Hyperintricate matrices, Foundations

© Jim H. Adams

jim-adams@supanet.com

7th October 2012, revised 19th July 2013

Abstract. In this work I introduce a representation of 2 X 2 matrices called the *intricate* representation, which contains the complex numbers as a subalgebra, and for 2^n X 2^n matrices a corresponding representation called the *hyperintricate* representation.

On the chapter on intricate numbers we develop some of their properties, such as a proof of non-uniqueness of intricate factorisation and non-commutation aspects.

Our investigations are further developed to include a description of hyperintricate numbers – with a section on compression and expansion, a discussion of norms (determinants) and traces, and a method for computing 'J-abelian' hyperintricate determinants, and inverses when these exist. We also discuss roots of intricate numbers, which can be hyperintricate.

Nonassociative algebras are characterised using the hyperintricate methodology.

We discuss division algebras from the point of view of the hyperintricate representation of [Ad12i] and [Ad12j]. A frequently asked question is "what is the relationship between intricate numbers and quaternions?" I answer this, supplying multiple representations of quaternions. We prove using hyperintricate methods that the only associative division algebras are the reals, complex numbers and quaternions. We extend the discussion to algebras which do not have complete division.

We extend Fermat's little theorem to matrices, using the hyperintricate representation. The little theorem is simply extended in the real case and a proof is then given of its extension to intricate numbers (2 × 2 matrices). The intricate Euler totient formula is developed. The Fermat and Euler theorems for n-hyperintricate numbers are introduced, derived simply from the determinant. More explicit formulas are available here in the 'J-abelian' case.

We present the outlines of a theory of polynomial equations for non-commutative algebras, essentially for matrix variables, all in characteristic zero. We deduce that solvability results of Galois theory carry over to 'J-abelian' $2^n \times 2^n$ matrices [Ad12j], and more generally for those hyperintricate polynomials with abelian coefficients.

We explore exponential algebras from the point of view of the intricate and hyperintricate binomial theorem for real powers, and likewise the Euler relations – formulae of $e^{i\theta}$ type, including Euler relations with an arbitrary determinant, a further generalisation. We compare the intricate number $e^{p + (bi + c\alpha + d\phi)K}$ with $e^w e^{xi} e^{y\alpha} e^{z\phi}$.

The classical 2×2 matrix for $i = \sqrt{-1}$ and its intricate extension is the basis of the models with which we will be working. The intricate algebra may be partitioned into those numbers for which the Euler relation

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (*)$$

is of a similar type, chosen to obtain i^i, alternatively this specific Euler relation may not exist within the partition, when models may use 4×4 matrices (hyperintricate numbers), and the complex exponential algebra discussed has $i^i \neq e^{-\pi/2 + 2\pi z}$ and differs from the conventional algebra. The initial proposals, except the non-multifunction non $e^{i\theta}$ models which we will describe by the formulas $g^4 = 1$ and $g = i^i$, exhibit the same multivalued inconsistencies as the conventional approach. None remove the equation $i^i = e^{-\pi/2 + 2\pi z}$ as a possibility by using (*).

Algebras of a completely new type are next introduced. These include the limited proposal $\mathbb{B}$, which corresponds to operations in the projective general linear group PGL(2), but to adopt this would be an admission of failure. The proposal $\mathbb{D}1$ is a complex algebra and eliminates the above i^i possibility. In order to find the natural intricate extension of $\mathbb{D}1$ we introduce $\mathbb{D}3$ but it is a multifunction algebra and must be modified, which gives $\mathbb{D}4$. We then discuss the hyperintricate exponential algebra under proposal $\mathbb{D}1$.

Of significance in the hyperintricate methodology is the existence of two emergent phenomena developed here. The first is the change of intricate basis i, α, ϕ to $\mathcal{JAF}$ format – the triple $\mathcal{J}$, $\mathcal{A}$, $\mathcal{F}$ of Chapter I and [Ad12f]. The second is the important property of being or not being J-abelian. Two hyperintricate numbers which are J-abelian have the property of being consequently relatively abelian. The J-abelian property is shared by all sums with real coefficients of powers of constant intricate numbers, but not all such constant hyperintricate ones, and has an application in the extension of Galois theory to matrix variables [Ad12d].

Keywords: matrices, hyperintricate number, Euler relation, division algebra

Contents

Chapter X
Hyperintricate exponential algebra $\mathbb{D}1$

Chapter XI
Superexponentiation

Foreword

The hyperintricate representation is detailed in this work. It is a representation of real $2^n \times 2^n$ matrices, the 2×2 matrix version of which (intricate numbers) contains as a subalgebra the complex numbers. The confluence of these two ideas – complex numbers and matrices – allows the development of mathematics from both these subject areas and their interaction. Its use in Galois theory for matrix variables, where we have shown that certain solvability criteria carry over from the complex case, is given here and in [Ad12k], an application to Fermat's little theorem for matrices in [Ad12c] is provided here, and applications to division algebras from [Ad12b]. Hyperintricate exponentiation is given in two papers [Ad12e], [Ad12f], the second of which introduces a novel hyperintricate algebra for exponentiation and is recorded here. Hyperanalysis, which extends the ideas of complex analysis, is developed in [Ad12d]. The hyperintricate research programme intersects with what appears to be new ideas on probability sheaves introduced in [Ad12l].

A list of further areas to which I believe hyperintricate numbers can have an influence in the development of topics of interest is: Zeta functions, L series, Fourier analysis, hyperintricate analysis, classification of groups, and noncommutative rings. Areas I have not mentioned are practical applications. It follows that since hyperintricate numbers contain complex numbers on one end and matrices on the other, and these mathematical tools abound in science and engineering, that this is fertile ground for further developments and applications.

CHAPTER I

Intricate numbers

1.1 Introduction.

Complex numbers may be represented by matrices. We develop a full representation of 2×2 matrices, which have 4 components rather than the two of complex numbers. These are called *intricate numbers*, which contain the complex numbers as a subalgebra. We explore the properties of this representation.

1.2 Basic properties of intricate numbers.

A complex number, represented by $g = a1 + bi$, where $i = \sqrt{-1}$, can also be represented by matrices, where

$$1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \qquad i = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

This representation follows all the rules for a *field*, including the existence of a multiplicative inverse g^{-1} of a complex number, satisfying

$$g^{-1} = (a1 - bi)/(a^2 + b^2).$$

If we wish to extend this algebra to include all possible 2×2 matrices with real elements, then we can introduce two more *basis elements* – the *actual* matrix

$$\alpha = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$$

and the *phantom* matrix

$$\phi = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

Just as for complex numbers where we can represent the (a, b) pair of real and imaginary components as vectors in what is called an *Argand diagram*, we can also have a 4-dimensional diagram representing what I call an *intricate number*

$$h = a1 + bi + c\alpha + d\phi.$$

The linearly independent intricate basis elements satisfy

$$\begin{aligned} &1^2 = 1,\ i^2 = -1,\ \alpha^2 = 1,\ \phi^2 = 1, \\ &1i = i = i1,\ 1\alpha = \alpha = \alpha 1,\ 1\phi = \phi = \phi 1, \\ &i\alpha = -\phi = -\alpha i,\ i\phi = \alpha = -\phi i \text{ and } \alpha\phi = i = -\phi\alpha. \end{aligned} \qquad (1)$$

An intricate number can represent uniquely any real 2×2 matrix.

A 2×2 real matrix

$$\begin{Vmatrix} p & q \\ r & s \end{Vmatrix}$$

does not have an inverse if its *determinant* $ps - rq = 0$, in which case it is called a *singular* matrix. For a complex number the basis elements 1 and i have determinant 1 – so all matrices except the zero matrix for a complex number have multiplicative inverses. We can see in contrast that non-zero intricate numbers may have no multiplicative inverse.

In more detail, the matrix above has the intricate representation

$$h = a1 + bi + c\alpha + d\phi$$
$$= ½(p + s)1 + ½(q - r)i + ½(p - s)\alpha + ½(q + r)\phi.$$

The intricate conjugate is $(a1 - bi - c\alpha - d\phi)$. If the multiplicative inverse exists, it is

$$h^{-1} = (a1 - bi - c\alpha - d\phi)/(a^2 + b^2 - c^2 - d^2),$$

so the denominator (the determinant) is non-zero.

A matrix is called *nilpotent* if its nth power is zero, when it is necessarily singular, by determinant multiplication. The singular matrices $(\phi + i)$ and $(\alpha + i)$ have zero square.

1.3 Factorisation of intricate numbers. [Ad12k]

An intricate number may be represented as $1(p1 + qi + r\alpha + s\phi)$, and 1 may be factorised intricately in an infinite number of distinct ways.

All integers have integer valued intricate factorisations in an infinite number of ways. This follows because

$$(a^2 + b^2 - c^2 - d^2) = (a1 + bi + c\alpha + d\phi)(a1 - bi - c\alpha - d\phi)$$

and any integer may be represented for integers a, b, c, d by $(a^2 + b^2 - c^2 - d^2)$, since

$$(a^2 - c^2) = (a + c)(a - c)$$

and if $(a - c) = 1$, $(a + c)$ can have any odd value, thus considering $(b^2 - d^2)$, this can have any odd value, and varying over $(a + c)$ any even integer can be accommodated. Likewise, if $(a - c) = 2$, the product with $(a + c)$ forms an arbitrary multiple of 4, and keeping $(b^2 - d^2)$ odd, any odd integer can be accommodated. □

1.4 Intricate products under non-commutation.

Say we wished to evaluate

$$(a + bi + c\alpha + d\phi)(p + qi + r\alpha + s\phi) = (t + ui + v\alpha + w\phi)(a + bi + c\alpha + d\phi).$$

Then multiplying both right hand sides by the intricate conjugate $(a - bi - c\alpha - d\phi)$ gives, on putting $G = (a^2 + b^2 - c^2 - d^2)$ and equating intricate parts,

$$t = p,$$
$$Gu = q(a^2 + b^2 + c^2 + d^2) + 2[r(-bc - ad) + s(ac - bd)],$$
$$Gv = r(a^2 - b^2 - c^2 + d^2) + 2[q(bc - ad) + s(ab - cd)]$$

and

$$Gw = s(a^2 - b^2 + c^2 - d^2) + 2[q(ac + bd) - r(ab + cd)].$$ □

1.5 The J representation. [Ad12e]

An intricate number $p1 + qi + r\alpha + s\phi = p1 + JK$ satisfies

$$(qi + r\alpha + s\phi)^2 = (\pm qi \pm r\alpha \pm s\phi)^2 = -q^2 + r^2 + s^2.$$ □

When $J^2 = 0$ we obtain for J the parameterisation

$$e^{\rho}[\pm i \pm \cos\sigma\,\alpha \pm \sin\sigma\,\phi],$$

when $J^2 = -1$

$$\pm\cosh\rho\, i \pm \sinh\rho\cos\sigma\,\alpha \pm \sinh\rho\sin\sigma\,\phi,$$

and when $J^2 = 1$

$$\pm\sinh\rho\, i \pm \cosh\rho\cos\sigma\,\alpha \pm \cosh\rho\sin\sigma\,\phi.$$ □

If $J_1^2 = \pm1$ and $J_2^2 = \pm1$, where $J_1 = bi + c\alpha + d\phi \neq J_2 = qi + r\alpha + s\phi$, then it is possible to write

$$J_1J_2 = a + J_3f,$$

where $J_3^2 = 0$ or ±1 and J_3 is intricate. Then $J_2J_1 = (J_1J_2)^*$, the intricate conjugate, and

$$(J_1J_2)(J_2J_1) = \pm1 = (a + J_3f)(a - J_3f)$$
$$= a^2 \pm f^2, \text{ or } a^2 \text{ if } J_3^2 = 0.$$ □

1.6 Composites.

To determine additive composite basis elements, first let

$$J_1 = ui + v\alpha + w\phi, \qquad J_1^2 = -1,$$
$$J_2 = xi + y\alpha + z\phi, \qquad J_2^2 = -1.$$

Then for $1 \geq L, M \geq 0$, provided the denominator is positive

$$J = (J_1L + J_2M)/\sqrt{[(uL + xM)^2 - (vL + yM)^2 - (wL + zM)^2]}$$

satisfies

$$J^2 = -1.$$

If $u > 0$, $x < 0$, a smooth mapping

$$[L, M]: [1, 0] \xrightarrow{t} [0, 1] \qquad (2)$$

carries $(uL + xM)$ inadmissibly through zero, which is also a feature of the complex case $v = w = y = z = 0$, but if u and x are of the same sign, then the positive constraint on the denominator is superfluous. □

Next, multiplicatively, let $\mathcal{J}^2 = -1$, $\mathcal{A}^2 = 1$ and $\mathcal{F}^2 = 1$, where we put

$$\mathcal{J} = qi + r\alpha + s\phi,$$
$$\mathcal{A} = bi + c\alpha + d\phi,$$
$$\mathcal{F} = ei + f\alpha + g\phi,$$

and we allocate

$$\mathcal{A}\mathcal{F} = \mathcal{J}. \qquad (3)$$

Since $\mathcal{J}$ does not have a real part, it follows from the relations

$$-be + cf + dg = 0,$$
$$cg - df = q,$$
$$bg - de = r$$

and

$$-bf + ce = s$$

that

$$\mathcal{A}\mathcal{F} = -\mathcal{F}\mathcal{A} = \mathcal{J}. \qquad (4)$$

Multiplying (3) on the left by $\mathcal{A}$

$$\mathcal{F} = \mathcal{A}\mathcal{J},$$

and multiplying on the right by $\mathcal{F}$

$\mathcal{A} = \mathcal{J}\mathcal{F}$.

Correspondingly, multiplying (4) on the right by $\mathcal{A}$ and the left by $\mathcal{F}$ gives

$\mathcal{F} = -\mathcal{J}\mathcal{A}$,

$\mathcal{A} = -\mathcal{F}\mathcal{J}$,

and we have established an equivalence of algebras for

$\mathcal{J} \leftrightarrow i$,

$\mathcal{A} \leftrightarrow \alpha$

and

$\mathcal{F} \leftrightarrow \phi$.□

CHAPTER II

Hyperintricate numbers

2.1 Introduction.

We develop a representation of $2^n \times 2^n$ matrices by analogy with the intricate case, which is a regular extension of this idea. These are called *hyperintricate numbers*, and we investigate some of the properties of this representation.

2.2 Construction and properties of hyperintricate numbers.

We can define n-*hyperintricate* numbers recursively, by building up starting from intricate ones. Consider a 2^n X 2^n matrix. Let "+" be a chosen 2^{n-1} X 2^{n-1} matrix which is a hyperintricate basis element of lower dimension, for example an intricate basis element 1, i, α or ϕ. Let "–" be the corresponding matrix with all negative entries from "+".

Consider the set of 2^n X 2^n hyperintricate basis elements

$$\begin{Vmatrix} + & 0 \\ 0 & + \end{Vmatrix} \qquad \begin{Vmatrix} 0 & + \\ - & 0 \end{Vmatrix} \qquad \begin{Vmatrix} + & 0 \\ 0 & - \end{Vmatrix} \qquad \begin{Vmatrix} 0 & + \\ + & 0 \end{Vmatrix}$$

Any 2^n X 2^n matrix can be represented uniquely by a linear combination of these.

All n-hyperintricate basis elements beyond the intricate have determinant +1, since the "+" and "–" components both have the same determinant, ±1. The n-indexed determinant is derived from the product of two (n – 1)-hyperintricate determinants, which multiplied together to form the higher dimensional one, has value always +1. We caution that for n > 1, general determinants are *not* additive functions.

A j X j matrix may be extended both right and below with zero entries to give a larger 2^n X 2^n matrix, or main diagonal entries of 1 may be substituted here to maintain determinants. By this means matrix theorems may be expressed hyperintricately.

I now introduce some notation. I will do this by giving examples of 4 X 4 matrices. Write

$$1_1 = \begin{Vmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{Vmatrix} \qquad \alpha_i = \begin{Vmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{Vmatrix}$$

$$i_1 = \begin{Vmatrix} 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} & 0 \end{Vmatrix} \qquad \phi_i = \begin{Vmatrix} 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \end{Vmatrix}$$

So "+" corresponds with the subscript, which will be described as an example of a layer, for example in α_i. Mnemonically 'subscripts are the little part'.

If in general each of the 16 real 4 X 4 matrices are represented by e.g. $\alpha_i = A_B$, then

$(A_B) + (A_C) = A_{(B + C)}$,
$(A_B) + (C_B) = (A + C)_B$,
$(A_B)(C_D) = (AC)_{BD}$,
$A_{-B} = -(A_B) = (-A)_B$.

For further nesting of matrices, consider instead of stepping down a further layer, introducing (possibly) a comma, thus: $A_{B,C}$, so that matrix multiplication becomes

$(AB)_{CD,EF} = (A_{C,E})(B_{D,F})$.

The *layers* of a basis element $m_{n \dots p}$, are the vectors m, n, ... p, and its *layer dimension* is the number of layers.

Intricate and hyperintricate numbers appear in four avatars – as scalars, satisfying a non-commutative algebra, as vectors with a linearly independent basis, as matrices – where the first instance is intricate numbers, and in the hyperintricate case, say as the object similar to a tensor, $m_{n,p}$, where m, n and p are vectors.

Layers may be *permuted*, and except for interior coefficient algebras, and compression and expansion studied later, uniformly applied the resulting algebraic relations under addition and multiplication are the same.

We define, in violation of normal usage, an n-*hypercomplex* number to be an n-hyperintricate number with each layer restricted to the set {1, i}. We can also define *hyperactual* numbers, containing elements of {1, α} in all layers and *hyperphantom* numbers for which every layer ∈ {1, ϕ}. A (hyper)actual or (hyper)phantom number is not a member of a field. Another way of putting this is that complex numbers constitute the only algebra of the three which is analytic. This arises because $(1 + \alpha)$ and $(1 + \phi)$ have determinant zero, and so are singular with no inverse and $(a1_1 + bi_i)$ has inverse $(a1_1 - bi_i)/(a^2 - b^2)$, which does not exist for a = b.

An n-*hyperintricate number is* J-*abelian if* U, V, ... W *are intricate numbers for the layers of an* n-*hyperintricate number* $\Sigma U_{V \dots W}$, *where for each layer the value of* J *is constant* (*but* J *can vary over different layers*), J *is not real and* $J^2 = 0$ *or* ± 1.

2.3 Deriving hyperintricate basis element coefficients from a matrix.

The trace (the sum of diagonal entries) of an intricate 1 is 2, and of α is zero. If we take consecutively the diagonal entries of α, which are (1, -1), and subtract the second entry, -1, we get 2 for α, and applied to 1 the same process now gives 0. For each diagonal matrix, the component of say 1_α, represented by (1, -1, 1, -1) is not zero over its average only on multiplying, for negative values, by -1, and this procedure applied identically to 1_1, α_1 and α_α gives zero. This idea can be generalised to non-diagonal hyperintricate values to obtain general coefficients from the defining matrix.

For example, let Я$_2$ be the 2-hyperintricate number

$$
\begin{aligned}
Я_2 = {} & a_{11}1_1 + a_{1i}1_i + a_{1\alpha}1_\alpha + a_{1\phi}1_\phi \\
& + b_{i1}i_1 + b_{ii}i_i + b_{i\alpha}i_\alpha + b_{i\phi}i_\phi \\
& + c_{\alpha 1}\alpha_1 + c_{\alpha i}\alpha_i + c_{\alpha\alpha}\alpha_\alpha + c_{\alpha\phi}\alpha_\phi \\
& + d_{\phi 1}\phi_1 + d_{\phi i}\phi_i + d_{\phi\alpha}\phi_\alpha + d_{\phi\phi}\phi_\phi.
\end{aligned}
$$

For $Я_2 = r_{jk}$ as elements of a matrix, we consider r_{11}, r_{22}, r_{33} and r_{44} in sequence

$$r_{11} = [a_{11} + c_{\alpha 1} + a_{1\alpha} + c_{\alpha\alpha}]/4$$
$$r_{22} = [a_{11} - c_{\alpha 1} + a_{1\alpha} - c_{\alpha\alpha}]/4$$
$$r_{33} = [a_{11} + c_{\alpha 1} - a_{1\alpha} - c_{\alpha\alpha}]/4$$

and

$$r_{44} = [a_{11} - c_{\alpha 1} - a_{1\alpha} + c_{\alpha\alpha}]/4.$$

Substituting in sequence r_{12}, r_{21}, r_{34} and r_{43} for the above expressions, we obtain equivalent results, by substituting in the second subscript above $1 \to \phi$ and $\alpha \to i$.

Likewise for r_{13}, r_{24}, r_{31} and r_{42}, equivalent results are obtained via $a \to d$, $c \to b$, and for the first subscript $1 \to \phi$ and $\alpha \to i$.

For r_{14}, r_{23}, r_{32} and r_{41} in sequence, the substitutions are $a \to d$, $c \to b$, and for both the first and second subscripts $1 \to \phi$ and $\alpha \to i$.

To obtain a_{11}, $c_{\alpha 1}$, $a_{1\alpha}$ and $c_{\alpha\alpha}$ respectively, for respective terms of r_{11}, r_{22}, r_{33} and r_{44}, there is an inverse type of relationship maintaining the signs:

$$a_{11} = [r_{11} + r_{22} + r_{33} + r_{44}]/4$$
$$c_{\alpha 1} = [r_{11} - r_{22} + r_{33} - r_{44}]/4$$
$$a_{1\alpha} = [r_{11} + r_{22} - r_{33} - r_{44}]/4$$

and

$$c_{\alpha\alpha} = [r_{11} - r_{22} - r_{33} + r_{44}]/4$$

extendable to the other cases.

For a general n-hyperintricate matrix, each term is divided by 2^n. □

2.4 Exterior, interior and relative coefficient algebras.

A real number, r, multiplied by a hyperintricate basis element A_B multiplies each element of the matrix by r. Then

$$rA_B = (aA)_{(bB)} \qquad (1)$$

where $ab = r$.

Hyperintricate basis elements may have coefficients acting on the left or right (or both) which are themselves hyperintricate. These coefficients may be considered as a sum of terms of real values multiplied by hyperintricate basis elements.

Generally speaking, there is more than one type of algebra in which the coefficients are multiplied by hyperintricate basis elements. In all cases real components of the coefficient bases are treated as in (1).

An *exterior coefficient algebra* takes the layer dimension, n, of the coefficients and to the m-hyperintricate basis element to which it is attached, appends a basis element of the coefficient to the (say) trailing 1 layers of the m-hyperintricate basis element. The exterior coefficient algebra is commutative with respect to a coefficient and an m-hyperintricate basis element.

