Note on partitions

by Jim Adams, 2nd June, 2011

(1) Definition.

A partition is a not necessarily strict monotonically decreasing sequence of natural numbers

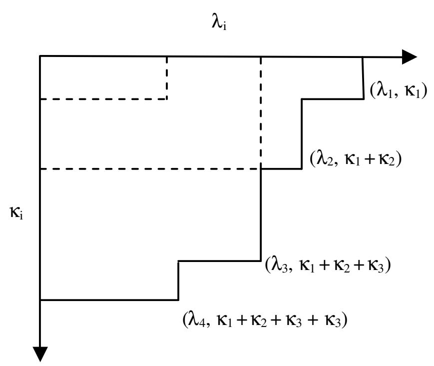
$$\lambda_1, \lambda_2, \ldots \lambda_n.$$

It is a lattice with meet λ_n and join λ_1.

(2) Partition pairs.

We can redefine the above monotonically decreasing partition λ_i to a strictly monotonically decreasing partition defined as the pairs $(\lambda_i, \kappa_i) = (\lambda, \kappa)_i$, where the λ_i are now a strictly decreasing sequence of natural numbers and κ_i is the number of occurrences of λ_i.

For i = 1, ... n, we now define the minimal meet of the $(\lambda, \kappa)_i$ lattice to be the product $\lambda_n\kappa_1$. This can be shown in the diagram



There is a sequence of meets up to the integer part of (n + 1)/2 = int {(n + 1)/2)} = j, given by

$$\lambda_n\kappa_1, \lambda_{n-1}(\kappa_1 + \kappa_2), \ldots \lambda_{n-j}[\Sigma(i = 1 \text{ to } j)\kappa_i].$$

The maximal meet is $\lambda_{n-j}[\Sigma(i = 1 \text{ to } j)\kappa_i]$.

Likewise there is a sequence of joins. The maximal join is $\lambda_1[\Sigma(i = 1 \text{ to } n)\kappa_i]$ and the minimal join is $\lambda_{n-j}[\Sigma(i = 1 \text{ to } j)\kappa_i]$. Thus maximal meet = minimal join.

(3) Notation.

Duality in category theory is: interchange domain and codomain, and reverse arrows. Thus the dual map of a dual map is the original mapping. For partitions, the dual partition of, say, this matrix representation

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

is the transposed matrix.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

To distinguish from the usage with category theory, we will call a dual partition the *transposed partition* here, although a matrix does indeed represent a map.

(4) Homological partition theory.

Partition theory specifies a series of numbers with an order relation. Since these numbers have magnitude, strictly speaking the order relation satisfying

$\lambda_p \geq \lambda_p$,

$\lambda_p \geq \lambda_q$ or $\lambda_q \geq \lambda_p$,

$\lambda_p \geq \lambda_q$ and $\lambda_q \geq \lambda_p$ implies $\lambda_p = \lambda_q$

and

$\lambda_p \geq \lambda_q$ and $\lambda_q \geq \lambda_r$ implies $\lambda_p \geq \lambda_r$

is irrelevant, since this can be provided ready-made by the magnitude of λ_i. Thus rearrangement of λ_i can be put in the same equivalence class as the standard arrangement $\lambda_1, \lambda_2, \ldots \lambda_n$.

For partition pairs (λ_i, κ_i) the $\geq$ relation is replaced by >. We now have a mapping from partition pairs to simplical homology theory, provided we do not put each rearrangement in the same equivalence class. In particular, there is the possibility of oriented partitions, where the number of rearrangements is of even parity, and oppositely oriented partitions, where the number of rearrangements is of odd parity.

(5) Invariants.

Consider an n x n matrix, fully occupied by either a row or a column of 1's. The determinant of the matrix is zero unless the boundary between 1's and zeros is the antidiagonal, in which case it is 1 for n = 4m or 4m + 1 and -1 otherwise.

There is another invariant, the trace of the matrix, which is the sum of the diagonal components. This is maintained under transposed partitions. [Ad12a]

These ideas will be revisited later in terms of the hyperintricate representation.

(6) Possible 2 x 2 partitions.

Consider a 2 x 2 matrix. The possible partitions are

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \quad \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \quad \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

(0) (A) (B) (C) (D) (E)

The null matrix (0), and the (A), (D) and (E) matrices are symmetric. Their intricate representation consists of no i's. The non-symmetric part of (B) is

$$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

which is ½(i + ϕ). Since ½ϕ is symmetric, we can represent the antisymmetry by ½i. We can similarly represent the antisymmetry of (C) by -½i.

(7) Representation of partitions by hyperintricate numbers.
A $2^n \times 2^n$ matrix may represent any partition for large enough n, just add zeros to the right and bottom, that is, fill in the reverse L shape here with zeros.
Its non-symmetric part is a hyperintricate number with an odd number of i's in its basis elements, representing the antisymmetric part of the partition.

