^{op}\ g^{op} = (gf)^{op},$$

inverting the order because $f^{op}\ g^{op}\ g\ f$ is the unit morphism 1.

We can now go on to the next stage in category theory, which will allow us to talk about covariance and contravariance. It is the idea of a *functor*. A functor is a *morphism between categories*. So for example, the morphisms in a category C can be mapped to morphisms in a different category, B.

A *covariant functor*, S, maps morphisms f, g in C to (say) morphisms f, g in C so that

$$S(fg) = S(f)S(g).$$

If a *functor*, U, maps morphisms $f^{op}\ g^{op}$ in C^{op} to (say) morphisms f, g in C so that

$$U(f^{op}\ g^{op}) = U(f)U(g),$$

then considered as a mapping from C rather than to C^{op} to C, then the new functor T rather than U is a *contravariant functor*. Because $f^{op}\ g^{op} = (gf)^{op}$, inverting the order, the contravariant functor inverts the order also:

$$T(gf) = T(f)T(g),$$

and this is the key descriptive fact about contravariant functors, they invert the order of composition.

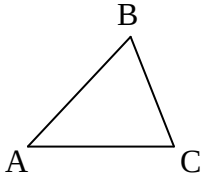
(9) **Varieties and manifolds.**

The axioms of flat Euclidean space contain a rule that is complicated to state precisely, it is the axiom that says that parallel lines do not meet, or meet at infinity. The ancients tried to develop an equivalent geometry without this axiom, but were unable to do so. Gauss, Bolyai and Lobachevsky developed geometries where the axiom fails, and these are not equivalent to Euclidean Geometry.

An example is a sphere in which straight lines – the shortest (or longest!) distance between two points – are great circles dividing the sphere into two equal hemispheres. Then a triangle on this surface consists of segments of three great circles, and the sum of the angles at the tangent plane of each vertex of the triangle sums to greater than the two right angles of Euclidean space.

These ideas were developed further by Riemann and Poincaré.

The idea of a metric space is that there is something called the distance between two points. It satisfies the triangle inequality:



For a metric space the distance AC is less than the distance AB + the distance BC. The metric does not have to be Pythagorean.

For a space given by coordinates (introduced by Descartes) of real numbers (a, b, ... c) we can always introduce a Euclidean metric if we wish. This means the product space satisfies an n-dimensional Pythagoras theorem – we call the space without necessarily the metric the Cartesian product R x R x ... x R, where a belongs to the real set R, etc. Spaces locally R^n are called manifolds (after the French variété, meaning manifold).

An embedding theorem due to Hassler Whitney states that, as we saw before slightly differently with the tangent space to a circle, any n-dimensional surface may be embedded in a space like the Cartesian product space above, with 2n dimensions. This is true however warped the space is, and even if it has singularities like the one below:

This means we can always have an induced metric on a curve or n-dimensional surface. Spaces with hypersurfaces given by equations, like say:

$$x^3 + y^4 + z^2 = c,$$

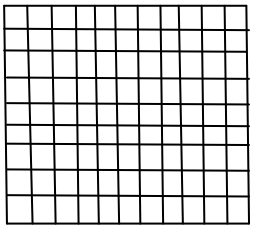
are called varieties – perhaps deliberately mistranslating the French. Sometimes we want something not like, say, a circle, but a disk, in which case we can describe the variety by an inequality rather than an equals sign:

$$x^2 + y^2 \leq c.$$

Intersections of hypersurfaces given by equations or inequalities like the above are known as equalisers.

Conversely, for a variety with a metric, we can apply a forgetful mapping and just end up with the curve or hypersurface without a distance function.

A space does not have to be locally flat. If we describe a space by a mesh



then a handle can be glued to each pair of squares of the mesh. It is then possible to contract the gridlines so that in the limit the squares tend to zero length. Likewise a non-oriented Möbius strip, which has boundary a circle, can be attached to each such square and the gridlines made finer to a limit.

A non-Archimedean number is a number which cannot be boosted by multiplication with a whole number to be greater than a chosen whole number. It therefore acts as an infinitesimal, and such a number can be chosen for the length of such squares.

For locally non-flat manifolds in higher dimensions, consider a hypermesh produced by R^n divided into hyperplanes of dimension R^{n-1}. Each resulting hypergrid element then has a boundary of a hypersphere S^{n-1} with corners, with other S^{n-1} intersecting elsewhere. In the oriented case remove in R^n an interior disk D^n in a hypergrid element, leaving a retract of S^{n-1} consisting of two copies at the ends, attach each copy to elements of the hypermesh, and take the limit.

For the non-oriented case, twisted spheres exist, but I am unclear of the general circumstance. Since spheres S^1, S^3 and S^7 are able to contain parallel flows, it is possible to form a retract of those spheres and identify a parallelisation with the parallelisation reversed. This is then an equivalent of a Möbius strip.