The *interior coefficient algebra* extracts layers from the basis element of the coefficient, permutes them in a uniform way and multiplies corresponding layers to those in the m-hyperintricate basis to which it is attached. For identity permutations and coefficients with layer dimension equal to that of basis elements, this corresponds to normal matrix multiplication. The interior coefficient algebra is not commutative in general.

The exterior coefficient algebra may be considered as a special case of the interior coefficient algebra, in which the coefficient is multiplied by trailing layers of 1 in the m-hyperintricate basis element.

The *relative coefficient algebra* operates on all layers rather than selective ones and treats r, a and b in (1) as intricate or hyperintricate numbers. Consider the example $A_B = 1_1$, $r = i$, $a = \alpha$, $b = \phi$, and real numbers $t = uv$. We would have

$$tiA_B = (u\alpha A)_{(v\phi B)} = t\alpha_\phi$$
$$= -(u\phi A)_{(v\alpha B)} = -t\phi_\alpha,$$

which is not the case. A solution not directly involving equivalence classes is to treat r, a and b as conforming to the scalar algebra given in (1.1) and to perform operations relative to the basis A_B. We write to indicate this

$$tiA_B = (uaA)_{(vbB)} \qquad (\text{rel } A_B),$$

then we have for example

$$ti1_1 = (u\alpha 1)_{(v\phi 1)} \qquad (\text{rel } 1_1),$$
$$ti1_1 = -(u\phi 1)_{(v\alpha 1)} \qquad (\text{rel } 1_1).$$

The relative coefficient algebra is not commutative in general.

2.5 Compression and expansion.

The *compression* of a v-hyperintricate number from $2^v \times 2^v$ matrix basis elements to $2^w \times 2^w$ basis elements, where we are compressing $v - w + 1$ vectors, consists in multiplying together in order the vectors to be compressed in the v-hyperintricate algebra.

The compression operation, κ, with abelian addition and non-commutative multiplication, satisfies for basis elements, and correspondingly for composites (we may use here real numbers r and s, although we can incorporate r and s as intricate numbers via an interior or relative coefficient algebra)

$$\kappa(rA_B) = r^2AB,$$
$$\kappa(rA_B + sC_D) = (r^2AB) + (s^2CD),$$

as may be verified using basis element universal objects.

Where B or C are 1 or B = C, we connect compression with matrix multiplication via

$$\kappa(rA_BsC_D) = \kappa(rA_B)\kappa(sC_D) \qquad (2)$$

otherwise for distinct non-real B and C, by non-commutation of basis elements we obtain

$$\kappa(rA_BsC_D) = -\kappa(rA_B)\kappa(sC_D),$$

for example

$$\kappa[(\alpha_\phi)^2] = -\kappa(\alpha_\phi)\kappa(\alpha_\phi) = -i^2 = 1.$$

The zero matrix is compressed to a zero matrix, and the unit matrix to a unit matrix. However $\kappa(rA_0) = 0$, and A_0 is 0, but $\kappa(\alpha_\alpha) = 1$ and $\alpha \neq 1$. Compression is an additive epimorphism from the v-hyperintricate algebra to the w-hyperintricate algebra.

The hypercomplex, hyperactual and hyperphantom algebras commute, so for hypercomplex, hyperactual and hyperphantom numbers, κ is commutative, and (2) always holds.

The equation

$$\kappa(A_BC_D) = \kappa(A_C)\kappa(B_D),$$

which is not multiplicative in the usual sense for homomorphisms, and the definition

$$\kappa(A_B + C_D) = \kappa(A_B) + \kappa(C_D)$$

define a type of ring structure. There is a unit:

$$\kappa(1_1C_D) = 1_1\kappa(1_C)\kappa(1_D) = 1_1\kappa(C_1)\kappa(1_D)$$

and the algebra is distributive:

$$\kappa[U_V(W_X + Y_Z)] = \kappa[U_VW_X] + \kappa[U_VY_Z].$$

There is an opposite operation, κ^{op}, called *expansion*, so that for expansion

$$\kappa^{op}(r^2AB) = rA_B,$$
$$\kappa^{op}[(r^2AB) + (s^2CD)] = (rA_B) + (sC_D).$$

Let B be an intricate number of the form

$$B = b + fJ_1,$$

where $J_1^2 = 0$ or ± 1, and C be of the form

$$C = c + gJ_2,$$

where

$$J_1J_2 = d - J_2J_1.$$

Then

$$BC = bc + fcJ_1 + bgJ_2 + fgJ_1J_2$$
$$CB = bc + fcJ_1 + bgJ_2 + fg(d - J_1J_2)$$

so that for some v, w and pure intricate J_3

$$\kappa(A_BC_D) = \kappa(A_CB_D) + v\kappa(A_D) + w\kappa[(AJ_3)_D].$$

We have proved that for intricate A, B, C and D there exists a v and intricate X such that

$$\kappa(A_BC_D) = \kappa(A_CB_D) + vX, \qquad (3)$$

conversely that for given A, D a selection of B and C can be made so that for arbitrary v and X, (3) holds. □

We can extend this type of notion of compression *for intricates down to reals* by taking the determinant of the matrix basis elements. First note that the number 1 we have been using is in fact a diagonal 2 X 2 matrix. To distinguish this from its real value elements, denote the latter occasionally by 1~.

We can now compress intricate basis elements down to ±1~. We have the mappings

$$1 \to 1\sim,\ i \to 1\sim,\ \alpha \to -1\sim \text{ and } \phi \to -1\sim,$$

where we denote this compression mapping by λ, so that the determinant

$$\lambda(r1 + s\alpha + t\phi + ui) = [(r + s)(r - s) - (t + u)(t - u)1\sim$$
$$= (r^2 - s^2 - t^2 + u^2)1\sim,$$

and therefore we have proved

$$\lambda(r1 + s\alpha + t\phi + ui) = \lambda(r1) + \lambda(s\alpha) + \lambda(t\phi) + \lambda(ui).$$

However, for the real component, and except possibly for sign similarly for i, α and ϕ

$$\lambda[(r_1 + r_2)1] = \lambda(r_1 1) + \lambda(r_2 1) + 2r_1r_2 1\sim = (r_1 + r_2)^2 1\sim.$$

The expansion λ^{op} is defined in like manner to κ^{op}.

2.6 The hyperintricate trace, norm and layer algebra.

Let 1, T = a1 + bi + cα + dϕ and U, V, W be intricate matrices. The norm or determinant, det, which is additive for intricate numbers when every basis element for T is linearly independent of those for U: det(T + U) = det(T) + det(U), and trace operators, tr1 = 2a, tri = 2bi, trα = 2cα and trϕ = 2dϕ, satisfy

det(a = 1, b = 0, c = 0, d = 0) = 1, det(a = 0, b = 1, c = 0, d = 0) = 1,
det(a = 0, b = 0, c = 1, d = 0) = -1 and det(a = 0, b = 0, c = 0, d = 1) = -1
tr1(a = 1, b = b, c = c, d = d) = 2, tri(a = a, b = 1, c = c, d = d) = 2i,
trα(a = a, b = b, c = 1, d = d) = 2α and trϕ(a = a, b = b, c = c, d = 1) = 2ϕ
det(TU) = det(T)det(U)
tr1(T + U) = tr1(T) + tr1(U), etc.
det(tr1(T)) = tr1(T), det(tri(T)) = -tri(T)i,
det(trα(T)) = -trα(T)α, det(trϕ(T)) = -trϕ(T)ϕ
tr1(det(T)) = det(T), tri(det(T)) = 0
trα(det(T)) = 0 and trϕ(det(T)) = 0. □

The hyperintricate layer operator $\veebar$ behaves as a tensor product with extra structure, where $T_U = T\veebar U$ satisfies [JL09]

tr1(T$\veebar$U) = tr1(T)tr1(U),
tri(T$\veebar$U) = -tri(T)tri(U)i, etc.
(T$\veebar$1)(1$\veebar$U) = T$\veebar$U
det(T$\veebar$1) = det(T)
det(1$\veebar$U) = det(U),

so

det(T$\veebar$U) = det(T)det(U),

whereas the relation

(T + U) $\veebar$ (V + W) = (T$\veebar$V) + (T$\veebar$W) + (U$\veebar$V) + (U$\veebar$W)

can be used with the determinant of the above for linearly independent basis elements between T and U, and also between V and W respectively, or to extend this further, if the basis elements 1, $\mathcal{J}$, $\mathcal{A}$, $\mathcal{F}$ of [Ad12g] between T and U and similarly 1, $\mathcal{J}'$, $\mathcal{A}'$, $\mathcal{F}'$ for V and W are linearly independent. Then

det[(T + U) $\veebar$ (V + W)] = [det(T) + det(U)][det(V) + det(W)],

and by enumeration in the same circumstances

det[T$\veebar$V + U$\veebar$W] = det(T)det(V) + det(U)det(W),

the above generalised accordingly for k layers. □

2.7 J-abelian hyperintricate determinants and inverses.

Bourbaki writes [Bo73] in the historical note in *Algebra* I, “Toeplitz ... makes the fundamental observation that the theory of determinants is not needed to prove the principal theorems of linear algebra”. We now introduce the J-layered approach to the hyperintricate representation, developing this representation to accommodate the determinant structure of linear algebra in the J-abelian case.

For intricate numbers A, B, ... H, the determinant of $A = a1 + bi + c\alpha + d\phi$ satisfies

$$\det A = AA^*,$$

where A* is the intricate conjugate

$$A^* = a1 - bi - c\alpha - d\phi.$$

Since the 2-hyperintricate

$$A_B C_D = (AC)_{(BD)},$$

the compression of these satisfies

$$\kappa(A_B C_D) = ACBD.$$

Thus the determinant of the above satisfies

$$\det[\kappa(A_B C_D)] = \det A \det C \det B \det D$$
$$= \det[\kappa(A_B)]\det[\kappa(C_D)].$$

Compression also satisfies

$$\kappa(E_F + G_H) = \kappa(E_F) + \kappa(G_H)$$
$$= EF + GH,$$

which may be thought of as a definition, so that κ defines a ring epimorphism.

Determinants are not additive and also in general

$$\det(E_F + G_H) \neq \det[\kappa(E_F) + \kappa(G_H)]. \;\square$$

A J-abelian hyperintricate number is not always representable by $A_{B \ldots D}$. But since

$$A_B = (A_1)(1_B),$$

it follows in this case

$$\det(A_B) = \det[\kappa(A_B)]$$
$$= \det A \det B.$$

For such numbers

$$(A_B)(A^*_{B^*}) = (AA^*)(BB^*)1_1,$$

which implies

$$(A_B)^{-1} = (A^*_{B^*})/[(AA^*)(BB^*)].$$

For a 2-hyperintricate number that is representable by $A_B + C_D$, then

$$[A_B + C_D]^{-1} = [A_B(1_1 + A^{-1}C_{B^{-1}D})]^{-1}$$
$$= (1_1 + A^{-1}C_{B^{-1}D})^{-1}(A_B)^{-1},$$

for which the first factor can be evaluated by a standard binomial expansion, given that terms commute.

This idea may be extended to more terms and to n-hyperintricate numbers. □

The second idea is to represent each J-abelian n-hyperintricate number as a sum of products, where each layer of the product, $1 \leq k \leq n$, is given by the intricate number $(a_{rk}1 + J_kL_{rk})$, where $J_k = b_ki + c_k\alpha + d_k\phi$ and $J_k^2 = 0$ or ± 1.

The n-hyperintricate representation has 4^n independent components, but for $n > 1$ a J-abelian number has less than this, so for this representation to work, a general n-hyperintricate satisfies 4^n relations and the number of independent variables in J_kL_{rk} is 3 for each layer, and a_{rk} makes 4, thus the number of sums is ~ $4^{n-1}/n$.

We will represent this n-hyperintricate number by

$$Я_n = \Sigma(r = 1 \text{ to } \lceil 4^{n-1}/n \rceil) \underline{\vee}(k = 1 \text{ to } n)(a_{rk}1 + J_kL_{rk}),$$

where we are using the ceiling function, $\lceil 4^{n-1}/n \rceil$, and the composite layer operator $\underline{\vee}$.

For each layer we select the value of $J_k \in \{i, \alpha, \phi\}$, or its corresponding $\mathcal{JAF}$ format $J_k \in \{\mathcal{J}, \mathcal{A}, \mathcal{F}\}$, and with $J_k \neq \pm 1$ for any layer, where the J_k are identical over r and independent over k.

We are now able to introduce, for an intricate number $X = (a1 + bJ_k) + (c1 + dJ_k)$ a type of conjugate, $X^{k\sim}$, so that

$$X^{k\sim} = (a1 - bJ_k) + (c1 - dJ_k),$$

which implies that $XX^{k\sim}$ is real. On expanding out $Я_n$, the kth layer is selected so that

$$Я_n^{k\sim} = \Sigma(r = 1 \text{ to } \lceil 4^{n-1}/n \rceil)$$
$$\underline{\vee}\,(m = 1 \text{ to } n)(a_{rk}1 + J_kL_{rk})(\text{except } m = k \text{ with})(a_{rk}1 - J_kL_{rk}).$$

For each k, $Я_nЯ_n^{k\sim}$ is real in layer k.

Let $Y^1 = Я_nЯ_n^{1\sim}$ and $Y^k = Y^{k-1}(Y^{k-1})^{k\sim}$. We are able to form a real value from the product

$$Y^n = Я_nЯ_n^*,$$

where we have introduced the n-hyperintricate conjugate $Я_n^*$.

The mth power Δ^m of the determinant Δ of $Я_n$, is a multiplicative function. We will allocate

$$Я_nЯ_n^* = \Delta g,$$

where g is a factor.

When $Я_n$ is a real number times a hyperintricate basis element, or a $\mathcal{JAF}$ format extension of this, the value of this factor is 1. If this factor is a constant it is always 1, as can be shown since Δg is a multiplicative function. But if g is not a constant, then the degree of the powers of compressed hyperintricate components in $Я_nЯ_n^*$, 2^n, is not equal to the degree of Δ in its hyperintricate matrix representation, which is not the case. Thus the inverse of $Я_n$ when it exists is $Я_n^*/\Delta$. □

2.8 The general hyperintricate inverse.

If a block diagonal of a 2-hyperintricate number is $1_{Я1} + \alpha_{Я2}$, where $Я_1$ and $Я_2$ are intricate numbers, then represented by a matrix

$$\begin{Vmatrix} A & 0 \\ 0 & D \end{Vmatrix},$$

$A = Я_1 + Я_2$ and $D = Я_1 - Я_2$, similarly an antidiagonal $\phi_{Я3} + i_{Я4}$ for the matrix

$$\begin{Vmatrix} 0 & B \\ C & 0 \end{Vmatrix},$$

satisfies $B = Я_3 + Я_4$ and $C = Я_3 - Я_4$.

By an algorithm of Boltz-Banachiewicz [Be05], the inverse of the matrix

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix}$$

given by

$$\begin{Vmatrix} E & F \\ G & H \end{Vmatrix}$$

satisfies

$$E = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1},$$
$$F = A^{-1}B(D - CA^{-1}B)^{-1},$$
$$G = -(D - CA^{-1}B)^{-1}CA^{-1},$$
$$H = (D - CA^{-1}B)^{-1},$$

where A, B, C and D are matrix sub-blocks of arbitrary size, and A and D are square matrices so that they can be inverted. $D - CA^{-1}B$ is the Schur complement of A. Thus for n-hyperintricate numbers this operation can be defined recursively. □

We can define the hyperintricate conjugate X* of a hyperintricate number X by the formula

$$XX^* = \det X,$$

and this works when X is singular.

To go into more detail, the determinant of a matrix A satisfies

$$\det A = \textstyle\sum_{\sigma}(-1)^{N_\sigma}\prod_{i=1}^{n} A(i,\sigma(i)),$$

where the sum is taken over all N! permutations σ, (σ(1), ... σ(n)) of the column indices 1, ... n, and where Nσ is the minimal number of pairwise transpositions needed to transform σ(1), ... σ(n) to 1, ... n.

The determinant is multiplicative:

$$(\det P)(\det Q) = \det (PQ)$$

and we will need from the definition of the column expansion of a determinant

$$\det\begin{Vmatrix} 1 & X \\ Y & Z \end{Vmatrix} = \det(Z - YX).$$

Then since

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} A & 0 \\ 0 & 1 \end{Vmatrix} \begin{Vmatrix} 1 & A^{-1}B \\ C & D \end{Vmatrix},$$

$$\det\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = (\det A)(\det (D - CA^{-1}B)).$$

We can also write

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} A & BD^{-1} \\ C & 1 \end{Vmatrix} \begin{Vmatrix} 1 & 0 \\ 0 & D \end{Vmatrix},$$

so

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} A & 0 \\ 0 & 1 \end{Vmatrix} \begin{Vmatrix} 1 & A^{-1}BD^{-1} \\ C & 1 \end{Vmatrix} \begin{Vmatrix} 1 & 0 \\ 0 & D \end{Vmatrix}.$$

If the intricate conjugate of XY is (XY)*, where A, B, C, D, X, Y are intricate numbers, then

$$Z = \begin{Vmatrix} 1 & X \\ Y & 1 \end{Vmatrix} \begin{Vmatrix} 1 & -X \\ -Y & 1 \end{Vmatrix} = \begin{Vmatrix} 1 - XY & 0 \\ 0 & 1 - YX \end{Vmatrix},$$

and $\det Z = \begin{Vmatrix} 1 - XY & 0 \\ 0 & 1 - YX \end{Vmatrix} \begin{Vmatrix} 1 - (XY)^* & 0 \\ 0 & 1 - (YX)^* \end{Vmatrix}.$

Since $(PQ)^{-1} = Q^{-1}P^{-1}$, putting $X = A^{-1}BD^{-1}$ and $Y = C$

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix}^{-1} = (\det Z)^{-1} \begin{Vmatrix} 1 & 0 \\ 0 & D^{-1} \end{Vmatrix} \begin{Vmatrix} 1 & -A^{-1}BD^{-1} \\ -C & 1 \end{Vmatrix} \times$$
$$\begin{Vmatrix} 1-(CA^{-1}BD^{-1})^* & 0 \\ 0 & 1-(A^{-1}BD^{-1}C)^* \end{Vmatrix} \begin{Vmatrix} A^{-1} & 0 \\ 0 & 1 \end{Vmatrix}. \square$$

We can define the hyperintricate conjugate X* of a hyperintricate number X by the formula

$XX^* = \det X$,

and this works for an equivalence class of X* when X is singular. □

2.9 Symmetric, antisymmetric and upper triangular matrices.

A matrix U is *symmetric* when its elements satisfy $u_{jk} = u_{kj}$, and the elements of a matrix V are *antisymmetric* when $v_{jk} = -v_{kj}$. Any matrix W may be represented uniquely as W = U + V.

A $2^n \times 2^n$ matrix V is antisymmetric when all of its hyperintricate components are antisymmetric. This follows from the uniqueness of the hyperintricate representation. The square of a symmetric basis element is 1 and of an antisymmetric basis element is -1. So this antisymmetry happens when a search of the layers of a basis element finds an odd number of i's, otherwise the basis element is symmetric. □

The *transpose* W^T of a matrix W, swapping rows and columns, is obtained under the map i → -i for all i layers of its component basis elements. □

A $2^n \times 2^n$ *upper triangular* matrix has all zero entries below the main diagonal. This diagonal is represented entirely by hyperactual components.

A necessary condition for the remainder outside the diagonal to be upper triangular is that each antisymmetric component where a ϕ or i layer is ranked earliest is summed equally with the symmetric component in which this ϕ or i is interchanged.

This condition is sufficient. Main diagonal symmetries, including antisymmetries, are symmetries with the largest such scope, only determined by the leading ϕ or i layer. The lower triangular region will then only be zero if a leading ϕ or i layer for each component is summed with an interchanged i or ϕ layer of equal value. □

CHAPTER III

Nonassociative algebras

3.1 Introduction.

We develop further the concept of a matrix product. Let A and B be hyperintricate matrices. The hyperintricate methodology of [Ad12a] and [Ad12b] is firstly redefined by the introduction for a permutation P of the hyperintricate interlayer operator $\underline{\vee}_P$, which permutes layers on the right in the matrix product A $\underline{\vee}_P$ B.

Next we introduce operations in the intricate case which in combination allow each element of A and B in these products to be expressed uniquely in the given variables as bilinear expressions. These operations, the diamond operator ◊, the right roll operator $°_S$ and the left roll operator $_S°$, are in general nonassociative. Their algebras are expressed in the intricate formalism, and this is extended to the hyperintricate case by the $\underline{\vee}_P$ operator.

We develop the interrelationship between these ideas, by discussing combined and other operations, and discuss split products which reduce these operations to more primitive ones.

3.2. The hyperintricate interlayer operator, $\underline{\vee}_P$.

Let P be a permutation on n objects, which can be represented in permutation cycle notation, as in the example

(1 2 3)(4 5)(6 7 8 ... n).

Let Я_n and Ч_n be two n-hyperintricate numbers. The usual matrix product for layer t, where $1 \leq t \leq n$ corresponds to multiplying for each t the intricate number r_t, which is at layer t for Я_n, and h_t which is also at intricate layer t for Ч_n.

For Я_n and Ч_n, define the operation $\text{Я}_n \underline{\vee}_P \text{Ч}_n$ as the matrix operation which corresponds on layer t to the intricate product of r_t with $h_{P(t)}$. Then for two permutations P1 and P2 involving a third hyperintricate number И_n

$$\text{Я}_n \underline{\vee}_{P1} (\text{Ч}_n \underline{\vee}_{P2} \text{И}_n) = (\text{Я}_n \underline{\vee}_{P1} \text{Ч}_n) \underline{\vee}_{P1\,P2} \text{И}_n.$$

The interlayer operator is distributive.

$$\text{Я}_n \underline{\vee}_P (\text{Ч}_n + \text{И}_n) = (\text{Я}_n \underline{\vee}_P \text{Ч}_n) + (\text{Я}_n \underline{\vee}_P \text{И}_n).$$
$$(\text{Я}_n + \text{Ч}_n) \underline{\vee}_P \text{И}_n = (\text{Я}_n \underline{\vee}_P \text{И}_n) + (\text{Ч}_n \underline{\vee}_P \text{И}_n).$$

3.3 The intricate diamond operator, ◊.

We explain the diamond product firstly in terms of 2×2 matrices

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \lozenge \begin{Vmatrix} e & f \\ g & h \end{Vmatrix} = \begin{Vmatrix} ae + bg & af + bh \\ -ec - dg & cf + dh \end{Vmatrix}.$$

The diamond operation is not expressible in any way by the normal matrix product.