(8) Transposed partitions and hyperintricate antisymmetry.
The antisymmetric transposed part of a partition is minus the antisymmetric part of the original partition. This follows from the definition of transpose and the representation of the antisymmetric part of a $2^n \times 2^n$ matrix by a sum of n-hyperintricate basis elements with an odd number of i's.

(9) Invariance of the sum of partition numbers for intricate morphisms.
For intricate numbers, apart from identity mappings, the only morphism of (0) to (E) which preserves the number of 1's of this mapping has symmetric part the meet (A). The antisymmetric part is

$$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \rightarrow \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

or vice versa, so this may be represented, according to the above formalism, by
½i → -½i,
or as the dual mapping.

For the symmetric part, the decomposition into the 1, α and ϕ components may be a relevant calculation.

(10) Antisymmetry for 2-hyperintricate representations.
For 2-hyperintricates, the top left 2 x 2 block can contain (0) to (E) with other blocks the null block.

We can adjoin to (B) on the left matrices (0), (A) or (B) on the right, to (C) on the top left and matrices (0), (A) or (C) on the bottom left, to (D) on the top left matrices (0), (A) or (B) on the top right and (0), (A) or (C) on the bottom left, to block (E) on the top left matrices excluding (C) on the top left and matrices excluding (B) on the bottom left.

There is the question of allocations to the bottom right. Block (0) will always serve, block (A) whenever (C), (D) or (E) is on the top right and at the same time (B), (D) or (E) is on the bottom left.

It is possible to continue with this sort of discussion, and see what the antisymmetries amount to in the 2-hyperintricate representation, and whether there is any systematic generalisation.

(11) The transpose formula.

Consider as an example the partition represented by the matrix

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

There is 1 occurrence of $\lambda_1 = 5$, which in the transpose becomes (5 – 3) occurrences of $\lambda^T{}_5 = 1$. There are 2 occurrences of $\lambda_2 \equiv \lambda_3 = 3$, which transposed becomes (3 – 1) occurrences of $\lambda^T{}_4 \equiv \lambda^T{}_3 = 3 = 1 + 2$, and 1 occurrence of $\lambda_5 = 1$, which becomes (1 – 0) occurrences of $\lambda^T{}_1 = 1 + 2 + 1$.

If we write occurrence of by the multiplication symbol **.**, then the above observations can be written as a transposition mapping

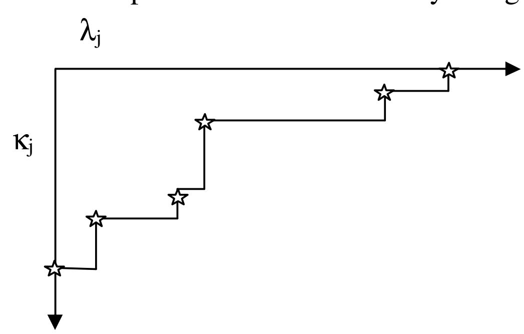
$$\kappa_j.\lambda_j \to (\lambda_j - \lambda_{j+1}).\Sigma(i = 1 \text{ to } j)\kappa_i,$$

and since the number of partitions is invariant under transpositions

$$\Sigma(j = 1 \text{ to } n)\kappa_j.\lambda_j = \Sigma(j = 1 \text{ to } n)(\lambda_j - \lambda_{j+1}).\Sigma(i = 1 \text{ to } j)\kappa_i.$$

(12) Representation of a partition by polynomials.

Partitions can be represented as the boundary of a graph, shown in the diagram below.



We can interpolate the left boundary nodes shown as a star above using a polynomial. The left boundary nodes are one more in number than the right boundary nodes, which they determine. For a partition of 5 objects, if the polynomial is

$$y = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F,$$

then

$$0 = A\lambda_1^5 + B\lambda_1^4 + C\lambda_1^3 + D\lambda_1^2 + E\lambda_1 + F,$$
$$\kappa_1 = A\lambda_2^5 + B\lambda_2^4 + C\lambda_2^3 + D\lambda_2^2 + E\lambda_2 + F,$$
....
$$\kappa_4 + \kappa_3 + \kappa_2 + \kappa_1 = A\lambda_5^5 + B\lambda_5^4 + C\lambda_5^3 + D\lambda_5^2 + E\lambda_5 + F,$$

which we can solve by Gaussian reduction. However, it is possible that the polynomial might attain negative values, even within the (non-strictly) monotonically decreasing range of values of λ_i.

(13) Normal distributions.

It is possible to consider the λ_i occurring as a random distribution, say a normal distribution, so the λ_i would be rearranged to fall within such a sequence.

A normal distribution is symmetric, so it may be possible to monitor the deviation from symmetry by the methods given above – say by representing the standard deviation (although this would represent symmetric deviations also). One might expect ½ of the deviations to be symmetric and ½ antisymmetric. There are also skew-symmetric distributions.

References

Ad12a J.H. Adams, *Hyperintricate rings*, www.jimhadams.com/math/HyperintricateRings.pdf.