Since we have broached the subject, I want to discuss limits, and their relationship to non-Archimedean numbers.

It is sometimes stated that the postulate of the existence of non-Archimeadean numbers can be removed from the real number system, and the result is a consistent standard analysis. However, the statement that non-Archimedean numbers are not present is an axiom in itself.

The Gödel incompleteness theorem asserts that the consistency of the real number system is unprovable, and Gentzen's theorem, which allows infinite proofs, asserts the contrary. I will avoid such questions and develop examples.

For the example of non-Archimedean handles on a manifold given above, we have not yet constructed a manifold which is connected in this way across Archimedean numbers, but we can do so. Let d be a real Archimedean number, and δ_r be non-zero and non-Archimedean. Then connect each end of a handle in position $\delta_1 \times \ldots \times \delta_n$ to the other end in position $(\delta_1 + d) \times \ldots \times (\delta_n + d)$. Then this is non-locally connected.

In the instance in which non-Archimedean numbers are removed, the question can be asked whether the resulting manifold can be defined as connected. Certainly there

exists an interpretation in which the Archimedean real number system is totally disconnected.

We can specify some of the properties of real numbers in two ways. Firstly, they can be represented by non-terminating decimals. Secondly, which we will have to prove, some of them can be represented in no other way than by adding non-Archimedean numbers, which we will call infinitesimals, to other numbers.

In fact, many years ago I argued in the following way. If, between every two real numbers there is a rational number, then the number of real numbers is equal to the number of rationals, which by Cantor's diagonal argument is not the case.

It would be rather bizarre if between every set of real numbers there was a rational number, and that still the number of real numbers is greater than that of the rationals, yet this is what we must infer if we are to discard non-standard analysis.

The problem may be approached as follows. The set of rational numbers is countable. Another way of putting this is that a countable set can be put in one-to-one correspondence with every element of the natural numbers. This is the basis on which finite determinate Turing machines work, and correspondingly algorithms. Algorithmic proofs depend on there being a halting process for such machines. However, what is true is not dependent on there being a proof. Rather than use countability, we can define a mapping of a set to the real numbers in which every element of the set is mapped to every element of the real numbers. Such a mapping is not countable, yet such maps exist, provided we admit the existence of the real numbers at all, for example the identity map of the real numbers to themselves. If we define a new term 'supercountable' for this enumeration, then we cannot prove that it is a contradiction that between every two real numbers there is a rational number, if we use countability, but we can prove a contradiction if we use supercountability. However, the existence of the contradiction is not dependent on the method of proof. Therefore, the non-existence of infinitesimals is a contradiction.

If two real numbers have no rational number between them, they are separated by an infinitesimal, since they are not separated by an interval which can be boosted greater than a rational number. Another way of putting this is that a real number plus an infinitesimal is not separated from the original number by a rational number in between. Thus, if real numbers exist, so do infinitesimals.

The question arises as to whether infinitesimals can be represented by non-terminating decimals. Thus do the recurring decimals 1.0000 ... minus 0.9999 ... equate to an infinitesimal? This clearly satisfies the condition, provided we do not state this is zero, in which case two differing decimal representations equate to the same value. If we state this is not zero, then there exist other real numbers with the same countable decimal representation, but which differ by an infinitesimal.

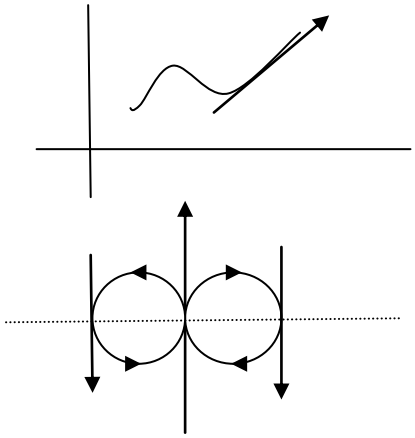
For each non-Archimedean number δ_1, there exists a second level non-Archimedean number δ_2, so that no natural number times δ_2 is greater than δ_1, and so on, by recursion. This was known to Poincaré. A number satisfying this is $\delta_2 = \delta_1^2$.

There are numbers $\Delta_1 = +1/\delta_1$ and $\Delta_r = +1/\delta_r$. No natural number divisor of Δ_1 is less than any given natural number > 0.

These infinitesimals cannot satisfy a generic property. It is not possible to choose all such numbers, because if a choice of all δ_1's were made, by the non-Archimedean property, this would satisfy $\delta_1 \in \{\delta_2\}$, but δ_2 is not included in δ_1. Thus the δ_1's do not form a set formed by a property, which is not always possible anyway.