Under the transformation $c \rightarrow -c$, $d \rightarrow -d$ the right hand side becomes

$$\begin{Vmatrix} ae + bg & af + bh \\ ec + dg & -cf - dh \end{Vmatrix}.$$

A reversal of sign between bottom left and top right elements may be found under the transformation $a \rightarrow -a$, $e \rightarrow -e$, $b \rightarrow -b$ and $g \rightarrow -g$.

We have the following relations for intricate basis elements 1, i, α, ϕ.

$$1 \lozenge 1 = 1,\ i \lozenge i = -1,\ \alpha \lozenge \alpha = 1,\ \phi \lozenge \phi = 1,$$
$$1 \lozenge i = \phi = i \lozenge 1,\ 1 \lozenge \alpha = \alpha = \alpha \lozenge 1,\ 1 \lozenge \phi = i = \phi \lozenge 1,$$
$$i \lozenge \alpha = -i = -\alpha \lozenge i,\ i \lozenge \phi = \alpha = -\phi \lozenge i,\ \alpha \lozenge \phi = \phi = -\phi \lozenge \alpha.$$

The equations above show that 1, i, α and ϕ have inverses under $\lozenge$, with the left inverse equal to the right inverse.

For an intricate number $M = m1 + ni + p\alpha + q\phi$, define the conjugate $M^{\lozenge *} = M^*$ by

$$M^{\lozenge *} = m1 - ni - p\alpha - q\phi.$$

We obtain

$$M^{\lozenge *} \lozenge M = m^2 + n^2 - p^2 - q^2 = M \lozenge M^{\lozenge *}$$

and thus

$$M^{-1} = M^{\lozenge *}/(m^2 + n^2 - p^2 - q^2),$$

when not divided by zero.

The diamond operator is distributive. For intricate A, B and C

$$(A + B) \lozenge C = (A \lozenge C) + (B \lozenge C)$$
$$A \lozenge (B + C) = (A \lozenge B) + (A \lozenge C).$$

The diamond operation is not associative in general, for example

$$(1 \lozenge i) \lozenge \phi = 1 \neq 1 \lozenge (i \lozenge \phi) = \alpha.$$

Operations of multiplication on the left or right by i, α and ϕ under the usual matrix product interchange whole rows and columns, and multiply each row or column everywhere by 1 or -1. The diamond operation multiplies just one element by -1, or under transformation of signs of elements an odd number by -1. It follows that two such operations by i, α or ϕ can sometimes convert to usual matrix multiplication. For instance

$$(A \lozenge 1) \lozenge 1 = A = 1 \lozenge (1 \lozenge A).$$

3.4 The intricate left and right roll operators, $_S°$ and $°_S$.

The right roll operator $°_1$ rotates clockwise the entries of the intricate matrix on the right, and then the usual matrix product is formed. Thus

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \ °_1 \ \begin{Vmatrix} e & f \\ g & h \end{Vmatrix} = \begin{Vmatrix} ag + bh & ae + bf \\ cg + dh & ce + df \end{Vmatrix}$$

This is not expressible in terms of the usual matrix product, nor the diamond product, nor a combination of these.

If two rolls take place, we will denote the operation by $°_2$. In general if S rolls take place and $S' = S \pmod 4$ then $°_{S'} = °_S$. Since

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \;°_1\; \begin{Vmatrix} e & f \\ g & h \end{Vmatrix}$$

amounts to one roll, the iteration

$$1 \, °_1 \, (1 \, °_1 \, A) = 1 \, °_2 \, A$$

leads to the result that $°_S$ can be derived from $°_1$ operations. For notational convenience on occasion we may drop the S suffix in $°_1$ to form °.

In terms of the intricate operation, $°_1$ has the following algebra

$$1 \, ° \, 1 = \phi, \;\; i \, ° \, i = \phi, \; \alpha \, ° \, \alpha = \phi, \; \phi \, ° \, \phi = \phi,$$
$$1 \, ° \, i = -\alpha = -i \, ° \, 1,$$
$$1 \, ° \, \alpha = i = \alpha \, ° \, 1,$$
$$1 \, ° \, \phi = 1 = \phi \, ° \, 1,$$
$$i \, ° \, \alpha = -1 = \alpha \, ° \, i,$$
$$i \, ° \, \phi = i = \phi \, ° \, i,$$
$$\alpha \, ° \, \phi = \alpha = -\phi \, ° \, \alpha.$$

From these relations it can be seen that the $°_S$ algebra has inverses. The left and right inverse are identical.

The conjugate $M°*$ of $M = m1 + ni + p\alpha + q\phi$ is

$$M°* = m\phi - n\alpha + pi - q1,$$

so that

$$M \, ° \, M°* = m^2 + n^2 - p^2 - q^2 = M°* \, ° \, M.$$

For intricate A, B and C the $°_S$ algebra is distributive

$$(A + B) \, °_S \, C = (A \, °_S \, C) + (B \, °_S \, C)$$
$$A \, °_S \, (B + C) = (A \, °_S \, B) + (A \, °_S \, C).$$

The left roll operator $_S°$ rotates clockwise the entries of the intricate matrix on the left, then forming the usual matrix product. The intricate algebra for $_S°$ maps bijectively

$$A \, {}_S° \, B \leftrightarrow B \, °_S \, A,$$

the order in the product in terms of usual matrix multiplication now being reversed in the mapping. For the case of a simultaneous $_S°$ and $°_{S'}$ operation, we denote this by $_S°_{S'}$.

3.5 Combined and other operations.

The usual, diamond and roll products can be applied in different combinations on each layer of a hyperintricate number, possibly including the hyperintricate layer operator. On occasion we may denote the layers in a column, with the layer operators for each layer matched with the column. For example

$$\begin{matrix} 1 & \Diamond & \phi & = i = -i_1. \\ \alpha & °_1 & i & -1 \end{matrix}$$

There are a number of other operations. The tilde operator, ~, converts the hyperintricate diagonal $1 \to 1$, and all other hyperintricate numbers over all layers are multiplied collectively by minus one. Thus if $A = 1 + ai_i$, then $A^{\sim} = 1 - ai_i$.

Since the usual product is not commutative, in general

$$A(B^{\sim}) \neq B(A^{\sim}).$$

We will denote $A(B^{\sim})$ under usual multiplication by A ~ B. Because of multiplicative associativity of minus signs, but not additive associativity, these features carry over to the ~ operation.

The conjugate * acts on individual layers, but otherwise acts like ~.

The transpose T acts on individual layers, and converts $i \rightarrow -i$. We have seen that for hyperintricate numbers the transpose on all layers corresponds to the matrix transpose as commonly understood.

Under usual matrix multiplication T is contravariant

$$(AB)^T = B^T A^T.$$

We may combine the usual covariant product AB = AB with the T operator to form the right transpose operation

$$A_{\ TR}\ B = A(B^T).$$

The left transpose operator LT satisfies

$$A_{\ LT}\ B = (A^T)B.$$

Normally, the transpose is associative, but $A_{\ LT}\ B$ and $A_{\ TR}\ B$ are not.

We may introduce transposes (as $i \rightarrow -i$) on a layer with diamond and roll operators. For instance

$$1 \Diamond_{TR} \phi = 1 \Diamond (\phi^T) = 1 \Diamond \phi = i.$$

3.6 Split products.

Usual matrix multiplication satisfies

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} . \begin{Vmatrix} e & f \\ g & h \end{Vmatrix} = \begin{Vmatrix} ae + bg & af + bh \\ ec + dg & cf + dh \end{Vmatrix} = \begin{Vmatrix} ae & af \\ ec & cf \end{Vmatrix} + \begin{Vmatrix} bg & bh \\ dg & dh \end{Vmatrix}$$

where we will write this as

$$A.B = C + D.$$

The matrix C is the left split product and we also write

$$A|.B = C.$$

The right split product is

$$A.|B = D.$$

These operations can be extended to the diamond and roll products, where we write

$A|\Diamond B$,

$A\Diamond|B$,

$A|{\circ}_1 B$, etc.

We define the split transpose of A.B by

$$(A|.B)^T = B^T|.A^T.$$

The split products may be carried over to layers, with different operations on each layer.

CHAPTER IV

Division algebras

4.1 Introduction.

We use the hyperintricate representation for $2^n \times 2^n$ matrices documented in [Ad12i] and [Ad12j] to develop representations of the quaternions, and other algebras without complete division. We show by hyperintricate methods that the reals, complex numbers and quaternions are the only associative division algebras.

4.2 Representations of quaternions by hyperintricate numbers.

The quaternions are extensions of the complex numbers with 3 'imaginary' – or quaternionic – parts. So we can represent a quaternion by

$$a1 + bi + cj + dk$$

where

$$1^2 = 1,\ i^2 = j^2 = k^2 = -1,$$
$$1i = i = i1,\ 1j = j = j1,\ 1k = k = k1,$$
$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik \qquad (1)$$

and the inverse is

$$(a1 - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2).$$

This (1, i, j, k) basis is representable by four hyperintricate numbers – in fact the four given above – 1_1, α_i, i_1 and ϕ_i. An alternative representation, under swapping of layer levels, is 1_1, i_α, 1_i and i_ϕ. Some other representations are 1_{11}, $i_{\alpha\phi}$, $\alpha_{\phi i}$ and $\phi_{i\alpha}$ or 1_{1111}, $i_{11\alpha\phi}$, $\alpha_{ii\phi i}$ and $\phi_{iii\alpha}$. □

4.3 Non-existence of new standard associative division algebras.

The only standard associative division algebras are the reals, complex numbers and quaternions [AdJF60]. We will represent the basis elements of these associative division algebras by hyperintricate numbers.

In category theory, a basis of a vector space is an example of a universal arrow, which shows this result is independent of basis. Nevertheless, we can also prove this is so in the case where the hyperintricate basis elements are transformable to the generalised case studied next.

These basis elements have square ±1, and any other representation can be reduced to a linear combination of these basis elements, for which the basis element squares are also ±1. In detail, any representation of the set {1, i, α, ϕ} under a change of basis which preserves squares maps each element to the set

$$\{1, \pm\sqrt{(\gamma_i^2 + \delta_i^2 + 1)}i + \gamma_i\alpha + \delta_i\phi, \pm\sqrt{(\gamma_\alpha^2 + \delta_\alpha^2 - 1)}i + \gamma_\alpha\alpha + \delta_\alpha\phi,$$
$$\pm\sqrt{(\gamma_\phi^2 + \delta_\phi^2 - 1)}i + \gamma_\phi\alpha + \delta_\phi\phi\}$$

with the coefficients $\gamma_i \neq \gamma_\alpha \neq \gamma_\phi$ etc. real. This extends to all layers.

It is not initially clear, for example, whether 1_{11}, $i_{\alpha 1}$, 1_{i1}, $i_{\phi 1}$, 1_{1i}, $i_{\alpha i}$, 1_{ii} and $i_{\phi i}$ can form a normed division algebra in which more than one square of a basis element is 1, e.g.1_{ii}.

For any hyperintricate basis element, the inverse is known. For a basis element A whose square is 1, the inverse $A^{-1} = A$. These A amount to all basis elements which have an even number (including zero) of i's in their hyperintricate representation. For any basis element, B, whose square is -1, the inverse $B^{-1} = -B$. The set of all B's is those basis elements which have an odd number of i's in their hyperintricate representation.

There are only two possibilities for basis elements, they either commute, AB = BA, or they anticommute, AB = -BA.

Consider finding the inverse of $aA_1 + bA_2$, $A_1 \neq A_2$, where A_1^2 and $A_2^2 = 1$. This is then

$$(aA_1 - bA_2)/(a^2 - b^2) \qquad (2)$$

when A_1 and A_2 commute and

$$(aA_1 + bA_2)/(a^2 + b^2)$$

when A_1 and A_2 anticommute. If we incorporate the fact that 1 is always present amongst such A's, then for some values of a and b, (2) holds, which implies that there exist a's and b's for which (2) includes the possibility of dividing by zero. The statement that we can do division is incorporated in the definition of a division algebra (although we have to specifically exclude division by zero, as for a field), therefore there exists in such division algebras only one basis element with square 1, and this must be the real basis element. We will extend these considerations later.

To find the inverse of $a1 + bB_1$, where $B_1^2 = -1$, then this is

$$(a1 - bB_1)/(a^2 + b^2),$$

which introduces no further problems.

To find the inverse of $aB_1 + bB_2$, for $B_1^2 = 1$ and $B_2^2 = -1$, then this is the permissible

$$-(aB_1 + bB_1)/(a^2 + b^2),$$

when B_1 and B_2 anticommute, which is now the only possibility.

The above argument may be generalised for more B_r's, and it becomes necessary to stipulate that all $B_1, B_2, \ldots B_n$ mutually anticommute.

We know there are solutions for B_1, B_2, B_3 given by basis elements for the quaternions. Now assume the existence of four such basis elements, B_1, B_2, B_3, B_4, all mutually anticommuting and distinct, so that $B_rB_s = -B_sB_r$. We will use associativity of these basis elements in computing from $B_1B_2B_3B_4$ its mirror reflection in two separate ways. So

$$\begin{aligned} B_1B_2B_3B_4 &= -B_1B_2B_4B_3 \\ &= B_1B_4B_2B_3 \\ &= -B_4B_1B_2B_3 \\ &= B_4B_1B_3B_2 \\ &= -B_4B_3B_1B_2 \\ &= B_4B_3B_2B_1. \end{aligned}$$

However

$(B_1B_2)(B_3B_4) = -(B_3B_4)(B_1B_2)$
$= -(B_4B_3)(B_2B_1)$,

a contradiction.

Thus the maximum number of dimensions for a standard associative division algebra is 4. ☐

4.4. Wedderburn's little theorem.

Lagrange's theorem states that *any natural number is the sum of four squares*.

As background, if $w^2 + x^2 + y^2 + z^2 = n$, then so is $w'^2 + x'^2 + y'^2 + z'^2$, where

$w' = ½(w + x + y + z)$
$x' = ½(w + x – y – z)$
$y' = ½(w – x + y – z)$
$z' = ½(w – x – y + z)$,

and the condition that w', x', y' and z' are whole numbers is that the number of even w, x, y and z is even.

Further, Jacobi found an exact formula for the total number of ways a given positive integer n can be represented in this way. Two representations are considered different if their terms are in different order or if the integer being squared (not just the square) is different.

The number of ways to represent n as the sum of four squares is given by the symbol $r_4(n)$ and is eight times the sum of the divisors of n if n is odd, and 24 times the sum of the odd divisors of n if n is even, written

$r_4(n) = 8\Sigma(\text{for } m|n)m$

if n is odd and

$r_4(n) = 24\Sigma(\text{for } m|n, m \text{ odd})m$

if n is even. Equivalently, it is eight times the sum of all its divisors which are not divisible by 4, so that for a prime number p we have the explicit formula

$r_4(p) = 8(p + 1)$.

Wedderburn's little theorem states that *any finite division ring is commutative*. This invites some further remarks. If the coefficients of the basis elements of a division algebra belong to an integer arithmetic (mod n), or if each individual basis element in the case of the quaternions is multiplied by a coefficient (mod n_1) for 1, (mod n_i) for i, (mod n_j) for j and (mod n_k) for k, so that say

$ai.bj = abk \qquad (\text{mod } n_k)$,

and if n is the least common multiple of n_1, n_i, n_j and n_k, giving

$ai.bj = abk \qquad (\text{mod } n)$,

so that there are epimorphisms

$(\text{mod } n) \to (\text{mod } n_1)$,

etc., then in general the division algebra (mod n) is a finite subalgebra of the complex numbers, the quaternions or octonions.

One or more of the components i, j, k of the quaternions can be zero, otherwise

$$a1 + bi + cj + dk \qquad \text{(mod n)}$$

has inverse

$$a1 – bi – cj – dk/(a^2 + b^2 + c^2 + d^2) \qquad \text{(mod n)},$$

and by Lagrange's theorem, putting $n = (a^2 + b^2 + c^2 + d^2)$ means the inverse does not exist, since dividing by n is equivalent to dividing by zero.

For quaternions $n_1 = n_i = n_j = n_k$ is possible under the condition that this can only have one *equivalent* division by zero. We have seen that this gives rise to the non-existence of certain non-zero inverses.

If we have a torus of Gaussian integers a + bi, with coefficients (mod n) for 1 and (mod n) for i, the case for complex numbers is analogous.

A theorem of Fermat states that a prime p is expressible as the sum of two squares if and only if $p \equiv 1 \pmod 4$ or p = 2, and the Brahmagupta identity

$$(s^2 + mt^2)(u^2 + mv^2) = (su – mtv)^2 + m(sv + tu)^2$$
$$= (su + mtv)^2 + m(sv – tu)^2$$

implies that if all of n's odd prime factors congruent to 3 modulo 4 occur to an even exponent, it is expressible as a sum of two squares. The converse also holds [Co89]. This more detailed result will not be needed subsequently.

Since some numbers are not the sum of two squares, when this sum is absent (mod n) such non-zero inverses are always present for Gaussian integers (mod n).

The situation on one component, so that $a \equiv 0$ or $b \equiv 0$ but not both, is simpler. The condition now for the existence of all inverses $\neq 0$ is that n is prime, otherwise for g|n, there exists an h with $gh \equiv 0 \pmod n$, so that g or h have no inverse. Thus, overall, a necessary and sufficient condition is that $n \equiv 3 \pmod 4$ is prime. ☐

4.5 Extension of the reasoning to possibly singular matrices.

If we were to allow equation (2) to operate, this means that we can divide by $(a^2 – b^2)$. There exists the possibility that this is zero, but we could treat this situation on the same footing as dividing explicitly by zero, excluded as a number, but see [Ad13b]. We will describe a non-standard division algebra as one in which the number of singular occurrences, divided by the total number of occurrences, is a not well-ordered infinitesimal [Ad13a], a number ε such that for any number $n \in N_{\neq 0}$ there does not exist an $m \in N$ with $\varepsilon m > n$. We will now incorporate these circumstances where $(a^2 – b^2) \neq 0$, which allows more than one timelike square, that is, we permit multiple basis element squares of 1.

If we write the intricate components possible for a basis as columns along a row and the layers as each row, then a quaternion basis is represented for instance by

1 i α ϕ
1 1 i i.

These basis elements are irreducible. To take an example

$$(i + \alpha)_{(1 + i)} = i_1 + \alpha_1 + i_i + \alpha_i,$$

where α_1 and i_i are not in this algebra.

Now consider the algebra which has some singularities as already described

$$\begin{array}{l} 1\ i\ \alpha\ \phi \\ 1\ 1\ i\ i \\ 1\ i\ \alpha\ \phi \\ 1\ 1\ i\ i \\ \hline 1\ a\ b\ c, \end{array}$$

where

$$a^2 = b^2 = c^2 = 1,$$
$$1a = a = a1,\ 1b = b = b1,\ 1c = c = c1,$$

and

$$ab = c = ba,\ bc = a = cb,\ ca = b = ac.$$

It is now possible to form a 16-dimensional associative algebra with a limited set of singularities, given by

$$\begin{array}{lllll} 1\ i\ \alpha\ \phi & 1\ 1\ 1 & i\ i\ i & \alpha\ \alpha\ \alpha & \phi\ \phi\ \phi \\ 1\ 1\ i\ i & 1\ 1\ 1 & 1\ 1\ 1 & i\ i\ i & i\ i\ i \\ 1\ 1\ 1\ 1 & i\ \alpha\ \phi & i\ \alpha\ \phi & i\ \alpha\ \phi & i\ \alpha\ \phi \\ 1\ 1\ 1\ 1 & 1\ i\ i & 1\ i\ i & 1\ i\ i & 1\ i\ i \\ 1\ 1\ 1\ 1 & i\ \alpha\ \phi & i\ \alpha\ \phi & i\ \alpha\ \phi & i\ \alpha\ \phi \\ 1\ 1\ 1\ 1 & 1\ i\ i & 1\ i\ i & 1\ i\ i & 1\ i\ i, \end{array}$$

and this process can be continued to produce 2^n-dimensional associative algebras with restricted division.

If we generalise the example for which we began the last section, we note that $1_{p,q,\ \ldots\ 1} + 1_{p,q,\ \ldots\ \alpha}$ is a matrix with a zero bottom row and therefore corresponds to a singular matrix. Similarly $1_{p,q,\ \ldots\ 1} + 1_{p,q,\ \ldots\ \phi}$ has two equal rows and is consequently also singular.

For the remainder of this section we will be considering $\{1_{p,q,\ \ldots\ 1}, \ldots\} \cup \{1_{p,q,\ \ldots\ i}, \ldots\}$, but we will see here too that a singular matrix can be derived with the trailing layer, in the example which follows by setting a = 1, g = 1 and all other coefficients zero.

To deal with the case considered next, first note that

$$P + Q1_{ii} + Ri_{\alpha i} + Si_{\phi i}$$

has inverse (another method of finding hyperintricate inverses is given in [Ad12e])

$$(P - Q1_{ii} - Ri_{\alpha i} - Si_{\phi i})/(P^2 - Q^2 - R^2 - S^2). \qquad (3)$$

Let us now investigate the properties of

$$\{1_{11}, 1_{i1}, i_{\alpha 1}, i_{\phi 1}, 1_{1i}, 1_{ii}, i_{\alpha i}, i_{\phi i}\}^{+,X} \qquad (4)$$

under addition and multiplication. The above example is closed under multiplication. Does it form a multiplicative group?

We will write these 3-hyperintricate numbers as

$$a1_{11} + b1_{i1} + ci_{\alpha 1} + di_{\phi 1} + f1_{1i} + g1_{ii} + hi_{\alpha i} + ki_{\phi i}. \qquad (5)$$

If we change (5) to an expression with inverses of basis elements substituted, we get

$$a1_{11} - b1_{i1} - ci_{\alpha 1} - di_{\phi 1} - f1_{1i} + g1_{ii} + hi_{\alpha i} + ki_{\phi i}. \qquad (6)$$

Multiplying (5) by (6), a little manipulation gives the expression

$$[a^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2 + k^2]$$
$$+ 2[(ag - fb)1_{ii} + (ah - fc)i_{\alpha i} + (ak - fd)i_{\phi i}], \qquad (7)$$

which is precisely of the form (3).

Thus

$$(5)\text{x}(6)\text{x}(3) = 1$$

provided

$$P = [a^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2 + k^2],$$

$$Q = 2(ag – fb),$$

$$R = 2(ah – fi)$$

and

$$S = 2(ak – fd),$$

so that (5) does indeed have a multiplicative inverse. This is not a *normed* division algebra in the usual sense, since the denominator contains terms of degree 4. ☐

Further, we can continue such a process recursively, considering n-hyperintricate examples with trailing layer 1 or i, for instance derived from the above example. Let us look at this next.

We will consider both the next stage up, and indicate how we can generalise in an induction procedure, and describe these in parallel. It is possible to be more formal, but then we can lose the thread of the idea.

The case corresponding to (4) is

$$\{1_{111}, 1_{i11}, i_{\alpha 11}, i_{\phi 11}, 1_{1i1}, 1_{ii1}, i_{\alpha i1}, i_{\phi i1}, 1_{11i}, 1_{i1i}, i_{\alpha 1i}, i_{\phi 1i}, 1_{1ii}, 1_{iii}, i_{\alpha ii}, i_{\phi ii}\}^{+,\text{x}}. \quad (8)$$

In an induction procedure, we assume a set of basis elements, and append as a trailing layer both 1 and i to those elements, thereby doubling the number of basis elements from its previous instance. By the induction procedure, there are 2^{n-1} elements to begin with, doubled to 2^n in the next stage.

Corresponding to (5), in the specific example we have chosen we consider the hyperintricate number

$$a1_{111} + b1_{i11} + ci_{\alpha 11} + di_{\phi 11} + f1_{1i1} + g1_{ii1} + hi_{\alpha i1} + ki_{\phi i1}$$
$$+ a'1_{11i} + b'1_{i1i} + c'i_{\alpha 1i} + d'i_{\phi 1i} + f'1_{1ii} + g'1_{iii} + h'i_{\alpha ii} + k'i_{\phi ii}, \quad (9)$$

whereas corresponding to (6), we generate the hyperintricate

$$a1_{111} – b1_{i11} – ci_{\alpha 11} – di_{\phi 11} – f1_{1i1} + g1_{ii1} + hi_{\alpha i1} + ki_{\phi i1}$$
$$– a'1_{11i} + b'1_{i1i} + c'i_{\alpha 1i} + d'i_{\phi 1i} + f'1_{1ii} – g'1_{iii} – h'i_{\alpha ii} – k'i_{\phi ii}. \quad (10)$$

In the general situation we will have coefficients in lower case of hyperintricate numbers with a trailing 1 layer, minus, in the format corresponding to (10), hyperintricates in primed lower case coefficients each with basis element with a trailing i layer.

Multiplying (9) and (10) together gives

$$[a^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2 + k^2]$$
$$+ [a'^2 + b'^2 + c'^2 + d'^2 + f'^2 + g'^2 + h'^2 + k'^2]$$
$$+ 2[(ag – fb – a'g' + f'b')1_{ii1}$$
$$+ (ah – fc – a'h' + f'c')i_{\alpha i1} + (ak – fd – a'k' + f'd')i_{\phi i1}]$$
$$+ 2[(ab' – a'b – fg' + gf')1_{i1i}$$
$$+ (ac' – a'c – hf' + fh')i_{\alpha i1} + (ad' – a'd – kf' + fk')i_{\phi 1i}]. \quad (11)$$

In general there is a set of positive squares of coefficients both primed and unprimed, followed by twice a number of coefficients times basis elements with an even number of i's.

The inverse of

$$P + Q1_{ii1} + Ri_{\alpha i1} + Si_{\phi i1} + T1_{i1i} + Ui_{\alpha 1i} + Vi_{\phi 1i}$$

is

$$(P - Q1_{ii1} - Ri_{\alpha i1} - Si_{\phi i1} - T1_{i1i} - Ui_{\alpha 1i} - Vi_{\phi 1i}) \\ /(P^2 - Q^2 - R^2 - S^2 - T^2 - U^2 - V^2), \qquad (12)$$

this being a generalisation of (3), involving in its typical characteristic basis elements an even number of i's, so that once again, in its general form, the inverse of (11) can be obtained, as is also derived using the result for a general hyperintricate inverse in Chapter II, section 8. ☐

CHAPTER V

Fermat's little theorem and Euler's totient formula for matrices

5.1 Introduction.

We use the hyperintricate representation for $2^n \times 2^n$ matrices documented in [Ad12i] and [Ad12j] to develop Fermat's little theorem for matrices.

5.2 The intricate version of Fermat's little theorem.

Let r be an odd prime number.

We first consider the intricate number (2×2 matrix)

$$Я = a1 + bi + c\alpha + d\phi = a1 + J,$$

where the coefficients a, b, c and d are *integers*. Not all instances of integer 2×2 matrices adhere to this criterion, but for example all even 2×2 matrices satisfy it.

The intricate conjugate [Ad12i] of Я is

$$Я^* = a1 - bi - c\alpha - d\phi.$$

If $J^2 = (-b^2 + c^2 + d^2)$ *is non-zero* (mod r)*, if it is a quadratic residue* (mod r), *then*

$$Я^r - Я \equiv 0 \pmod r$$

otherwise

$$Я^r - Я^* \equiv 0 \pmod r.$$

Note. Considerations below will show $(Я^r)^* = (Я^*)^r$.

Proof. The trivial case b = c = d = 0 is equivalent to $2^0 \times 2^0$ matrices, where the following demonstration is often given.

$$0^r - 0 \equiv 0 \pmod r,$$

so if

$$y^r - y \equiv 0 \pmod r,$$

then by the binomial theorem, since the denominators are not divisible by r,

$$(y + 1)^r - (y + 1) \equiv 0 \pmod r.$$

For intricate numbers, since a1 commutes with $J = (bi + c\alpha + d\phi)$, by the binomial theorem

$$Я^r = (a1)^r + [\text{terms multiplied by r}] + (bi + c\alpha + d\phi)^r.$$

Now

$$(bi + c\alpha + d\phi)^r = \{i(b1 + c\phi - d\alpha)\}^r.$$

The first two terms on the right expanded out become

$$-1(b1 - c\phi + d\alpha)(b1 + c\phi - d\alpha)1 = -(b^2 - c^2 - d^2)1.$$

Continuing the process gives

$$(bi + c\alpha + d\phi)^r = (-1)^{(r-1)/2}(b^2 - c^2 - d^2)^{(r-1)/2} i(b1 + c\phi - d\alpha)$$
$$= (-b^2 + c^2 + d^2)^{(r-1)/2}(bi + c\alpha + d\phi). \square$$

A standard result we will later prove for totients, is that $h^{(r-1)/2} \equiv 1 \pmod r$ when h is a positive perfect square, more generally a quadratic residue (mod r), and ≡ -1 (mod r) when h is not a positive perfect square, or generally not a quadratic residue (mod r). Further [Ad12a], with equivalent statements substituting 1 → -1, -1 → 1 on the right of the next equivalence signs, if r = 4k – 1 with $h^{(r-1)/2} \equiv 1 \pmod r$ then $(-h)^{(r-1)/2} \equiv -1 \pmod r$, and if r = 4k + 1 with $h^{(r-1)/2} \equiv 1 \pmod r$ then $(-h)^{(r-1)/2} \equiv 1 \pmod r$. □

Whenever a prime r = 4k – 1 then because 1 is a square, if $-b^2 + c^2 + d^2 = 1 \pmod r$ then $(r + b^2 - c^2 - d^2)^{(r-1)/2} = (-1)^{(r-1)/2} \pmod r = -1 \pmod r$ which is not a quadratic residue. Likewise if $-b^2 + c^2 + d^2 = -1 \pmod r$ then $r + b^2 - c^2 - d^2 = 1 \pmod r$. But if r = 4k + 1 then when $-b^2 + c^2 + d^2 = 1 \pmod r$ then $r + b^2 - c^2 - d^2 = 1 \pmod r$, and when $-b^2 + c^2 + d^2 = -1 \pmod r$ then $r + b^2 - c^2 - d^2 = -1 \pmod r$. Squares (mod r), i.e. quadratic residues, are of the form $1.B = 1.A^2 \pmod r$, and thus it can be determined from this case ≠ 0 (mod r) whether $-1.A^2 \pmod r$ is or is not a quadratic residue.

Note that if a real number $A^2 = -b^2 + c^2 + d^2$, this implies

$$(A + c)(A - c) = (d + b)(d - b)$$

for the linearly independent (A + c) and (A – c), and likewise for (d + b) and (d – b) respectively.

Corollary. For all integer 2×2 *matrices with an intricate representation* Я′ *where* $J^2 = (-b^2 + c^2 + d^2)$ *is non-zero* (mod r)*, if* J^2 *is a quadratic residue* (mod r)*, then*

$$(2Я')^r - 2Я' \equiv 0 \pmod r \qquad (1)$$

otherwise

$$(2Я')^r - 2Я'^* \equiv 0 \pmod r. \ \square \qquad (2)$$

Example. Let g be a Gaussian integer of the form a + ib, and g* be its complex conjugate. Then for prime p = 4k – 1

$$g^p - g^* \equiv 0 \pmod p,$$

since c = d = 0 and a and b are integers in the intricate representation, with $-b^2$ the negative of a perfect square, and therefore not a quadratic residue for this p.

5.3 Some synthetic generalisations of Fermat’s little theorem.

We have for integer y and distinct odd primes p, q, r

$$(y^p - y)^q - (y^p - y) \equiv 0 \pmod{pq}.$$

This can be written, ignoring binomial terms containing q, as

$$y^{pq} - y^q - y^p + y \equiv 0 \pmod q.$$

Swapping p and q gives an identical expression on the left, so

$$y^{pq} - y^q - y^p + y \equiv 0 \pmod{pq}. \qquad (3)$$

Suppose p is 2. Then still

$$y^p - y \equiv 0 \pmod 2.$$

Thus

$$y^{pq} - y^q - y^p + y \equiv 0 \pmod q$$

continues to hold. However

$$(y^p - y)^2 - (y^p - y) \equiv 0 \pmod 2,$$

which gives

$y^{2q} - y^q + y^2 + y \equiv 0 \pmod 2$,
with a different sign for y^2, which is obviated by the equivalence
$y^2 \equiv -y^2 \pmod 2$,
so that equation (3) continues to hold even for distinct p or q = 2.

Equation (3) can be generalised for distinct primes p, q, r to
$$y^{pqr} - [y^{qr} + y^{pr} + y^{pq}] + [y^p + y^q + y^r] - [y] \equiv 0 \pmod{pqr}, \qquad (4)$$
with an obvious extension to n distinct odd primes, when the expressions in square brackets alternate in sign. ☐

Corollary. For all integer 2×2 *matrices with an intricate representation* Я′ *where* $J^2 = (-b^2 + c^2 + d^2)$ *is non-zero* (mod p), (mod q) *and* (mod r)*, if* J^2 *is a quadratic residue* (mod p), (mod q) *and* (mod r), *then*
$$(2Я')^{pqr} - [(2Я')^{qr} + (2Я')^{pr} + (2Я')^{pq}]$$
$$+ [(2Я')^p + (2Я')^q + (2Я')^r] - [2Я'] \equiv 0 \pmod{pqr}. ☐$$

When we wish to determine a square (mod p^m), then
$(y^p - y)^m \equiv 0 \pmod{p^m}$,
where the left hand side is given by a binomial expansion. ☐

If $y^p - y \equiv 0 \pmod p$, then
$y^{n(p-1)+1} - y \equiv 0 \pmod p$.

Proof. Since $y^{2p-1} - y^p = y^{p-1}(y^p - y)$ is divisible by p, the sum $(y^{2p-1} - y^p) + (y^p - y)$ is also, with the general result following by recursion. ☐

Example. Putting p = 3, we have for any odd natural number q, $y^q - y \equiv 0 \pmod 6$.

5.4 The intricate version of Euler's totient formula.

The following theorem can be deduced from pages 163-167 of [Bu89], although the material is evident in the earlier [Hec33]. An insightful work is [Ma1886].

Let ↑ be the exponential operator.

Let s > 1 *be a natural number and* y *an integer with* gcd(y, s) = 1 *or* lcm(y, s) = s. *Then if* s *has the prime factorisation*
$s = (q_1 \uparrow j_1)(q_2 \uparrow j_2) \ldots (q_n \uparrow j_n)$
for primes q_i, *and where the totient*
$\varphi(s) = s(1 - 1/q_1)(1 - 1/q_2) \ldots (1 - 1/q_r)$
then
$y^{\varphi(s)+1} - y \equiv 0 \pmod s$.

Proof. The above factorises as
$y(y^{\varphi(s)} - 1) \equiv 0 \pmod s$
so either $y \equiv 0 \pmod s$, which is the same way as putting lcm(y, s) = s, or otherwise gcd(y, s) = 1 and
$y^{\varphi(s)} - 1 \equiv 0 \pmod s$. ☐

Corollary. For gcd(y, s) = 1, y^2 *belongs to the* $y^{\varphi(s)/2} \equiv 1$ (mod s) *equivalence class. Otherwise for* gcd(y, s) = 1, *non quadratic residues satisfy* $y^{\varphi(s)/2} \equiv -1$ (mod s).

Proof. φ(s) is even and

$$(y^2)^{\varphi(s)/2} - 1 \equiv (y^{\varphi(s)/2} - 1)(y^{\varphi(s)/2} + 1) \equiv 0 \text{ (mod s)}. \square$$

Corollary. If r is prime, so φ(r) = r – 1, *when* J *is a quadratic residue* (mod r), $J^{(r-1)/2} \equiv 1$ (mod r), *whereas* $J^{(r-1)/2} \equiv -1$ (mod r) *when* J *is not a quadratic residue* (mod r). □

To prove the intricate version of Euler's totient formula, for elements with lcm(y, t) = t or gcd(y, t) = 1 with j > 0, q prime and $q^j = t$, we first argue that, *if* $J^2 = -b^2 + c^2 + d^2$ *is a quadratic residue* (mod t), *then*

$$Я\uparrow[\varphi(q^j) + 1] - Я \equiv 0 \text{ (mod t)} \quad (5)$$

otherwise

$$Я\uparrow[\varphi(q^j) + 1] - Я^* \equiv 0 \text{ (mod t)}. \quad (6)$$

Proof. If all elements are equivalent to 0 (mod t), then Я ≡ 0 (mod t). So assume gcd(y, t) = 1 for at least one y = a, b, c or d with otherwise a, b, c or d ≡ 0 (mod t). Using the binomial theorem,

$$Я\uparrow[\varphi(q^j) + 1] = (a1)^{\varphi(s)+1} + (\varphi(s) + 1)[\text{intermediate binomial terms}] + (bi + c\alpha + d\phi)^{\varphi(s)+1}.$$

An argument similar to that for Fermat's little theorem gives

$$(bi + c\alpha + d\phi)^{\varphi(s)+1} = (-b^2 + c^2 + d^2)^{\varphi(s)/2}(bi + c\alpha + d\phi).$$

We have previously proved that if $(-b^2 + c^2 + d^2)$ is a quadratic residue ≠ 0 (mod t), then it belongs to the $(-b^2 + c^2 + d^2)^{\varphi(s)/2} \equiv 1$ (mod t) equivalence class, otherwise if ≠ 0 (mod t) to the -1 (mod t) equivalence class.

Notice that

$$\varphi(q^{j+1}) = q^{j+1} - q^j = q(q^j - q^{j-1}) = q\varphi(q^j),$$

so that for j = 1 the terms

$$(\varphi(t) + 1)[\text{intermediate binomial terms}]$$

are ≡ 0 (mod t), and if assumed for j, they likewise satisfy this for j + 1.

Now *assume* $-b^2 + c^2 + d^2$ is a perfect square ≠ 0 (mod $q_i\uparrow j_i$) for each i, then

$$Я\uparrow\varphi(q_i\uparrow j_i) \equiv 1 \text{ (mod } q_i\uparrow j_i).$$

Noting that φ(s) is divisible by $\varphi(q_i\uparrow j_i)$, raising both sides to the power $\varphi(s)/\varphi(q_i\uparrow j_i)$, we arrive at

$$Я^{\varphi(s)} \equiv 1 \text{ (mod } q_i\uparrow j_i).$$

Inasmuch as the moduli are relatively prime, this leads to the relation

$$Я^{\varphi(s)} \equiv 1 \text{ (mod } (q_1\uparrow j_1)(q_2\uparrow j_2) \ldots (q_n\uparrow j_n))$$

or

$$Я^{\varphi(s)} \equiv 1 \text{ (mod s)}.$$

If Я ≡ 0 (mod s), then this possibility is incorporated in

$$Я^{\varphi(s)+1} \equiv Я \text{ (mod s)}. \square$$

The following result is also proved in [Ad13b]. *Let* s > 1 *be a natural number and* y *an integer. Then if* s *has the prime factorisation*

$$s = (q_1 \uparrow j_1)(q_2 \uparrow j_2) \ldots (q_n \uparrow j_n)$$

for primes q_i, *and where the totient*

$$\varphi(s) = s(1 - 1/q_1)(1 - 1/q_2) \ldots (1 - 1/q_r)$$

then

$$\varphi(s)[y^{\varphi(s)+1} - y] \equiv 0 \pmod{s}. \qquad (7)$$

Proof. When s = s′s″, where only s′ is coprime to y then

$$y(y^{\varphi(s)} - 1) \equiv 0 \pmod{s'},$$

and from the definition of φ(s″) when s″ contains a subselection of primes in s, and y contains all the factors of s″, then

$$\varphi(s'')y \equiv 0 \pmod{s''},$$

so that on combining these two results, in general

$$\varphi(s)[y^{\varphi(s)+1} - y] \equiv 0 \pmod{s}. \ \square$$

5.5 Some hyperintricate versions of Fermat's little theorem.

Let $Я_n$ be an n-hyperintricate number.

The inverse $Я_n^{-1}$ of a non-singular $2^n \times 2^n$ matrix $Я_n$ exists and is unique. Its denominator may be expressed as the determinant, det $|Я_n|$, of the matrix. If we wish to obtain a hyperintricate conjugate, $Я_n$*, so that even in the singular case det $|Я_n| = 0$

$$Я_n(Я_n^*) = \det |Я_n|,$$

then provided det $|Я_n| \neq 0$ the inverse is

$$Я_n^{-1} = (Я_n^*)/\det |Я_n|$$

with

$$Я_n Я_n^{-1} = 1.$$

Since each individual element of $Я_n$ is expressible hyperintricately [Ad12b], in all cases the determinant can be computed. A formula for determining the hyperintricate inverse when it is expressed in a 'J-abelian' form similar to the one provided below is given in that same reference in section 7. This states how the determinant, det $|Я_n|$, is derived from $Я_n$*. The general inverse has also been computed.

Then since det $|Я_n|$ is a scalar, the standard Fermat and Euler formulas apply to it.

When an n-hyperintricate number is J-abelian, that is expressed in the form $\Sigma U_{V \ldots W}$, where U, V ... W are intricate numbers, with all values of J identical for a particular layer, then powers of this entity are abelian, the n-hyperintricate conjugate is readily available, and considerations obtained next apply.

We will derive from [Ad12b] the definition of a J-abelian n-hyperintricate number given by

$$Я_n = \Sigma(r = 1 \text{ to } \lceil 4^{n-1}/n \rceil) \otimes (k = 1 \text{ to } n)(a_{rk}1 + J_k L_{rk}),$$

where we are using the ceiling function, $\lceil 4^{n-1}/n \rceil$, and the composite layer operator ⊗, and convert it for the Fermat little theorem case. The intricate $J_k \in \{i, \alpha, \phi\}$ or J_k in integer *JAF* format, with J_k^2 an integer, are identical over r and independent over k, so that such numbers are abelian in their J_k components, and the standard binomial theorem holds. This means the arguments we have been using for Fermat's little theorem as is generalised from equations (1) or (2) and the Euler totient formula generalised from (5), (6) and (7), carry over to this case. □

The intricate algebra expresses the 2×2 noncommutative matrix

$$x = a1 + bi + c\alpha + d\phi$$

as the matrix $x = a1 + JK$ where $J^2K^2 = -b^2 + c^2 + d^2$ and

$$J^2 = 0 \text{ or } \pm 1.$$

Then for $J^2 = -1$ the algebra works under the substitution $J \rightarrow i$, and the considerations for complex x carry over to these matrices.

If we take K^2 as an integer, then we can represent $\pm K$ in congruence arithmetic in the manner we devised in section 8 of [Ad13b]. Further, the intricate conjugate of x is

$$x^* = a1 - bi - c\alpha - d\phi,$$

so that we can adopt equations in the intricate case with J replacing i, that is, in an intricate, or more generally J-abelian, congruence arithmetic.

Outside this case, we cannot apply the reduction of an n-hyperintricate number to an intricate one. □

CHAPTER VI

Hyperintricate Galois theory

6.1 Introduction.

Galois theory [Ed84], [Ga62], [Ro90] provides a no-go theorem for solutions by radicals of polynomials of degree > 4 with variables and coefficients in the complex number field. One way of looking at this is to show that the existence of a solution depends on inverting a set of polynomial equations, where such polynomials remain invariant under permutation of the variables. For permutations of five or more variables corresponding to the quintic or higher degree polynomials, the group structure of these permutations does not allow this inversion to take place within the algebraic number system. Therefore no general inversion algorithm of this type exists.

Further, if a solution by radicals of a general polynomial of degree n > 4 were found 'by chance' within the algebraic number system, then this would imply the existence of such an inversion algorithm [Ar59].

Despite this, Galois theory is configured to deal with only one equation for a solution by radicals, if present. For equations of degree 5 or higher there is always a constraint on the coefficients which means solutions are not the most general. But it has to be proved that a set of equations with different formulas for these solutions cannot cover completely the state of general coefficients. We do not prove this here.

For non-commutative algebras, where the multiplicative structure is non-abelian, Galois arguments need enhancement. The general noncommutative case is more extensive than that of mere quaternions, being available in the hyperintricate methodology for general matrix rings, the matrices of which may be singular. In this work we investigate and specify how Galois theory transforms under the extension to these algebras.

Intricate and hyperintricate numbers were designed to investigate what happens to Galois theory for non-commutative algebras.

For intricate and hyperintricate polynomials unique factorisation is invalid in general. We provide an example, demonstrating that there exist at least three distinct intricate factorisations of the polynomial equation $x^2 - a^2 = 0$. There are more.

We show for polynomials in x that are not described by multifunctions, that is, differing values of x are not used in the same polynomial, that the constraints in the commutative case carry over to the noncommutative one, provided coefficients are abelian. An extension to this uses 'J-abelian' algebra for hyperintricate numbers. We have not investigated a general hyperintricate polynomial with intricate coefficients.

6.2 Polynomials with integer solutions.

If a polynomial has all integer solutions, say

$$(x + A)(x + B) \dots (x + D) = 0, \qquad (1)$$

expanded out as

$$x^n + (A + B + \ldots + D)x^{n-1} + \ldots + AB\ldots D = 0, \quad (2)$$

then its solution set can be determined by factorising AB…D, and checking each set of multiplicative factors for a solution.