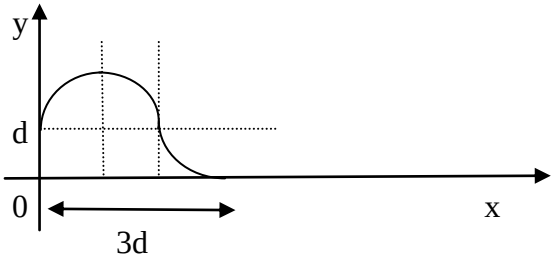
Whereas it is not possible to remove all δ_1's by a property, if one is selected, and a real number r is chosen as a non-terminating decimal, then all other such real numbers are separated by such a δ_1, and the real number system formed from these is discrete.

We will now look at some ways in which tangent vectors have awkward properties on taking the limit. This is important to know, because the gradient of a tangent vector to a curve, shown below, is sometimes said to be evaluated using non-standard analysis.



The above critical point – if extended with many flow lines, that of a magnetic field – has tangents to its flow. All the points of such circles at the horizontal straight line have downwards-pointing tangent vectors, except at the critical point itself, where it is upwards. All flows are continuous, and at the limit all tangents are downwards except one. The question then arises whether the flow is continuous at the limit.

Now consider this function:



The function above is an upper half-circle of radius d at x = 0 to 2d, a lower circle quadrant at x = 2d to x = 3d and a value y = 0 thereafter.

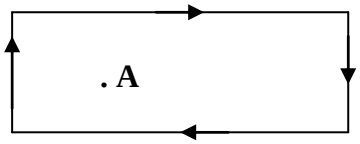
Choosing a positive flow from x = 0 to x > 0, the tangent vector is upwards at x = 0, covers intermediate values from up to down from x > 0 to x = 2d, then smoothly transforms and is finally horizontal from x = 3d onwards. Take the limit $d \to 0$. The vector is upwards at x = 0, not upwards and ranging over values, in non-Archimedean values of d, and is horizontal for x Archimedean > 0.

The only way to get round this difficulty is to say the infinitesimal region is flat, that is, a continuation of the local non-infinitesimal neighbourhood around it. In this case, in order to do differential calculus, we are reduced to using standard methods.

(10) **Stokes' and the Divergence Theorem.**

The theorems of Green, Stokes and Gauss, and the higher dimensional analogues of de Rham cohomology are, I am going to assert, theories of physics and not theorems of mathematics, although they are presented as resulting from axioms. These axioms, which in their differential format are the exterior algebra of Eli Cartan, assume what they set out to prove – that integrals are contour independent. These axioms do not generally hold.

I will note for you the following concern. I assign the following vector diagram for Stokes' Theorem, enclosing a singularity at A.



Stokes' theorem asserts the line integral around a closed curve to a vector field emanating from A, and any other singularities like A internal or external to this loop, or superposed with parallel flows, is zero.

Gauss's theorem is often stated to be a consequence of Stokes' theorem in higher dimensions. It says that the surface integral of a vector field emanating from an interior point A, or by superposition with other internal such points, is independent of the surface, and for a topological sphere with a 2-dimensional surface is a constant related to A times 4π. External points to the sphere or external parallel flows superposed on these flows contribute nothing.

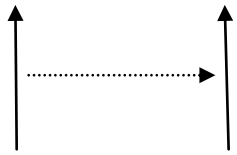
In order to obtain these results there must be assumptions equivalent to rotational symmetry of the flow from A, there being no component of the flow at right angles to it, and to assume the diminution of the flow is reduced by $1/r^{n-1}$ for a distance r from A in a flat Euclidean space with dimension n.

It is possible to add to the vector flows described above flows with other characteristics.

First note that spheres with n-dimensional surfaces, denoted S^n, are parallelisable in the cases when n = 1, 3 and 7 – that is a vector flow can be described on them with no singularities. These parallel flows correspond to multiplication with a constant radius for the complex numbers (n = 1), the quaternions (n = 3) and the octonions (n = 7).

For a central singularity A in two dimensions there can be superimposed concentric circles with constant flow along them, and correspondingly for the 3-sphere and the 7-sphere. The integral along these n-surfaces will not then be zero, violating Stokes' and Gauss's theorems in these dimensions.

There are spaces which are parallelisable which are not S^n. The torus is an example. In general for a vector shown below we can obtain a deformation retract.



This retract can become a square with parallel flows in which the left and right edges, and the top and bottom edges can respectively be identified – this is a parallel flow on a torus. The idea can be extended to an n-dimensional torus. The flow can also wind around the torus, even an infinite number of times.

To conclude, here is an example. Let a central point A have a vector flow in 2 dimensions given by a number r, the distance from A, and an angle of rotation $\theta = 1/(1 - r)$ radians. The flow can be of constant magnitude, it is of infinite extent, yet never enters the region $r = 1$ around which the line integral is zero and is in fact null, but the integral is not zero for $0 < r < 1$.