If we adjoin to (1) by multiplication *any* quartic polynomial in integer coefficients, this also is solvable by solvability of the quartic. □

6.3 Complex polynomials of degree n.

For a polynomial of degree n, let

$$X^n + a_{n-1}X^{n-1} + \ldots + a_0 = 0, \quad (3)$$

where X is a complex number and the a_i, i = 0 … (n – 1), are fixed complex numbers. Suppose we set for complex b_1

$$a_{n-1}X^{n-2} + a_{n-2}X^{n-3} + \ldots + a_2X + b_1 = 0, \quad (4)$$

then for any substitution of the variable X, a value of b_1 can be found.

Suppose

$$X^n + (a_1 - b_1)X + a_0 = 0. \quad (5)$$

Then if (5) is solvable we can find a b_1 so that (3) holds. Conversely, if we can solve (3) then we can find a b_1 such that (5) obtains. □

6.4 Solvability criteria for fields may not extend to matrix algebras.

Note that the equation (3) if written as

$$X^n + Tx^{n-1} + Ux^{n-2} + \ldots + W = 0, \quad (6)$$

reduces to the equations

$$A + B + \ldots + D = T \quad (7)$$
$$AB + AC + \ldots + BA + BC + \ldots = U$$
$$ABC\ldots D = W,$$

which are invariant under permutations of A and B, A and C, B and C etc., i.e. of n objects. Galois theory states there is no equation to convert A, B, C etc. in terms of combinations of T, U, … W for n > 4.

However, the argument for reducing to groups of permutations breaks down in the more complicated situation of

$$AB \neq BA$$

etc. – non-commutative algebras. If we expand solutions to belong to non-commutative algebras, we can represent these by hyperintricate numbers.

6.5 Intricate zeros.

As is proved in [Ad12i] the number of integer intricate factorisations of any integer, including zero, is countably infinite. If we multiply two intricate numbers and set them to zero

$$(a1 + bi + c\alpha + d\phi)(p1 + qi + r\alpha + s\phi) = 0, \quad (8)$$

then on equating intricate parts to zero and eliminating a in two ways

$$b(p^2 + q^2 - r^2 - s^2) = 0.$$

A similar argument can be made on eliminating p.

The expression $(p^2 + q^2 - r^2 - s^2)$ is the determinant of its intricate number, so we are dealing with a singular matrix; in other words, its intricate inverse does not exist. This may be reformulated by taking the determinants of both sides of (8) and noting that the determinant of a product is the product of the determinants. □

6.6 Non-unique factorisation for hyperintricate polynomials.

The equaliser of two polynomials f(x) and g(x) is their intersection, so a polynomial equation is an equaliser of f(x) with the constant zero polynomial.

We first note that hypercomplex, hyperactual and hyperphantom polynomials are commutative.

Theorem. *Unique factorisation fails for hyperintricate polynomials of degree* > 1.

Proof. We give examples of this. Let a be a real number. Both of the polynomial equations

$$(x - a)(x + a) = x^2 - a^2 = 0$$

and

$$(x - a\alpha)(x + a\alpha) = x^2 + ax\alpha - a\alpha x - a^2 = 0,$$

the latter treated as a commutative hyperactual polynomial equating to

$$x^2 - a^2 = 0,$$

amount to the same equation with non-unique solutions as intricates.

A third solution is

$$(x - a\phi)(x + a\phi) = 0.$$ □

6.7 J-abelian solutions in hyperintricate Galois theory.

If we allocate a polynomial of the form (5) where T, U, ... W are real coefficients, or are J-abelian, and we select an intricate value of x which is not a multifunction, then this polynomial may be represented, as indicated in reference [Ad12b], by using a J-abelian x of the form

$$x = e^{y + JK} + Je^{z + JL},$$

where y, z, K and L are real and J is a *constant* intricate number with $J = bi + c\alpha + d\phi$ and $J^2 = 0$ or ± 1.

If v and w are intricate, with v J_1-abelian where $J_1^2 = 0$ or ± 1 and w is similarly J_2-abelian, then for a polynomial in x we can set x = v + w where x is J-abelian, and we can put $x = x_1 + x_2$, where x_1 and x_2 are both J-abelian.

If a general solution by radicals on expanding out v + w were available, then there would be a solution for x, and therefore for expanding out in terms of $x_1 + x_2$, but as we shall see, no such solution exists for $n > 4$.

We have for $J \neq 0$ the polynomial in x acts as a 1-hypercomplex, 1-hyperactual or 1-hyperphantom variable which is abelian in J, and maps bijectively to i, α or ϕ respectively. For a 1-hypercomplex, i.e. complex variable, Galois solvability criteria apply. For a 1-hyperactual or 1-hyperphantom variable, x is not in a field, there not necessarily being a multiplicative inverse, but it is still a member of a polynomial ring and again Galois solvability criteria apply. When $J = 0$ the algebra is J-abelian and so cannot conform to non-commutative Galois solvability criteria for $n > 4$.

Thus there are no intricate solutions by radicals of the quintic or general higher degree polynomials with real coefficients that are not multifunctions.

Since no solution by radicals exists for J-abelian x with real coefficients for $n > 4$, no solution by these means exists for a J_1-abelian variable with general J_2-abelian coefficients, or with general intricate coefficients, or for general extensions with terms of degree r of the form

$$R_0x^r + xR_1x^{r-1} + ... + x^rR_r,$$

where $R_0 ... R_r$ are intricate.

For dimension n, n-hyperintricate numbers possess a maximal abelian subring. Thus for 2-hyperintricate numbers we can form abelian vectors with basis 1_1, J_1, 1_K and J_K, where $J^2 = 0, \pm1$ and $K^2 = 0, \pm1$, comprising

$$x = 1_1e^{y_1+J_1L_1} + J_1e^{z_1+J_1M_1} + 1_1e^{y_2+1_KL_2} + 1_Ke^{z_2+1_KM_2} + 1_1e^{y_3+J_KL_3} + J_Ke^{z_3+J_KM_3},$$

which is closed under addition and vector multiplication, so that in such a polynomial ring, Galois solvability criteria again apply.

A general hyperintricate *cannot* be represented by X_Y where X and Y are separately J-abelian and J′-abelian respectively (X_Y has 8 components whereas a general 2-hyperintricate has 16).

If $X = A_B$ satisfies a polynomial equation with real coefficients, then by symmetry so does $(A_B)^{\sim} = B_A$. The operator ~ is covariant: $(A_B)^{\sim}(C_D)^{\sim} = (A_BC_D)^{\sim}$.

We conclude this section with the remark that for J-abelian n-hyperintricate numbers, where we use a 2-hyperintricate variable X_Y for demonstration purposes, if A, B, C and D are constant intricate numbers, R, S, T and U are real constants and a, b, c, d, e, f, g and h are real variables $\neq 0$, then by radicals, when $n > 4$

$$X^n + R(X)^{n-1} + ... + T = 0 \qquad (9)$$

cannot be described by a formula

$$(X + aA)^n + ... + cC = 0 \qquad (10)$$

and by radicals

$$Y^n + S(X)^{n-1} + ... + U = 0$$

cannot be put in the form

$$(Y + bB)^n + ... + dD = 0. \;\square \qquad (11)$$

We need to prove the lemma that if $q_1^2 = 1$, then

$$(\Sigma_ru_r)^2 = [\Sigma_ru_rq_r][\Sigma_r(u_r/q_r)],$$

where for each r, $q_r^2 = 1$. This holds for $r = 1$. Assume it holds for $r = n$. We wish to prove

$$(\Sigma_ru_r + u_{n+1})^2 = [\Sigma_ru_rq_r + u_{n+1}q_{n+1}][\Sigma_r(u_r/q_r) + (u_{n+1}/q_{n+1})],$$

so

$$2(\Sigma_ru_r) = [\Sigma_ru_r(q_r/q_{n+1}) + \Sigma_ru_r(q_{n+1}/q_r)],$$

and if $q_r = 1$ then $q_n = 1$, if q_r -1, $q_n = -1$. □

For example, if the equation (12) to follow reduces in one instance to the equation

$$t1_\alpha + \Sigma_ru_r1_\alpha = 0_0 = 0$$

and t and u are real, then

$$t = -\Sigma_r u_r$$

and for variables p and q_r, on components this implies

$$t(p)1 + \Sigma_r u_r(q_r)1 = 0,$$

$$t(1/p)\alpha + \Sigma_r u_r(1/q_r)\alpha = 0,$$

then $p = \Sigma_r u_r q_r/\Sigma_r u_r$, $1/p = \Sigma_r u_r/q_r\Sigma_r u_r$, $p^2 = 1$, $q_1 = p$, $(\Sigma_r u_r)^2 = (\Sigma_r u_r q_r)(\Sigma_r u_r/q_r)$, so $q_r^2 = 1$, and equations valid on hyperintricates in this way transform to equations valid on intricate numbers.

This reasoning for real t and u is equivalent to the extension where t and u are themselves hyperintricate, say

$$t1_\alpha + \Sigma u\phi_\phi = 0,$$

in which case

$$\Sigma u = -t\phi_i.$$

Since taking layer components maintains additive and multiplicative structures, which cannot interfere with other layers except in the way to be specified,

$$(X_Y)^n + R_S(X_Y)^{n-1} + ... + T_U = 0 \qquad (12)$$

is satisfied exactly by the formulas

$$X^n + eR(X)^{n-1} + ... + gT = 0$$

and

$$Y^n + fS(Y)^{n-1} + ... + hU = 0,$$

where ef = ... = gh = 1, but if n > 4 these cannot be put in the form (10) and (11), and therefore not in the form

$$(X_Y + abA_B)^n + ... + cdC_D = 0,$$

otherwise they could be so allocated, thus (12) is unsolvable by radicals for n > 4.

For a general proof for J-abelian n-hyperintricate numbers, we proceed by a method of descent, separating out the first layer from the remainder of the layers. Then the remainder of the layers form an (n – 1)-hyperintricate polynomial with similar properties to the above case for the second layer. □

6.8 General solutions in hyperintricate Galois theory.

A polynomial in X with real coefficients has the formal properties of a polynomial in x, where x is real or complex, see [Ro90]. In particular commutative properties are satisfied by this polynomial. A distinction is that X may be nilpotent, whereas x cannot be. By the definition of nilpotent

$$X^n = 0,$$

so that by taking determinants and using the multiplicative property of det

$$\det(X^n) = 0,$$

giving

$$\det X = 0.$$

In the non-nilpotent case we now see from this reference that there is no solution by radicals for n > 4 to

$$X^n + aX + b = 0,$$

whereas in the nilpotent case

$$X = -b/a,$$

which is not possible in characteristic zero, where we will have det X = 0. □

CHAPTER VII

Explicit polynomial solutions

7.1 Introduction.

We obtain detailed new types of solution for Galois solvable polynomial equations in degree less than 5 extended to intricate variables. We discuss solutions by equating hyperintricate parts and the relaxation conditions arising from and necessary for such methods in order to obtain solutions.

7.2 Equating hyperintricate parts.

In the allocation of a coefficient, c, to a hyperintricate basis element, say A_B, it is natural to consider c as a real number. However, for polynomial equations in a J-abelian variable as a summation of distinct such terms

$$X_Y = \Sigma cA_B$$

to a power n,

$$(X_Y)^n + E_F(X_Y)^{n-1} + ... + G_H = 0,$$

the imposition of the constraint

$$cA_B = 0 \text{ implies } c = 0,$$

which we call equating hyperintricate parts to zero, may result in equations of the type

$$(c^2 + d^2)A_B = 0 \text{ implies } (c^2 + d^2) = 0, \qquad (1)$$

which are not satisfied by real c or d except both c = 0 and d = 0.

Classical solutions to polynomial equations, obtained by killing central terms in a linear substitution of variables to form a cyclotomic equation, may violate such assumptions.

We are then forced to allow

$$X_Y = \Sigma cA_B$$

in which c is a general intricate or hyperintricate variable. Thus the constraint of equating hyperintricate parts in order to obtain a solution may have to be relaxed if a solution by radicals is to be obtained.

We can bypass such considerations by appending a further index to X_Y and A_B of 1, and employ an exterior coefficient algebra by multiplying an intricate c by this layer of 1. Then in the first instance c is commutative with $A_{B,1}$, unless c is multiplied by other intricate coefficients.

7.3 Intricate roots of unity.

We denote ω_n as a complex n-th root of unity, and Ω_n as an intricate such root. Then from Chapter VI, suppose n = 2m + s, with m, n, k and s natural numbers. Let int be the integer part of a real number and 0! = 1. Then

$$\Omega_n{}^n = 1 = [a1 + bi + c\alpha + d\phi]^n,$$

$$= [a^2 - b^2 + c^2 + d^2 + 2a(bi + c\alpha + d\phi)]^m\Omega_n{}^s. \qquad (2)$$

This gives

$$\Omega_n^n = \sum(k = 0 \text{ to } m)[m!/(m - k)!k!]$$
$$\{2^k a^k (a^2 - b^2 + c^2 + d^2)^{m-k}(-b^2 + c^2 + d^2)^{int(k/2)}(bi + c\alpha + d\phi)^{k - 2int(k/2)}\}\Omega_n^s.$$

What is Ω_2?

$$\Omega_2^2 = 1 = [a^2 - b^2 + c^2 + d^2 + 2a(bi + c\alpha + d\phi)].$$

Equating intricate parts, there are two alternatives, a = 0 or b = c = d = 0, where a = 0 implies

$$1 = -b^2 + c^2 + d^2,$$
$$\Omega_2 = \pm\sqrt{(-1 + c^2 + d^2)}i + c\alpha + d\phi.$$

For an extension to an interior algebra, the square root in parentheses can be imaginary. The alternative b = c = d = 0 results in

$$\Omega_2 = \pm 1 = a.$$

More extensively, if we wished to find the square root of ($a1 + bi + c\alpha + d\phi$), say this is ($p1 + qi + r\alpha + s\phi$), then

$$p^2 - q^2 + r^2 + s^2 + 2p(qi + r\alpha + s\phi) = a1 + bi + c\alpha + d\phi$$

so that when p = 0 this implies b = c = d = 0, and otherwise on equating intricate parts and solving a quadratic (we will assume restrictively that p is complex. Real coefficients may be generalised as those in an interior or exterior intricate algebra.)

$$p = \pm\sqrt{[2a \pm\sqrt{(4a^2 + b^2 - c^2 - d^2)}]},$$
$$q = b/2p,\ r = c/2p \text{ and } s = d/2p.$$

We will select and define Ω_3 by the following method

$$\Omega_3^3 = 1 = [a^2 - b^2 + c^2 + d^2 + 2a(bi + c\alpha + d\phi)](a1 + bi + c\alpha + d\phi).$$
$$= a^3 + 3a(-b^2 + c^2 + d^2) + (3a^2 - b^2 + c^2 + d^2)(bi + c\alpha + d\phi).$$

Initially if we equate intricate parts, then

$$0 = 3a^2 - b^2 + c^2 + d^2$$

so

$$1 = -8a^3 = (-2a)^3.$$

If we now deviate from the allocation of a, b, c and d as real, but maintain the above relations, then for complex values of a:

$$-2a = 1,\ \omega_3 \text{ or } \omega_3^2,$$

and

$$-b^2 + c^2 + d^2 = -3a^2,$$
$$= -\tfrac{3}{4},\ -\tfrac{3}{4}\omega_3^2 \text{ or } -\tfrac{3}{4}\omega_3.$$

Hence the intricate cube roots of unity are

$$\Omega_3 = -\tfrac{1}{2} + (\tfrac{3}{4} + c^2 + d^2)^{1/2}i + c\alpha + d\phi,$$
$$\Omega_3 = -\tfrac{1}{2}\omega_3 + (\tfrac{3}{4}\omega_3^2 + c^2 + d^2)^{1/2}i + c\alpha + d\phi$$

or

$$\Omega_3 = -\tfrac{1}{2}\omega_3^2 + (\tfrac{3}{4}\omega_3 + c^2 + d^2)^{1/2}i + c\alpha + d\phi.\square$$

Alternatively, we can use the Euler relations of Chapter VI, in which

$$e^{i\theta} = \cos\theta + i\sin\theta,$$
$$e^{\alpha\theta} = \cosh\theta + \alpha\sinh\theta$$

and

$$e^{\phi\theta} = \cosh\theta + \phi\sinh\theta$$

imply in the first case that

$1 = \cos(2\pi k) + i \sin(2\pi k)$

and thus the complex nth root of unity is uniquely

$\omega_n = 1^{1/n} = e^{2\pi ki/n}$.

For intricate numbers, note that

$e^{fi + g\alpha + h\phi} \neq e^{fi}e^{g\alpha}e^{h\phi}$,

since the exponentiated sum on the left commutes, but by the Euler relations, the product on the right does not.

However, if we consider

$e^{(bi + c\alpha + d\phi)U}$,

where the square of $bi + c\alpha + d\phi$ satisfies $-b^2 + c^2 + d^2 = -1$, then a Taylor series expansion equates this to

$\cos U + (bi + c\alpha + d\phi)\sin U$.

Thus the intricate nth root of unity is

$\Omega_n = 1^{1/n} = e^{2\pi k(bi + c\alpha + d\phi)/n}$,

where $-b^2 + c^2 + d^2 = -1$.

Non-unique values of Ω_n may satisfy, say

$1 + \Omega_{3A} + \Omega_{3B}^2 \neq 0$,

since putting

$\Omega_{3A} = e\uparrow\{[2\pi k(\pm_A \sqrt{}(1 + c^2 + d^2)i + c\alpha + d\phi]/3\}$,

$\Omega_{3B} = e\uparrow\{[2\pi k(\pm_B \sqrt{}(1 + g^2 + h^2)i + g\alpha + h\phi]/3\}$

gives

$1 + \Omega_{3A} + \Omega_{3B}^2 = \{[\pm_A \sqrt{}(1 + c^2 + d^2) \pm_B \sqrt{}(1 + g^2 + h^2)]i$

$+ (c - g)\alpha + (d - h)\phi\}(\sqrt{}3)/2$.

Also in general, as may be computed

$(\Omega_{3A})(\Omega_{3B}^2) \neq 1$.

More extensively an intricate number may be represented by

$e^p e^{(bi + c\alpha + d\phi)U}$

so the nth root is, if $-b^2 + c^2 + d^2 = -1$, and not +1 or 0,

$e^{p/n}[\cos (U/n) + (bi + c\alpha + d\phi)\sin (U/n)]$. □

7.4 The classical complex quadratic.

We will now explore the consequences of equating complex parts in the classical solution of the quadratic.

Let X, R and S be complex numbers, where $X = a + bi$, $R = r + ti$, $S = s + ui$, and

$X^2 + RX + S = 0$. (3)

We will use the method of equating complex parts, so the real and imaginary parts are equated separately to zero, giving

$a^2 - b^2 + ra - tb + s = 0$, (4)

$2ab + rb + ta + u = 0$. (5)

The solution on putting all imaginary coefficients, t and u to zero, gives the classical solution

$$a = -r/2,$$
$$b = \pm\tfrac{1}{2}\sqrt{(4s - r^2)},$$

which can be extended to solutions with r and s complex.

On denying ourselves this luxury, we obtain the quartic equation

$$4a^4 + 4ra^3 + [5r^2 + 4s + (2r - 1)t^2]a^2$$
$$+ r[4(r^2 + s) + t^2]a + [r^2(r^2 + s) + rtu - u^2] = 0,$$

solvable also by classical means. Once again, these solutions can be extended to r, s, t and u complex. □

7.5 The quadratic in an intricate variable.

Method 1. Firstly, let us see what equating intricate parts means for the quadratic. We describe the case of a quadratic in intricate variables with real coefficients. This case subsumes that for a quadratic in a complex variable with real coefficients. Solutions for real coefficients may be extended to coefficients in an intricate algebra. The constraints on this are revealed in the discussion of *Method 2*.

We will set

$$X^2 + RX + S = 0, \tag{6}$$

with R and S *real* numbers, where

$$X = a1 + bi + c\alpha + d\phi,$$

so

$$X^2 = (a^2 - b^2 + c^2 + d^2)1 + 2abi + 2ac\alpha + 2ad\phi.$$

Thus the real part gives

$$(a^2 - b^2 + c^2 + d^2) + Ra + S = 0, \tag{7}$$

and each intricate part gives

$$a = -R/2, \tag{8}$$

provided one of $b \neq 0$, $c \neq 0$ or $d \neq 0$ and thus

$$b = \pm[\sqrt{(-(R^2/4) + S + c^2 + d^2)}].$$

Hence

$$X = -(R/2)1 \pm[\sqrt{(-(R^2/4) + S + c^2 + d^2)}]i + c\alpha + d\phi. \square \tag{9}$$

Method 2. If we wished to use more traditional methods, by killing central terms, then putting $X = Y + A$ gives under the assumption that Y and A mutually commute

$$Y^2 + (2A + R)Y + A^2 + RA + S = 0.$$

If we put

$$A = -R/2,$$

then

$$Y^2 = (R^2/4) - S.$$

The assumption that Y and A commute for

$$Y = p1 + qi + r\alpha + s\phi$$

and

$$A = f1 + gi + h\alpha + k\phi,$$

is

$$0 = YA - AY = 2\{(rk - sh)i + (qk - gs)\alpha + (gr - qh)\phi\}.$$

Eliminating r and q, on equating intricate parts, the ϕ coefficient is

$$g(sh/k) - (gs/k)\,h = 0,$$

which is equivalent to the two constraints of just

$$r = sh/k,$$
$$q = gs/k,$$

i.e. that the non-real parts of Y and A are in the same ratio. This is no more than that Y and A are in the same J-abelian format.

To obtain the intricate square root, we append

$$\Omega_2 = 1^{1/2} = e^{\pi k(bi + c\alpha + d\phi)},$$

where

$$b^2 = 1 + c^2 + d^2.$$

Then on putting $\pm_b$ as resulting from the square root of b^2 and $\pm_Y$ as resulting from the square root of Y^2, we have

$$X = -(R/2)1 \pm_b \pm_Y\sqrt{[(-(R^2/4) + S)(1 + c^2 + d^2)]}i \pm_Y [\sqrt{((R^2/4) - S)}][c\alpha + d\phi].$$

So transforming variables

$$c \rightarrow \pm_Y -d/[\sqrt{(-R^2/4) + S}],$$
$$d \rightarrow \pm_Y c/[\sqrt{(-R^2/4) + S}]$$

gives an equation equivalent to (9). □

Method 3. The method we have used for the quadratic as a standard method of killing central terms is in J-abelian format, which we have seen in Chapter VIII is the most general for J-abelian hyperintricate variables with real coefficients, other solutions of J-abelian type being supersets of this form. What we have done so far is assume Y and A mutually commute, and have extended the solution by means of intricate roots of unity.

We will now explore for the quadratic true non-commutative solutions, *ab initio*. The method will parallel in other respects the solution by standard methods.

Let

$$A = a1 + bi + c\alpha + d\phi,$$
$$B = f1 + gi + h\alpha + k\phi$$

and

$$X = A + B.$$

Generally $AB \neq BA$, and we will not assume otherwise. Then equation (6) becomes

$$A^2 + [AB + BA + RA] + B^2 + RB + S = 0.$$

Now AB is the sum of symmetric and antisymmetric terms:

$$\begin{aligned} AB = {} & a1(gi + h\alpha + k\phi) \\ & + f1(bi + c\alpha + d\phi) \\ & + af - bg + ch + kd \\ & + (ck - hd)i + (bk - gd)\alpha + (bh - cg)(-\phi) \end{aligned}$$

The first three lines being symmetric: AB = BA. Thus if we consider AB + BA it consists of these first three lines only, doubled.

We will now kill the central term [AB + BA + RA]. Assume as an example that R and S are real. We have, on equating intricate parts

$$2[af - bg + ch + kd] = -Ra,$$
$$2[ag + fb] = -Rb,$$
$$2[ah + fc] = -Rc,$$
$$2[ak + fd] = -Rd,$$

so

$$[(R/2) + f]^2 = -g^2 + h^2 + k^2. \qquad (10)$$

Then

$$A^2 + B^2 + RB + S = 0$$

becomes, on equating intricate parts

$$a^2 - b^2 + c^2 + d^2 + f^2 - g^2 + h^2 + k^2 + Rf + S = 0$$

or

$$a^2 - b^2 + c^2 + d^2 + 2[(R/2) + f]^2 - R^2/4 + S = 0$$

with

$$ab + [(R/2) + f]g = 0,$$
$$ac + [(R/2) + f]h = 0,$$
$$ad + [(R/2) + f]k = 0.$$

This leads to

$$a^2 + [(R/2) + f]^4/a^2 + 2[(R/2) + f]^2 - R^2/4 + S = 0,$$

which is solvable for a in terms of a free parameter f, which we can limit arbitrarily, if we do not want to solve by other methods, to $[(R/2) + f] = a$, where $b = -g$, $c = -h$ and $d = -k$, with k satisfying the constraint (10). □

7.6 The cubic in a complex and an intricate variable.

Method 1a. The method of equating intricate parts is useful because it does precisely that: it splits the solution into intricate parts. Again, the coefficients employed in the solution may be generalised as those in an intricate algebra. Let

$$X^3 + RX + S = 0,$$

with

$$X = a1 + bi + c\alpha + d\phi,$$
$$R = g1 + hi$$

and

$$S = t1 + ui.$$

Then

$$X^3 = (a^2 - 3b^2 + 3c^2 + 3d^2)a1$$
$$+ (3a^2 - b^2 + c^2 + d^2)bi$$
$$+ (3a^2 - b^2 + c^2 + d^2)c\alpha$$
$$+ (3a^2 - b^2 + c^2 + d^2)d\phi.$$

Then on equating intricate parts

$$(a^2 - 3b^2 + 3c^2 + 3d^2)a + ga - hb + t = 0, \qquad (11)$$
$$(3a^2 - b^2 + c^2 + d^2)b + gb + ha + u = 0, \qquad (12)$$
$$(3a^2 - b^2 + c^2 + d^2)c + gc - hd = 0, \qquad (13)$$
$$(3a^2 - b^2 + c^2 + d^2)d + gd + hc = 0, \qquad (14)$$

then if $h = 0$, (12) implies $v = 0$, so (11) gives

$$8a^3 + 2ga - t = 0, \qquad (15)$$

and if $h \neq 0$, (13) and (14) imply

$$(3a^2 - b^2 + c^2 + d^2 + g)(c^2 + d^2) = 0, \quad (16)$$

and assuming real values for c and d implies

$$8a^3 + 2ga + hb - t = 0, \quad (17)$$

$$gb + ha + u = 0, \quad (18)$$

$$gc - hd = 0, \quad (19)$$

$$gd + hc = 0, \quad (20)$$

with the contradiction from (19) and (20) of $(c^2 + d^2) = 0$. Thus we must abandon the assumption that both c and d are real. (17) and (18) now give

$$8a^3 + 2ga - (h/g)a - (u/g) - t = 0. \; \square \quad (21)$$

It is possible a solution of (21) for a, obtainable by standard methods itemised next, may not be real. However a cubic in real variables always has one real solution. In other cases we have the option of selecting complex values of a.

Put

$$q = [(t/4) - (u/8t)]$$

and

$$r = (1/8)[(w/t) - v],$$

then [Ro90] set

$$a = y + z,$$

so that

$$a^3 = y^3 + z^3 + 3ayz.$$

Therefore

$$y^3 + z^3 + (3yz + q)a + r = 0. \quad (22)$$

We now impose a second constraint:

$$yz = -q/3,$$

so that in (22) the linear term in a vanishes. We have

$$y^3 + z^3 + r = 0$$

and

$$y^3z^3 = -q^3/27.$$

These two equations can be solved for y^3 and z^3. In detail

$$y^3 - q^3/(27y^3) + r = 0,$$

and hence

$$y^6 + ry^3 - q^3/27 = 0,$$

with

$$z^6 + rz^3 - q^3/27 = 0.$$

The quadratic formula gives

$$y^3 = ½[-r + \sqrt{(r^2 + 4q^3/27)}]$$

$$z^3 = ½[-r - \sqrt{(r^2 + 4q^3/27)}].$$

If ω is a cube root of unity, the six cube roots available are

$$y, \omega y, \omega^2 y, z, \omega z \text{ and } \omega^2 z,$$

which may be paired to give a product $-q/3$

$$-q/3 = yz = (\omega y)(\omega^2 z) = (\omega^2 y)(\omega z).$$

We conclude that the roots of the cubic are

$$y + z, \omega y + \omega^2 z \text{ and } \omega^2 y + \omega z,$$

where

$$y = [\, ½(-r + \sqrt{(r^2 + 4q^3/27)})]^{1/3}$$

and

$$z = [\,½(-r - \sqrt{(r^2 + 4q^3/27)})]^{1/3}. \ \square$$

Method 1b. We now extend method 1a to the case where R and S are intricate numbers. X retains the names of its intricate coefficients, but now

$$R = g1 + hi + j\alpha + k\phi$$

and

$$S = t1 + ui + v\alpha + w\phi,$$

and again

$$X^3 + RX + S = 0,$$

so that on putting $J^2 = -b^2 + c^2 + d^2$ on equating intricate parts

$$(a^2 - 3J^2)a + ga - hb + jc + kd + t = 0, \qquad (23)$$

$$(3a^2 - J^2)b + gb + ha + ck - dj + u = 0, \qquad (24)$$

$$(3a^2 - J^2)c + gc + ja + bk - hd + v = 0, \qquad (25)$$

$$(3a^2 - J^2)d + gd + ka - bj + hc + w = 0, \qquad (26)$$

and we have four variables a, b, c and d in the four equations (23) to (26). For b, c and $d \neq 0$, on multiplying (23) by 3bcd, and subtracting acd × (24), abd × (25) and abc × (26) we obtain three equations in a^2, which enables us to derive two equations in a:

$$[(-8J^2 + 2g)(bj - ch) - jkc + j^2d + hkb - h^2d + ju + hv]a$$
$$+ 3[-hj(b^2 + c^2) + jkbd + j^2bc - hkcd - h^2bc - jtb - htc] = 0, \qquad (27)$$

and

$$[(-8J^2 + 2g)(bk - dh) - k^2c + jkd - hjb - h^2c - ku + hw]a$$
$$+ 3[-hk(b^2 + d^2) + k^2bd + jkbc - hjcd + h^2bd + ktb - htd] = 0. \qquad (28)$$

On eliminating a, we have an equation in b^5, and using (27) for a in (24), (25) and (26), we obtain four equations without a in b^5. □

CHAPTER VIII

Varieties

CHAPTER IX

Exponential algebra

9.1 Introduction.

We develop some elementary features of the intricate and hyperintricate *exponential algebra* and in particular the Euler relations – an extension of the $e^{i\theta}$ expansion idea. Hyperintricate roots and the hyperintricate binomial theorem are also addressed.

9.2 The intricate Euler relations.

Complex numbers satisfy the Euler relation with positive determinant

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which can be obtained using a Taylor series expansion, where

$$e^{\lambda} = 1 + \lambda + \lambda^2/2 + \lambda^3/3! + \ldots$$

For intricate basis elements α and ϕ, a similar argument gives

$$e^{\alpha\theta} = \cosh\theta + \alpha\sinh\theta$$

and

$$e^{\phi\theta} = \cosh\theta + \phi\sinh\theta.\square$$

There are many formulae derivable from the Euler relations. Since

$$\cosh\theta = 1 + \theta^2/2 + \theta^4/4! + \ldots$$

$$\sinh\theta = \theta + \theta^3/3! + \theta^5/5! + \ldots$$

we also have

$$\cos\theta = \cos\alpha\theta = \cos\phi\theta,$$

$$\cosh\theta = \cosh\alpha\theta = \cosh\phi\theta,$$

and for instance, using $\alpha\phi = i$,

$$i = e^{\alpha\theta}e^{\phi\theta} \pm 2^{1/2}e^{\alpha\theta} \pm 2^{1/2}e^{\phi\theta} + 2,$$

with $\theta = \sinh^{-1}\pm 1 = \pm\cosh^{-1}\pm 2^{1/2}$. □

If we choose to represent

$$e^{h} = e^{a1 + bi + c\alpha + d\phi} = e^{a1}e^{bi}e^{c\alpha}e^{d\phi}, \quad (1)$$

then multiplicative non-commutativity, e.g.

$$e^{bi}e^{c\alpha} \neq e^{c\alpha}e^{bi}$$

from the Taylor series expansions, immediately tells us that for abelian addition (1) cannot hold, although a1 commutes. In the example above the non-commutation expressed as $e^{bi}e^{c\alpha} - e^{c\alpha}e^{bi}$ depends on ϕ, b and c only.

The square of $J = (bi + c\alpha + d\phi)$ is $(-b^2 + c^2 + d^2)$. Setting $z = a1 + JK$, for K real, when $J^2 = -1$ the Taylor expansion gives

$$e^{z} = e^{a1 + (bi + c\alpha + d\phi)K} = e^{a1 + JK}$$

$$= e^{a1}(\cos K + J\sin K), \quad (2)$$

when $J^2 = +1$

$$e^{z} = e^{a1}(\cosh K + J\sinh K) \quad (3)$$

and when $J^2 = 0$

$$e^{z} = e^{a1}(1 + JK + J^2K^2/2 + J^3K^3/3! + \ldots) = e^{a1}\,(1 + JK). \quad (4)$$

We note for representations of this form that

$$e^{a1 + JL + JM} = e^{a1}e^{JL}e^{JM}. \quad \square \qquad (5)$$

It is useful to classify solutions. On putting

$$E = r\alpha + s\phi$$

so that

$$E^2 = r^2 + s^2,$$

then when

$$b^2 = E^2 + 1$$

we say the solution is trigonometric, implying we use trigonometric functions, when

$$b^2 = E^2,$$

the solution is linear and when

$$b^2 = E^2 - 1$$

the solution is hyperbolic, employing hyperbolic functions.

Incidentally, the expression $e^{p1 + (qi + r\alpha + s\phi)K}$ is equal to

$$\tfrac{1}{2}e^{p1}[(1 - q - s\alpha + r\phi)e^{-iK} + (1 + q + s\alpha - r\phi)e^{iK}]$$

when $J^2 = -1$,

$$\tfrac{1}{2}e^{p1}[(1 - qi - r\alpha - s\phi)e^{-K} + (1 + qi + r\alpha + s\phi)e^{K}]$$

when $J^2 = 1$, and evaluates as $e^{p1}[1 + (qi + r\alpha + s\phi)K]$ when $J^2 = 0$. $\square$

The inverse of $e^{p1+(qi + r\alpha + s\phi)K}$ is

$$e^{-p1-(qi + r\alpha + s\phi)K} = e^{-p1}[\cos K - (qi + r\alpha + s\phi)\sin K]$$

for $J^2 = -1$, with corresponding expressions for $J^2 = 1$ and $J^2 = 0$ (for the intricate conjugate we multiply by e^{p1} rather than e^{-p1}). Further, for $J^2 = -1$

$$e^{p1+(qi + r\alpha + s\phi)K} = -[e^{p1+(qi + r\alpha + s\phi)(K + \pi)}],$$

and similar considerations give

$$\sin K - (qi + r\alpha + s\phi)\cos K = e^{(qi + r\alpha + s\phi)(K + \pi/2)}. \quad \square$$

We can determine trigonometric products or products of other type, so that

$$\{e\uparrow[a1+ (bi + c\alpha + d\phi)K_1]\}\{e\uparrow[p1+ (qi + r\alpha + s\phi)K_2]\} =$$
$$\tfrac{1}{2}e^{a+p}\{(1 + M)\cos(K_1 + K_2) + (1 - M)\cos(K_2 - K_1)$$
$$+ [(q + b)i + (r + c)\alpha + (s + d)\phi]\sin(K_1 + K_2)$$
$$+ [(q - b)i + (r - c)\alpha + (s - d)\phi]\sin(K_2 - K_1)\},$$

where

$$M = (bq - cr - ds) + (cs - dr)i + (bs - dq)\alpha + (-br + cq)\phi. \quad \square$$

Chapter VI introduces g, where $g^2 = \pm J$, $J^2 = \pm 1$. The assignation $g^4 = 1$ with $g = i^i$ provides a non-multifunction complex algebra for exponentiation and is part of a study of hyperintricate exponentiation.

9.3 Comparing exponential products and the Euler relations.

If we wish to express

$$e^{p + (bi + c\alpha + d\phi)K} = e^p[\cos K + (bi + c\alpha + d\phi)\sin K]$$

where $-b^2 + c^2 + d^2 = -1$ as $e^w e^{xi} e^{y\alpha} e^{z\phi}$, then we can expand this out using the Euler relations for e^{xi}, $e^{y\alpha}$ and $e^{z\phi}$, as

$$e^w(\cos x + i\sin x)(\cosh y + \alpha\sinh y)(\cosh z + \phi\sinh z).$$

Putting X = cos x, Y = cosh y and Z = cosh z and equating intricate parts, we obtain

$$e^{w}[XYZ - (1 - X^2)^{1/2}(1 + Y^2)^{1/2}(1 + Z^2)^{1/2}] = e^{p}\cos K, \quad (6)$$
$$e^{w}[(1 - X^2)^{1/2}YZ + X(1 + Y^2)^{1/2}(1 + Z^2)^{1/2}] = e^{p}b\sin K, \quad (7)$$
$$e^{w}[X(1 + Y^2)^{1/2}Z + (1 - X^2)^{1/2}Y(1 + Z^2)^{1/2}] = e^{p}c\sin K, \quad (8)$$
$$e^{w}[XY(1 + Z^2)^{1/2} - (1 - X^2)^{1/2}(1 + Y^2)^{1/2}Z] = e^{p}d\sin K. \quad (9)$$

Squaring and adding these terms, but for (8) and (9) subtracting, gives

$$e^{2w} = e^{2p}, \quad (10)$$

whereas squaring (6) and substituting from (10) gives

$$\sin^2 K = X^2 - Y^2 - Z^2 + X^2Y^2 + X^2Z^2 - Y^2Z^2 + 2N, \quad (11)$$

where

$$N^2 = X^2Y^2Z^2(1 - X^2)(1 + Y^2)(1 + Z^2), \quad (12)$$

so correspondingly (6) and (7) squared and added give with (10)

$$(b^2 - 1)\sin^2 K = Y^2 + Z^2 + 2Y^2Z^2, \quad (13)$$

and similarly f or (7) squared, subtracted from (8) squared we obtain

$$(1 + d^2)\sin^2 K = X^2 - Y^2 + 2X^2Y^2. \quad (14)$$

We can devise solutions by expressing Z in terms of c, d, K, X and Y in the following linking equation. Put

$$T = (1 + c^2)\sin^2 K - X^2,$$
$$U = (1 + d^2)\sin^2 K - X^2 + Y^2,$$

then eliminating Y^2Z^2 and X^2Y^2 terms in (11) and (12) from (13) and (14) gives

$$[4U(2 - U) - (1 - 2X^2)^2]Z^4 + [8U(1 - X^2 + Y^2) - 2T(1 - 2X^2)]Z^2 - T^2 = 0. \ \square$$

9.4 Intricate zero determinants and negative determinants.

That the solutions already developed are not the most general may be deduced from the observation that the determinant of $e^{a1 + (bi + c\alpha + d\phi)K}$ is given by an expression of the form (2), (3) or (4) multiplied by its intricate conjugate, and in each case this is e^{2a1}, which cannot be zero or negative.

Since a determinant is a multiplicative function, that is for matrices C and D

$$\det CD = \det C \det D,$$

a representation with zero or negative determinant may be obtained on multiplying by an intricate number which itself has a zero or negative determinant. A non-unique intricate representation extending this type is

$$W = e^{x1 + J_1L} + \Delta.e^{y1 + J_2M},$$

with det Δ = -1.

When $J^2 = -1$, the determinant det J = $b^2 - c^2 - d^2 = +1$, and $Je^{a + JK} = e^{a + J(\pi/2 - K)}$ has positive determinant. The intricate number $Je^{a + JK}$ has respectively zero or negative determinant when det J is zero or negative respectively.

Consider two intricate numbers, U and V, where J is fixed and of the form $J^2 = 1$, where

$$U = e^{x_1 + JK_1} + Je^{y_1 + JL_1},$$
$$V = e^{x_2 + JK_2} + Je^{y_2 + JL_2},$$

then U and V commute, whereas if $J^2 = 0$ and

$$U = e^{x_1}(1 + JK_1) + Je^{y_1}(1 + JL_1)$$
$$= e^{x_1} + J(e^{x_1}K_1 + e^{y_1}),$$

$V = e^{x_2} + J(e^{x_2}K_2 + e^{y_2})$,

then again U and V commute. In neither case is the form for U and V the most general, so that the representations we have chosen, for fixed J, are abelianisations of the general case.

We now force a non-commutative algebra where $\mathcal{J} = bi + c\alpha + d\phi$, under the bijective mapping $\mathcal{J} \leftrightarrow i$, $\mathcal{A} \leftrightarrow \alpha$ and $\mathcal{F} \leftrightarrow \phi$ of Chapter I. The form is now

$W = e^{x_1 + \mathcal{J}K_1} + \mathcal{J}e^{x_2 + \mathcal{J}K_2} + \mathcal{A}e^{x_3 + \mathcal{J}K_3} + \mathcal{F}e^{x_4 + \mathcal{J}K_4}$.

This form is not restrictive up to sign since $e^{\mathcal{J}K} = -[e^{\mathcal{J}(K + \pi)}]$, $\mathcal{J} = -(bi + c\alpha + d\phi)$ also satisfies $\mathcal{J}^2 = -1$, and $\mathcal{A} \to -\mathcal{A}$, $\mathcal{F} \to -\mathcal{F}$ complies. However, there are two types of algebras here, with non-isomorphic handedness, that is chirality, for the triple $\mathcal{J}$, $\mathcal{A}$, $\mathcal{F}$.

The form is now projective to a general intricate number, which can have positive, zero or negative determinant. □

9.5 J-abelian intricate powers.

Let J_n for a variable n, where the J_n are distinct with $J_n^2 = 0$ or ± 1, satisfy

$$U_n = e^{x_1 + J_nK} + J_ne^{y_1 + J_nL}. \qquad (15)$$

Distinct U_n are non-commutative. This equation is of the most general form, since the determinant of U_n for $J_n^2 = -1$ is positive. Under $i \leftrightarrow J_n$ equivalence the determinant of $p1 + qi + r\alpha + s\phi$ is greater than zero with value $p^2 + q^2 - r^2 - s^2$ for $J_n^2 = -1$. □

Binomial expansions of (15) for real powers are abelian for fixed J using the binomial theorem for powers of intricate numbers, under i, α or φ ↔ J equivalence if $J^2 \neq 0$. □

9.6 The hyperintricate Euler relations.

The material of this section has been simplified and extended in Chapter II, where the J-layered approach to hyperintricate representations is discussed.

An n-hyperintricate number cannot be represented in general by a product of (n – 1)-equivalent hyperintricates, since to take the case n = 2, we may represent a product by

$$Я_2 = (A_1 + B_1i_1 + C_1\alpha_1 + D_1\phi_1)(T_1 + U_11_i + V_11_\alpha + W_11_\phi) \qquad (16)$$

and this has 8 variables whereas $Я_2$ has 16. The two factors above commute. For a full hyperintricate number, $Я_n$ can never be represented in the above way. However, we can represent $Я_2$ by a double sum of products

$$Я_2 = \Sigma(r = 1, 2)(A_r + B_ri_1 + C_r\alpha_1 + D_r\phi_1)(T_r + U_r1_i + V_r1_\alpha + W_r1_\phi). \qquad (17)$$

Extended alternatives to the form (16) are that T, U, V and W are complex of type T = (a + bi), or respectively of type T = (a + cα) or T = (a + dφ). These examples of interior coefficient algebras may be considered as a restatement of the form (17).

We now obtain a version of the n-hyperintricate Euler relations as an extension of form (12). The number of n-hyperintricate basis elements is 4^n, which may be represented as $(-1 + 5)^n = (1 + 3)^n = 1 + 3m$ for some m. So if n is even, m is divisible by 5, and if n is odd, (m – 1) is divisible by 20. We will use this to allocate an n-hyperintricate number as a real part and 3m other parts.

Let L range over the 3 values {i, α, ϕ}, $Ч_{rL}$ be an n-hyperintricate basis element, all layers of which are 1 except those taken from a triple L in position r, and h_{qrL} be a subscripted coefficient for $Ч_{rL}$. We will use u for r in context.

Then using the floor function $\lfloor\ \rfloor$, or integer part, for $Я_n$ such a product of intricates has 4n variables, therefore there must be $N = \lfloor 1 + [(4^{n-1} - 1)/n]\rfloor$ sums of these products, and if n is not a power of 2, the last product can contain less terms than the previous N – 1.

Thus an n-hyperintricate number may be represented by

$$Я_n = \Sigma(q = 1 \text{ to } N)\,\Pi(r = 1 \text{ to } n)[p_{qr} + \Sigma(\text{over } L)h_{qrL}Ч_{rL}]$$
$$= \Sigma(q = 1 \text{ to } N)\{\Pi(r = 1 \text{ to } n)p_{qr} + \{\Sigma(\#s = 1 \text{ to } n)$$
$$\Pi(\text{over } t = n \text{ except for } \#s \text{ subselections})p_{qt}$$
$$\Pi(\text{over } u = n \text{ except for } n - \#s \text{ subselections})[\Sigma(\text{over } L)h_{quL}Ч_{uL}]\}\},$$

where in products different values of u give commuting $Ч_{uL}$, and

$$[\Sigma(\text{over } L)h_{quL}Ч_{uL}]^2 = [-h_{qui}^2 + h_{qu\alpha}^2 + h_{qu\phi}^2].$$

We will write

$$[-h_{qui}^2 + h_{qu\alpha}^2 + h_{qu\phi}^2] = J_{qu}^2K_{qu}^2,$$

where $J_{qu}^2 = 0$ or ± 1. Then when the product $\Pi_uJ_{qu}^2 = 0$

$$e\uparrow(\Pi_uJ_{qu}K_{qu}) = [1 + \Pi_uJ_{qu}K_{qu}],$$

when the product $\Pi_uJ_{qu}^2 = -1$

$$e\uparrow(\Pi_uJ_{qu}K_{qu}) = [\cos(\Pi_uK_{qu}) + \Pi_uJ_{qu}\sin(\Pi_uK_{qu})]$$

and when the product $\Pi_uJ_{qu}^2 = 1$

$$e\uparrow(\Pi_uJ_{qu}K_{qu}) = [\cosh(\Pi_uK_{qu}) + \Pi_uJ_{qu}\sinh(\Pi_uK_{qu})].$$

Then for q = 1 to N, r = 1 to n

$$Я_n = \Sigma_q\{\Pi_rp_{qr} + \Sigma_{\#s}[\Pi_tp_{qt}][e\uparrow\Pi_u(J_{qu}K_{qu})]\}.$$

These are the n-hyperintricate Euler relations. □

Also for n = 2 a full representation is

$$Я_2 = A + J_1B + 1_{J'}C + J'''_{J''}D + J''''_{J''}E, \tag{18}$$

with 16 components, for which in general $J \neq J' \neq J'' \neq J''' \neq J''''$, where the first three terms amount to 7 variables and the last two to 9. We allocate $J_1^2 = 0, \pm 1$, $(1_J)^2 = 0, \pm 1$, $(J'''_{J''})^2 = 0, \pm 1$, with $(J''')^2 = G'''$, $(J'')^2 = H$, $G'''H = 0, \pm 1$ and $(J''')^2 = 0, \pm(J'''')^2$.

For fixed J_1, $1_{J'}$, $J'''_{J''}$, $J''''_{J''}$ and real variables A, B, C, D and E there is no analogue of J-abelian; for two such 2-hyperintricate numbers usually

$$Я_2Я'_2 \neq Я'_2Я_2,$$

since J_1 and $J'''_{J''}$, J_1 and $J''''_{J''}$, $1_{J'}$ and $J'''_{J''}$, $1_{J'}$ and $J''''_{J''}$, and $J'''_{J''}$ and $J''''_{J''}$ do not generally commute.

For n-hyperintricate numbers the representation related to (18) has 4^n components in $Я_n$, and the number of combinations of various J's is greater than this for n > 8, so such a representation exists either with multiple J's for specific indices as in (18) for $n \leq 8$ or without them for n > 8. □

9.7 Roots of intricate basis elements. [Ad12j]

The multiplicative inverses of the intricate basis elements are as follows.

$1^{-1} = 1$, $i^{-1} = -i$, $\alpha^{-1} = \alpha$, $\phi^{-1} = \phi$.

We shall see that for $\alpha^{1/2}$ and $\phi^{1/2}$, square roots of intricate basis elements can be represented by hyperintricate numbers. For square roots

$$1^{1/2} = \pm 1 \text{ or } \pm(ui + v\alpha + w\phi),$$

with $-u^2 + v^2 + w^2 = 1$, allowing us to expand the list of *possibilities* below, also

$$i^{1/2} = \pm(1 + i)/\surd 2,$$

giving the dependent relation (with the same positive or negative sign)

$$i^{-1/2} = \pm(1 - i)/\surd 2,$$

with

$$\alpha^{1/2} = \pm \begin{Vmatrix} 1 & 0 \\ 0 & \pm i \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \alpha & 0 \\ 0 & \pm i \end{Vmatrix}$$

or

$$\pm \begin{Vmatrix} \phi & 0 \\ 0 & \pm i \end{Vmatrix}$$

and

$$\phi^{1/2} = \pm(1/\surd 2) \begin{Vmatrix} i^{-1/2} & i^{1/2} \\ i^{1/2} & i^{-1/2} \end{Vmatrix}$$

For integers m and n we give the following roots.

$$1^{1/(2n+1)} = e^{i2\pi m/(2n+1)}$$
$$i^{1/(2n+1)} = e^{i\pi(4m+1)/[2(2n+1)]}$$
$$\alpha^{1/(2n+1)} = \alpha$$
$$\phi^{1/(2n+1)} = \phi.$$

$$1^{1/(2n)} = e^{i\pi m/n}$$
$$i^{1/(2n)} = e^{i\pi(4m+1)/(4n)}$$

with

$$\alpha^{1/(2n)} = \pm \begin{Vmatrix} 1 & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \alpha & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \phi & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$\phi^{1/(2n)}$ may be obtained recursively from roots with smaller n. For example, since

$$\phi^{1/2} = \pm(1/\surd 2)\, i^{1/2} \begin{Vmatrix} -i & 1 \\ 1 & -i \end{Vmatrix}$$

we have

$$\phi^{1/4} = \pm(1 \text{ or } i)[i^{1/4}/2^{1/4}] \begin{Vmatrix} a & b \\ b & a \end{Vmatrix}$$

where (see the next paragraph for an indication of how to obtain this)

$a^2 = (-½ \pm 1/\surd 2)i$

$b^2 = -i - a^2$.

More generally, for natural numbers $t = 2^n$ and $u = 2^{n-1}$, if

$$\phi^{1/t} = e^{i2\pi m/t}[i^{1/t}/2^{1/t}] \begin{Vmatrix} P & Q \\ Q & P \end{Vmatrix}$$

where we have previously determined that

$$\phi^{1/u} = e^{i2\pi m/u}[i^{1/u}/2^{1/u}] \begin{Vmatrix} p & q \\ q & p \end{Vmatrix}$$

then on squaring the matrix in P and Q

$P^2 + Q^2 = p$

$2PQ = q$

which reduces to a solvable quadratic equation, say in P^2, so $\phi^{1/t}$ is determined.

If a natural number T = ty, with y an odd number, is encountered instead of t, the determination of $\phi^{1/T}$ can be found from the relation, valid for natural numbers y and t

$\phi^{1/T} = [\phi^{1/y}]^{1/t} = \phi^{1/t}$. □

The results of section 2, where an intricate number may be represented by e^z for a positive determinant, can be used to give a root $e^{z/n}$, although for $J^2 = 1$, cosh cannot take the value zero.

These ideas may be combined. For example, to get around the cosh restriction on representations of α, we may write using an exterior coefficient algebra

$$\alpha^{1/n} = \begin{Vmatrix} e^{2\pi mi/n} & 0 \\ 0 & e^{\pi(2m+1)i/n} \end{Vmatrix}$$

$$\equiv ½(e^{2\pi mi/n} + e^{\pi(2m+1)i/n})1_1 + ½(e^{2\pi mi/n} - e^{\pi(2m+1)i/n})\alpha_1. \ \square$$

Further considerations are given in Chapter VII.

9.8 The intricate binomial theorem for real powers.

Suppose w = 2m + s, *with* m *and* k *natural numbers and* w *real*. *Let* int *be the integer part of a real number and* 0! = 1. *Then*

$Я_1{}^w = [a^2 - b^2 + c^2 + d^2 + 2a(bi + c\alpha + d\phi)]^m Я_1{}^s$.

This gives

$Я_1{}^w = \sum(k = 0 \text{ to } m)[m!/(m - k)!k!]$

$\{2^k a^k (a^2 - b^2 + c^2 + d^2)^{m-k}(-b^2 + c^2 + d^2)^{\text{int}(k/2)}(bi + c\alpha + d\phi)^{k - 2\text{int}(k/2)}\}Я_1{}^s$.

Proof. By the binomial theorem, we use the fact that real numbers commute and the relation

$(bi + c\alpha + d\phi)^2 = \{i(b1 + c\phi - d\alpha)\}^2$

$= -(b1 - c\phi + d\alpha)(b1 + c\phi - d\alpha)$

$= -b^2 + c^2 + d^2$. □

9.9 The n-hyperintricate binomial theorem for real powers.

Now consider the 2-hyperintricate

$Я_2 = A + P + Q + R,$

in which typically a_{1i} is a real coefficient and 1_i is an indexed basis element, where

$A = a_{11}1_1,$

$P = a_{1i}1_i + a_{1\alpha}1_\alpha + a_{1\phi}1_\phi,$

$Q = b_{i1}i_1 + c_{\alpha 1}\alpha_1 + d_{\phi 1}\phi_1$

and

$R = b_{ii}i_i + b_{i\alpha}i_\alpha + b_{i\phi}i_\phi + c_{\alpha i}\alpha_i + c_{\alpha\alpha}\alpha_\alpha + c_{\alpha\phi}\alpha_\phi + d_{\phi i}\phi_i + d_{\phi\alpha}\phi_\alpha + d_{\phi\phi}\phi_\phi.$

A is commutative with respect to P, Q and R, *also products are*:

$A^2 = a_{11}{}^2 1_1,$

$P^2 = (-a_{1i}{}^2 + a_{1\alpha}{}^2 + a_{1\phi}{}^2)1_1,$

$Q^2 = (-b_{i1}{}^2 + c_{\alpha 1}{}^2 + d_{\phi 1}{}^2)1_1,$

$R^2 = H + 2L,$

where

$H = (b_{ii}{}^2 - b_{i\alpha}{}^2 - b_{i\phi}{}^2 - c_{\alpha i}{}^2 + c_{\alpha\alpha}{}^2 + c_{\alpha\phi}{}^2 - d_{\phi i}{}^2 + d_{\phi\alpha}{}^2 + d_{\phi\phi}{}^2)1_1,$

$L = \{(b_{ii}c_{\alpha\alpha} - b_{i\alpha}c_{\alpha i})\phi_\phi - (b_{ii}c_{\alpha\phi} - b_{i\phi}c_{\alpha i})\phi_\alpha + (b_{i\phi}c_{\alpha\alpha} - b_{i\alpha}c_{\alpha\phi})\phi_i$

$\quad - (b_{ii}d_{\phi\alpha} - b_{i\alpha}d_{\phi i})\alpha_\phi + (b_{ii}d_{\phi\phi} - b_{i\phi}d_{\phi i})\alpha_\alpha - (b_{i\phi}d_{\phi\alpha} - b_{i\alpha}d_{\phi\phi})\alpha_i$

$\quad + (c_{\alpha\alpha}d_{\phi i} - c_{\alpha i}d_{\phi\alpha})i_\phi - (c_{\alpha\phi}d_{\phi i} - c_{\alpha i}d_{\phi\phi})i_\alpha + (c_{\alpha\alpha}d_{\phi\phi} - c_{\alpha\phi}d_{\phi\alpha})i_i\},$

$PQ = (a_{1i}b_{i1}i_i + a_{1i}c_{\alpha 1}\alpha_i + a_{1i}d_{\phi 1}\phi_i$

$\quad + a_{1\alpha}b_{i1}i_\alpha + a_{1\alpha}c_{\alpha 1}\alpha_\alpha + a_{1\alpha}d_{\phi 1}\phi_\alpha$

$\quad + a_{1\phi}b_{i1}i_\phi + a_{1\phi}c_{\alpha 1}\alpha_\phi + a_{1\phi}d_{\phi 1}\phi_\phi)$

$\quad = QP,$

$F = (PR + RP)/2$

$\quad = (-a_{1i}b_{ii} + a_{1\alpha}b_{i\alpha} + a_{1\phi}b_{i\phi})i_1$

$\quad + (-a_{1i}c_{\alpha i} + a_{1\alpha}c_{\alpha\alpha} + a_{1\phi}c_{\alpha\phi})\alpha_1$

$\quad + (-a_{1i}d_{\phi i} + a_{1\alpha}d_{\phi\alpha} + a_{1\phi}d_{\phi\phi})\phi_1$

and

$G = (QR + RQ)/2$

$\quad = (-b_{i1}b_{ii} + c_{\alpha 1}c_{\alpha i} + d_{\phi 1}d_{\phi i})1_i$

$\quad + (-b_{i1}b_{i\alpha} + c_{\alpha 1}c_{\alpha\alpha} + d_{\phi 1}d_{\phi\alpha})1_\alpha$

$\quad + (-b_{i1}b_{i\phi} + c_{\alpha 1}c_{\alpha\phi} + d_{\phi 1}d_{\phi\phi})1_\phi,$

so with different coefficients, F acts like Q, G like P, and L and PQ like R, then e.g.

$FG = GF.$ □

Thus

$Я_2{}^2 = A^2 + P^2 + Q^2 + R^2 + 2\{A(P + Q + R) + PQ + F + G\},$

and separating commutative and non-commutative terms

$Я_2{}^w = \sum(k = 0 \text{ to } m)[m!/(m - k)!k!]$

$\quad \{2^k(A^2 + P^2 + Q^2 + H)^{m-k}[(A(P + Q + R) + PQ + F + G + L]^k\}Я_2{}^s.$ □

The n-hyperintricate binomial theorem for real powers may also be considered. From

$Я_n{}^w = \{\Sigma(q = 1 \text{ to } N)\,\Pi(r = 1 \text{ to } n)[p_{qr} + \Sigma(\text{over } L)h_{qrL}Ч_{rL}]\}^w$

which is of the form

$Я_n{}^w = \{\sum(v = 1 \text{ to } N)[a_vA_v]\}^w,$

where the a_v are real, the terms A_v involve a real term and terms Σ(over L)$h_{qrL}Ч_{rL}$ which commute for different r, or correspond with $Ч_{rL}$ to a power for the same r. Therefore on taking the power of w, we can apply the multinomial theorem to obtain

$$Я_n^w = \sum(\text{for } k_1 + k_2 + \ldots + k_N = w)[w!/ k_1!k_2! \ldots k_N!]\Pi(1 \leq t \leq N)[a_tA_t]\uparrow k_t. \ \square$$

CHAPTER X

Hyperintricate exponential algebra $\mathbb{D}1$

10.1 Introduction.

We use the hyperintricate representation of matrices and explore exponentiation for these objects. Proofs by contradiction are employed to eliminate a number of possibilities for intricate exponentiation, and a novel hyperintricate exponential algebra is finally adopted.

10.2 The search for consistency for intricate exponential algebras.

For proposal $\mathbb{A}1$, the value of i^i is often derived as follows

$$i^i = [e^{i(\pi/2 + 2\pi z)}]^i = e^{-\pi/2 + 2\pi z}, \qquad (1)$$

with $z \in \mathbf{Z}$, which is a real multifunction. In the J-abelian format of [Ad12e], [Ad12f] and the $\mathcal{JAF}$ format of [Ad12i], for $\mathcal{J}^2 = -1$, $\mathcal{A}^2 = 1$ a general intricate number may be represented for $b^2 > c^2 + d^2$ by

$$[a + \mathcal{J}(b^2 - c^2 - d^2)^{½}]e^{\mathcal{J}\theta} \qquad (2)$$

where $e^{\mathcal{J}\theta} = \cos\theta + \mathcal{J}\sin\theta$, and for $b^2 < c^2 + d^2$ by

$$[a + \mathcal{A}(-b^2 + c^2 + d^2)^{½}]e^{\mathcal{A}\theta} \qquad (3)$$

where $e^{\mathcal{A}\theta} = \cosh\theta + \mathcal{A}\sinh\theta$. The determinant in cases (2) and (3) is $a^2 + b^2 - c^2 - d^2$, and may be negative for (3).

Putting $\theta = \pi/2$, taking the exponent of $\mathcal{J}$ of equation (2) and specialising to $\mathcal{J} = i$ indicates that (1) is the general form for i^i when $c = d = 0$. □

We will now investigate exponent zero solutions, and see whether they remain valid for intricate numbers under the equivalence class (3).

We call an evaluation of an expression derived from formulae $f_1, ... f_n$ *inconsistent* if two separate instances of composite formulae applied to the expression, including possibly the identity formula, give no values for the expression equal to itself.

To investigate assignations of m^n where now m and n are intricate basis elements, consider proposals of type $\mathbb{A}$ under assumptions which are documented as we proceed

$$1^i = (e^0)^i = e^{0i} = e^0 = 1,$$

so that

$$1 = 1^i = (i.i.i.i)^i = (i^i)(i^i)(i^i)(i^i) = (i^i)^4,$$

giving proposals $\mathbb{A}2$ and $\mathbb{A}3$ respectively

$$i^i = \pm 1, \pm(ti + u\alpha + v\phi),$$

with $-t^2 + u^2 + v^2 = \pm 1$, or more generally for proposal $\mathbb{A}4$ a number $g = i^i$ satisfying

$$g^4 = 1 = 1^i = (i^i)^4. \ □$$

We eliminate the possibility $i^i = \pm 1$. If $i^i = 1$, then

$$1 = 1^i = (i^i)^i = i^{-1} = -i,$$

and if $i^i = -1$, then

$$-i = (i^i)^i = (-1)^i = (i^i)(i^i),$$

so

$$-i = 1.$$

We can eliminate the specific circumstances $i^i = \pm i$, $\pm\alpha$ and $\pm\phi$. If $i^i = i$, then

$$-i = i^{-1} = (i^i)^i = i^i = i,$$

if $i^i = -i$, then

$$-i = i^{-1} = (i^i)^i = (-i)^i = (i.i.i)^i = (i^i)(i^i)(i^i) = i,$$

if our choice were $i^i = \alpha$, then since $\alpha = \alpha^{-1}$

$$(i^i)^{-i} = (i^i)^i,$$

so

$$(i^i)^{-i} = i = (i^i)^i = -i,$$

where a similar comment can be made for $i^i = \phi$, likewise if $i^i = -\alpha$, since $(-\alpha) = (-\alpha)^{-1}$,

$$(i^i)^{-i} = (i^i)^i,$$

leading to the same contradiction, and also for $i^i = -\phi$. □

The inverse of

$$T = \pm(ti + u\alpha + v\phi),$$

is

$$T^{-1} = \pm[(ti + u\alpha + v\phi)]/(t^2 - u^2 - v^2),$$

where the sign for T^{-1} is dependent solely on $(t^2 - u^2 - v^2)$, and is thus effectively independent of T. Thus if $i^i = T$, then since $T = \pm T^{-1}$, if $T = T^{-1}$, we obtain

$$(i^i)^{-i} = (i^i)^i,$$

$$i = -i.$$

However for $T = -T^{-1}$, apart from the allocation $t = 1$, $u = v = 0$, which we have already discounted, there appears to be no obstruction. For example

$$i^i = T = -T^{-1}$$

implies

$$-i = (i^i)^i = (-1)^i[(i^i)^{-i}] = (i^i)(i^i)i = T^2 i,$$

but

$$T^2 = (-t^2 + u^2 + v^2) = -1,$$

which gives a consistent result.

We now have a proposal of type A3 in which

$$i^i = \pm(ti + u\alpha + v\phi),$$

with $-t^2 + u^2 + v^2 = -1$ and $t > 1$ could be consistent. We must match this against what is known for the Euler formulae.

We note that

$$e^{TK} \neq e^{\pm tiK}e^{\pm u\alpha K}e^{\pm v\phi K},$$

since such exponential multiplication is non-commutative in general, and so does not map to addition in the usual sense. The correct argument is that $T^2 = -1$ implies

$$e^{T\theta} = \cos K + T\sin K,$$

$$= \cos K \pm (ti + u\alpha + v\phi)\sin K.$$

with T expressed in the first instance in terms of intricate variables as

$$T = i^i = \pm(ti + u\alpha + v\phi),$$

so $e^{p1+(qi + r\alpha + s\phi)}$ may be evaluated, for $K \in$ R and $j \in$ intricate И with $j \notin$ R, as

$$q = tK,$$

$$r = uK$$

and

$$s = vK.$$

For either u = 0 or v = 0, $T^2 = -1$ ascribes the value (putting, say, u = 0)

$$v = \pm(t^2 - 1)^{1/2}.$$

Thus an allocation of i^i compatible with the Euler relations is

$$i^i = T = ti \pm (t^2 - 1)^{1/2}\phi,$$

with |t| > 1.

We will show that the choice $T = i^i = \pm(ti + u\alpha + v\phi)$, leads to multivalued functions. This is evident in the relation

$$T^T = [e^{T(\pi/2 + 2\pi z)}]^T$$
$$= e^{-(\pi/2 + 2\pi z)},$$

under the assumption that

$$(a^b)^c = a^{(bc)}.$$

Further, a real valued t commutes as an additive exponential, so that for a real variable $x = -t(\pi/2 + 2\pi z)$, then

$$e^{x+y} = e^x e^y.$$

Putting again u = 0, so that y above is a term in a single basis element, then

$$T^i = e^{(-t - v\alpha)(\pi/2 + 2\pi z)}$$
$$= e^{-t(\pi/2 + 2\pi z)}[\cosh -v(\pi/2 + 2\pi z) + \alpha \sinh -v(\pi/2 + 2\pi z)],$$

which is clearly multivalued. Moreover, it does not satisfy

$$T^i = (i^i)^i = -i. \;\square$$

10.3 The exponential algebra A4 for $g = i^i$.

To generalise the Euler relations, define

$$\chi_{t,u} = \chi_{t,u}(\lambda) = \Sigma(n = 0 \text{ to } \infty)[\lambda^{tn+u}/(tn + u)!]$$

so that

$$\cos \lambda = \chi_{4,0} - \chi_{4,2},$$
$$\sin \lambda = \chi_{4,1} - \chi_{4,3},$$
$$\cosh \lambda = \chi_{4,0} + \chi_{4,2},$$
$$\sinh \lambda = \chi_{4,1} + \chi_{4,3}.$$

Taking $g^2 = J$, $J^2 = -1$, if

$$e^{g\lambda} = (\chi_{8,0} - \chi_{8,4}) + g(\chi_{8,1} - \chi_{8,5}) + g^2(\chi_{8,2} - \chi_{8,6}) + g^3(\chi_{8,3} - \chi_{8,7}),$$

then for $g^2 = -J$, $J^2 = -1$

$$e^{g\lambda} = (\chi_{8,0} - \chi_{8,4}) + g(\chi_{8,1} - \chi_{8,5}) - g^2(\chi_{8,2} - \chi_{8,6}) - g^3(\chi_{8,3} - \chi_{8,7}),$$

for $g^2 = J$, $J^2 = 1$

$$e^{g\lambda} = (\chi_{8,0} + \chi_{8,4}) + g(\chi_{8,1} + \chi_{8,5}) + g^2(\chi_{8,2} + \chi_{8,6}) + g^3(\chi_{8,3} + \chi_{8,7}),$$

and for $g^2 = -J$, $J^2 = 1$

$$e^{g\lambda} = (\chi_{8,0} + \chi_{8,4}) + g(\chi_{8,1} + \chi_{8,5}) - g^2(\chi_{8,2} + \chi_{8,6}) - g^3(\chi_{8,3} + \chi_{8,7}),$$

whereas for $g^4 = 0$

$$e^{g\lambda} = 1 + g\lambda + g^2\lambda^2/2 + g^3\lambda^3/6.$$

We will now explore the solution $g = i^i$ given in the third and fourth assignation of $e^{g\lambda}$ above, namely for $g^2 = \pm J$ and $J^2 = 1$. Clearly $g^4 = -1$ does not satisfy $(i^i)^4 = 1$.

If

$g^i = (i^i)^i = i^{-1} = -i$

then

$(g^i)(g^i) = (\pm J)^i = -1,$

which gives

$[(\pm J)^i]^2 = [(\pm J)^2]^i = 1^i = 1,$

and finally we have obtained an allocation for which as yet we have not derived any inconsistencies. □

We need to be clear that $g^2 \neq -1$, which otherwise would be an assignation with model $g = i$. The model with $g = i$ corresponds to the classical Euler relation

$e^{i\theta} = \cos\theta + i\sin\theta.$

Our solutions are non-classical, consistent with the Euler relations for cosh and sinh, and in particular we have selected a choice where

$i^i \neq [e^{i\pi/2}]^i = e^{-\pi/2 + 2\pi z}.$

10.4 A model for the intricate A4 $g = i^i$ exponential algebra.

Let us assume there is a model where

$g = i^i = \alpha^{1/2},$

which corresponds to a model with $g^4 = 1$, $g^2 \neq \pm 1$. Possible hyperintricate matrices for $\alpha^{1/2}$ are given in reference [Ad12a]. This gives directly

$i^\alpha = \alpha^{1/2\phi},$

$i^\phi = \alpha^{-1/2\alpha}.$

The reasoning applied also holds under the substitutions

$\alpha \rightarrow [c\alpha - (1 - c^2)^{1/2}\phi],$

$\phi \rightarrow [(1 - c^2)^{1/2}\alpha + c\phi],$

for which the algebra of i, α, φ remains true.

Multiplying $i^i = (\alpha^i)(\phi^i) = \alpha^{1/2}$ on the left by α^i, or on the right by ϕ^i gives

$\phi^i = \alpha^i\alpha^{1/2},$

$\alpha^i = \alpha^{1/2}\phi^i.$

What is α^i in this model?

$\alpha^i = (\alpha^{1/2})^{2i}$

$= (i^i)^{2i}$

$= (i^{-1})^2$

$= -1.$

Thus

$\phi^i = -\alpha^{1/2}.$

Taking a power of φ

$\alpha^\alpha = (-1)^\phi$

$\phi^\alpha = (-\alpha^{1/2})^\phi$

and for a power of α

$\alpha^{-\phi} = (-1)^{\alpha}$,

so that

$\alpha^{\phi} = (-1)^{-\alpha}$

$\phi^{\phi} = (-\alpha^{-½})^{\alpha}$.

The i^i root under non-commutation is now

$(\alpha\phi)^i = \alpha^i\phi^i = (-\phi\alpha)^i = (-1)^i\phi^i\alpha^i$.

Thus if we maintain an allocation of

$(-1)^i = (i^i)(i^i) = g^2$

we obtain the contradiction

$g = (\alpha\phi)^i = \alpha^{½}$

$= g^2\alpha^{½} = g^3$.

There are various ways of formulating a remedy, but we could comment that since roots are not unique, the roots confined to non-commutation

$(\alpha\phi)^i = \alpha^i\phi^i = (-\phi\alpha)^i = (-1)^i\phi^i\alpha^i$

can be allocated the value

$(-1)^i = 1$.

This is not strictly forbidden, but it is inelegant. □

10.5 The partition of intricate numbers under $\mathcal{JAF}$ format.

Under $\mathcal{JAF}$ transformations i ↔ $\mathcal{J}$, α ↔ $\mathcal{A}$ and ϕ ↔ $\mathcal{F}$ of [Ad12i], [Ad12j] and [Ad12k], each intricate number may be allocated to a $\mathcal{J}$, $\mathcal{A}$ or $\mathcal{F}$ equivalence class, or to $\mathcal{J}^2 = 0$, where $\mathcal{J}^2 = -1$, $\mathcal{A}^2 = 1$ and $\mathcal{F}^2 = 1$, to the equivalence class for $\mathcal{J}$ when Я = a + bi + cα + dϕ satisfies $-b^2 + c^2 + d^2 < 0$, to $\mathcal{A}$ when $-b^2 + c^2 + d^2 > 0$ and $c^2 > d^2$, and to $\mathcal{J}^2 = 0$ when $-b^2 + c^2 + d^2 = 0$.

Under the algebras so far considered, when Я has form (2) or (3), for partitions to the $\mathcal{J}$ equivalence class we obtain

$\mathcal{J}^{\mathcal{J}} = [e^{\mathcal{J}(\pi/2 + 2\pi z)\mathcal{J}}] = e^{(-\pi/2 + 2\pi z)}$,

a multivalued function. For the $\mathcal{A}$ equivalence class

$e^{\mathcal{A}\theta} = \cosh\theta + \mathcal{A}\sinh\theta$,

and we obtain here allocations of i^i under the $g^4 = 1$ format.

Formulas of form (1) have not disappeared, but we have now investigated solutions to i^i under the partition (3).

10.6 Some further proposals on intricate exponential algebras.

The question we now wish to raise is: are multivalued functions a necessity for intricate exponential algebras? We review some other alternatives.

In order to meet consistency conditions under the above conditions of equality, we can make the following proposal – proposal 𝔹. The relation of equality is not the same as equivalence. The relations deduced above under the = relation considered setting 1 = -1, i = -i, α = -α and ϕ = -ϕ under the exponential operations. Under an equivalence relation instead of these equalities, which corresponds to operations in the projective general linear group PGL(2), consistency returns.

In terms of intricate numbers, what is the identification algebra in $\mathbb{B}$? For a complex number, its norm is the determinant, and the complex number may be considered as a vector of unit norm, multiplied by its determinant, so that in reducing the complex number (mod 2), the norm (mod 2) is chosen which is multiplied by the vector of unit norm. For intricate numbers, the determinant may be zero or negative, but the same procedure works for non-zero norm. The operation is not defined for a singular matrix. □

For proposal $\mathbb{C}$, let Ш be the intricate number

$$Ш = p1 + qi + r\alpha + s\phi$$

so that its intricate conjugate is

$$Ш^* = p1 - qi - r\alpha - s\phi$$

so Ш** = Ш, in which

$$Ш^{-1} = Ш^*/(p^2 + q^2 - r^2 - s^2).$$

Then we define

$$Я^{nШ} = (Я^n)^{(Ш^*)}$$

where if $Я^{Ш}$ is wanted, the choice n = 1 will always be made, i.e. $(Я^1)^{Ш} = Я^{(Ш^*)}$.

As a consequence $(e^{\theta})^{í} = e^{-i\theta}$. If n = 1, then $Я^1 = Я$, 1Ш = Ш and 1Ш* = Ш*, so the immediate question arises whether this is consistent. A type of non-associativity is implied here with 1 as the middle term. Under what conditions is it meaningful to distinguish between $Я^{(nШ)}$ and $(Я^n)^{(Ш^*)}$? If we distinguish between the evaluation of $(e^{\theta})^{í}$ and $e^{-i\theta}$, the leftmost expression involves evaluating a power of i, and the expression on the right evaluates a polynomial in natural number powers.

For the complex number

$$A = re^{i\theta}$$

we have

$$\log A = \log r + i\theta,$$

and thus for a complex number B under proposal $\mathbb{C}$

$$(A^1)^B = e^{B^* \log A} = e^{B^*(\log r + i\theta)}.$$

If B is allocated as C + iD, then under the proposal the formula for $(A^1)^B$ can be written more explicitly as

$$(r^C e^{D\theta})e^{i(-D\log r + C\theta)} = (r^C e^{D\theta})[\cos(-D\log r + C\theta) + i\sin(-D\log r + C\theta)].$$

The Euler formula is the basis of the conventional argument on the assignation of i^i, namely, for integer z,

$$e^{i[(\pi/2) + 2\pi z]} = i,$$

so introducing our proposal $\mathbb{C}$

$$\{e^{i[(\pi/2) + 2\pi z]}\}^i = e^{(\pi/2) + 2\pi z} = i^i,$$

and i^i is multivalued.

The previous assignations of $(i^i)^i = -i$ are now excluded. No consistent allocation of a particular value can be given in general, although it is possible to choose a principal value. If we view the solutions as complex multivalued functions, in which a formula satisfies a set of solutions under equivalence, then we can allocate, for a complex number G* with integer coefficients,

$$i^i = e^{G^*(\pi/2)}$$

to encompass all the above complex solutions under proposal $\mathbb{C}$. □

Proposal $\mathbb{D}1$ is the following.

$(e^{\lambda})^{\mu} = e^{\lambda\mu}$,
$(e^{i\lambda})^{\mu} = e^{i\lambda\mu}$,
$(e^{\lambda})^{i\mu} = e^{i\lambda\mu}$,
$(e^{i\lambda})^{i\mu} = e^{i\lambda\mu}$.

Once again, there are questions of consistency, and there are variants of the above proposal. To the question of how this algebra might be extended, finally, to intricate numbers, a response is to convert to $\mathcal{JAF}$ format.

The multiplicative odd ($\mathcal{O}$)/even ($\mathcal{E}$) algebra satisfies

.	$\mathcal{O}$	$\mathcal{E}$
$\mathcal{O}$	$\mathcal{O}$	$\mathcal{E}$
$\mathcal{E}$	$\mathcal{E}$	$\mathcal{E}$

This Z_2 type of multiplication is precisely the type of exponential structure for proposal $\mathbb{D}1$, with 1 for $\mathcal{O}$ and i for $\mathcal{E}$.

As a variant, we can allocate, for basis elements m, n, p $\in \{1, i, \alpha, \phi\}$,

$(m^n)^{\theta p} = (m^{\theta n})^p = m\uparrow[\theta(n^p)]$,

with θ real. This differs in practice from proposal $\mathbb{B}$ where $(i^i)^1 \neq (i^1)^i$, in that now

$(i^i)\uparrow 1 = i\uparrow(i^1) = i$

and

$(i^1)\uparrow i = i\uparrow(1^i) = i$.

Directly under proposal $\mathbb{D}1$ we obtain the following relations in $\mathcal{JAF}$ format

$$
\begin{aligned}
[e^{k\pi/2}.e^{\mathcal{J}\pi/2}]^{\mathcal{J}} &= e^{(k+1)\mathcal{J}\pi/2} \\
&= e^{k\mathcal{J}\pi/2}.\mathcal{J} \\
&= [\cos(k\pi/2) + \mathcal{J}\sin(k\pi/2)].\mathcal{J} \\
&= -\cos[(k-1)\pi/2] - \mathcal{J}\sin[(k-1)\pi/2] \\
&= -e^{(k-1)\mathcal{J}\pi/2}.
\end{aligned}
$$

Putting k = 1 gives

$-e^0 = -1 = \cos\pi + \mathcal{J}\sin\pi = -1$,

so this aspect is consistent, which is also obtained under $[e^{\mathcal{J}\pi/2}]^{\mathcal{J}} = e^{-\mathcal{J}\pi/2}$. □

10.7 The $\mathbb{D}2$, $\mathbb{D}3$ and $\mathbb{D}4$ intricate exponential algebra proposals.

It is possible to introduce proposals where $(e^{i\lambda})^{i\mu} \neq e^{i\lambda\mu}$. These are of type $\mathbb{D}2$, where $(e^{i\lambda})^{i\mu} = e^{\alpha\lambda\mu}$ or $(e^{i\lambda})^{i\mu} = e^{\phi\lambda\mu}$, or involve hyperintricate variables, say allocating all initial variables as hyperintricates with a lower layer of 1, and an exponential of this as $e\uparrow[i_1\lambda 1_i\mu] = e\uparrow[i_i\lambda\mu]$. These are proposals of type $\mathbb{D}3$, a later modification being $\mathbb{D}4$, where the latter gives indications on the extension to an intricate algebra.

Proposals under the heading $\mathbb{D}2$ contain multifunctions since $(e^{i\lambda})^{i\mu} = e^{\alpha\lambda\mu}$ implies

$i^i = \cosh(\pi/2 + 2\pi z) + \alpha \sinh(\pi/2 + 2\pi z)$. □

Proposal $\mathbb{D}3$ is to consider exponential variables $e^{i\lambda}$ as elements of a 2-hyperintricate algebra with trailing layer 1 and that $(e^{i\lambda})^{i\mu}$ occurs where the second occurrence, the $i\mu$ variable, is represented by $1_i\mu$ with leading layer 1. Then

$$(e^{i\lambda})^{i\mu} = e\uparrow[(i_i)\lambda\mu]$$
$$= \cosh\lambda\mu + (i_i)\sinh\lambda\mu,$$

since $(i_i)^2 = 1$.

If it is understood that on forming expressions to the power i, all terms will be converted to exponential format for their evaluation, then we obtain multifunctions:

$$(-1)^i = (e^{i\pi + i2\pi z})^i = \cosh(\pi + 2\pi z) + i_i\sinh(\pi + 2\pi z)$$
$$= (i^i)(i^i) = e^{(i\pi/2 + i\pi/2)i} = \cosh^2(\pi/2) + \sinh^2(\pi/2) + 2i_i\cosh(\pi/2)\sinh(\pi/2). \ \square$$

The modification, $\mathbb{D}4$, we now introduce is formed from exponential variables $e^{i\lambda}$ as elements of a 3-hyperintricate algebra with trailing layers 1 and that the expansion of $(e^{i\lambda})^{i\mu}$ is implemented where the second occurrence, the $i\mu$ variable, is represented by $1_{ii}\mu$, which has a leading layer of 1. Then

$$(e^{i\lambda})^{i\mu} = e\uparrow[(i_{ii})\lambda\mu]$$
$$= \cos\lambda\mu + (i_{ii})\sin\lambda\mu,$$

since $(i_{ii})^2 = -1$.

The exponential operations involving i less than twice are defined by

$$(e^{\lambda})^{\mu} = e^{\lambda\mu},$$
$$(e^{i\lambda})^{\mu} = e\uparrow i_{11}\lambda\mu,$$
$$(e^{\lambda})^{i\mu} = e\uparrow i_{11}\lambda\mu,$$

so that $(e^{\lambda})^{i\mu}$ operates with identical formulas to $(e^{i\lambda})^{\mu}$.

Subsequent exponentiations have leading layer 1

$$\{(e^{i\lambda})^{i\mu}\}^{i\nu} = \{e\uparrow[(i_{ii})\lambda\mu]\}\uparrow i\nu$$
$$= e\uparrow\{[(i_{ii})\lambda\mu][1_{ii}\nu]\}$$
$$= e\uparrow[(i_{11})\lambda\mu\nu].$$

On forming expressions to the power i, all terms are converted to exponential format for their evaluation, so that for example

$$(-1)^i = (e^{i\pi + i2\pi z})^i = \cos(\pi + 2\pi z) + i_{ii}\sin(\pi + 2\pi z) = -1$$
$$= (i^i)(i^i) = e^{(i\pi/2 + i\pi/2)i} = \cos^2(\pi/2) - \sin^2(\pi/2) + 2i_{ii}\cos(\pi/2)\sin(\pi/2). \ \square$$

The extension to a general hyperintricate algebra is obtained under the relations

$$(\alpha_{\alpha\alpha})^2 = (\phi_{\phi\phi})^2 = (\alpha_{\phi\phi})^2 = (\phi_{\alpha\alpha})^2 = (\alpha_{ii})^2 = (\phi_{ii})^2 = 1$$

and

$$(i_{ii})^2 = (i_{\alpha\alpha})^2 = (i_{\phi\phi})^2 = -1,$$

with the understanding that

$$(e^{i\lambda})^{\alpha\mu} = e\uparrow[(i_{\alpha\alpha})\lambda\mu]$$
$$= \cos\lambda\mu + (i_{\alpha\alpha})\sin\lambda\mu,$$

and similarly for instance

$$(e^{\alpha\lambda})^{\phi\mu} = e\uparrow[(\alpha_{\phi\phi})\lambda\mu]$$
$$= \cosh\lambda\mu + (\alpha_{\phi\phi})\sinh\lambda\mu,$$

so that the complete $\mathbb{D}4$ exponential algebra can be generated. $\square$

The reason we have discussed $\mathbb{D}$4 is that the intricate algebra for $\mathbb{D}$1 is now indicated – compact the two lower layers together to 1, or effectively nullify them. Thus the intricate basis elements chosen in $\mathbb{D}$1 are the ones that are presented first in the exponentiation.

The algebra with these hyperintricate basis elements is of cosh/sinh form where the squares of these basis elements are 1, and of cos/sin form where the squares are -1, and a J-abelian 3-hyperintricate number has basis elements as a summation of these types, or of types with 1 in some layers. These representations may also be converted to more general $\mathcal{JAF}$ format.

Division is not in general available. This implies roots are not always present.

Under proposal $\mathbb{D}$4 we now obtain the following relations in $\mathcal{JAF}$ format

$$[e^{k\pi/2}.e^{\mathcal{J}\pi/2}]^{\mathcal{J}} = e^{k\mathcal{J}\pi/2}[e\uparrow\mathcal{J}_{\mathcal{JJ}}\pi/2]$$
$$= e^{k\mathcal{J}\pi/2}.\mathcal{J}_{\mathcal{JJ}}$$
$$= [\cos(k\pi/2) + \mathcal{J}_{11}\sin(k\pi/2)].\mathcal{J}_{\mathcal{JJ}}$$

which is not a contradiction, as is the similar case under proposal $\mathbb{D}$1. □

10.8 The $\mathbb{D}$1 hyperintricate exponential algebra.

A natural solution we adopt is to pursue the analogy of left nested exponentiation, $\uparrow$, with multiplication and define a type of $\uparrow$ distributivity, with the exponential term e, as follows:

$$[e\uparrow(ti + u\alpha + v\phi)]\uparrow(wi + x\alpha + y\phi) =$$
$$e\uparrow[t(w + x + y)]i + [u(w + x + y)]\alpha + [v(w + x + y)]\phi. \qquad (4)$$

This noncommutative ring with unit under + and × is explicit enough to generate all the relations we need. There is no division operation for $\uparrow$, because if a real value is absent, $\uparrow$ cannot generate 1. The hyperintricate exponential algebra becomes available through this structure by the means described subsequently. □

An important question is now whether it is decidable that the above suggestion is consistent. All binomial type exponential operations are generated and defined via

$$[e^{p1 + (qi + r\alpha + s\phi)K}]\uparrow(a1 + bi + c\alpha + d\phi),$$

so we conclude the operation $\uparrow$ is as consistent as rings in general. □

The equation (4) is not $\mathcal{JAF}$ invariant. If we transform i → $\mathcal{J}$, α → $\mathcal{A}$ and ϕ → $\mathcal{F}$, so that

$$ti + u\alpha + v\phi = t'\mathcal{J} + u'\mathcal{A} + v'\mathcal{F}$$

and

$$wi + x\alpha + y\phi = w'\mathcal{J} + x'\mathcal{A} + y'\mathcal{F}, \qquad (5)$$

then if the right hand side of (4) remains invariant,

$$ti + u\alpha + v\phi(w + x + y) = (t'\mathcal{J} + u'\mathcal{A} + v'\mathcal{F})(w' + x' + y') \qquad (6)$$

so we have

$$w + x + y = w' + x' + y', \qquad (7)$$

but taking the intricate conjugate of (5)

$$-w^2 + x^2 + y^2 = -w'^2 + x'^2 + y'^2,$$

for which (7) by the binomial theorem does not hold for

$w = 1, x = y = 0,$

$w' = \sqrt{(1 - x'^2 - y'^2)},$

with w′, x′ and y′ positive. □

J-abelian hyperintricate exponentiation is the natural extension of the intricate case. For example for 4 x 4 matrices we have 16 basis elements, 8 of which can be put in one combined J-abelian representation.

Since the $\mathcal{JAF}$ ring transformations map $i \to \mathcal{J}$, $\alpha \to \mathcal{A}$ and $\phi \to \mathcal{F}$, but do not provide an invariant description of D1 exponential and superexponential algebras, there is a spectrum of possible D1 algebras which may be chosen, and we will present the D1 exponential algebra spectrally, under the understanding that some specific $\mathcal{JAF}$ basis must be chosen throughout to maintain consistency.

For a hyperintricate number

$\Sigma_j [\underline{\vee}_k(a_{kj} + b_{kj}J_{kj})] = \Sigma_j [\underline{\vee}_k e\uparrow(\rho_{kj} + \sigma_{kj}J_{kj})],$

where $J_{kj}^2 = 0$ or ± 1, we evaluate

$[e\uparrow(\rho + \sigma J)]\uparrow(\rho' + \sigma' J')$

as the lower D1 exponential algebra expression

$e\uparrow(\rho\rho' + \sigma\rho' J + \sigma'\rho J' + \sigma\sigma' J),$

and the Σ terms by a binomial expansion under this rule. Note that the last term for the exponentiated sum is $\sigma\sigma' J$ and not $\sigma\sigma' JJ'$ – this is the specific difference between the D1 exponential algebra, which is minimally branched, and the standard exponential algebra.

As a $\mathcal{JAF}$ ring we select the lower expansion

$e^v = [e\uparrow(r + s\mathcal{J} + t\mathcal{A} + u\mathcal{F})]\uparrow(r' + s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}'),$

where

$v = rr' + r(s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}') + r'(s\mathcal{J} + t\mathcal{A} + u\mathcal{F})$

$\quad + (ss' + ts' + us')\mathcal{J} + (ts' + tt' + tu')\mathcal{A} + (us' + ut' + uu')\mathcal{F}.$ □

If we denote the exponential operator by ↑, then for J-abelian 2-hyperintricate numbers with selected intricate basis element components w, x and y, we define

$(w_x)\uparrow y = (w\uparrow y)_{(x\uparrow y)},$

with natural extensions to the J-abelian n-hyperintricate case. This structure is the $\mathbb{D}$1 hyperintricate exponential algebra.

This algebra is in accordance with what we would expect under the compression ring epimorphism κ of w_x, that is in this case

$[\kappa(w_x)]\uparrow y = (w\uparrow y)(x\uparrow y).$

10.9 Further reasoning on intricate binomials and Euler relations.

We will assume the *complex binomial theorem* is of the form

$(a + ib)^{(c + id)} = (a + ib)^c(a + ib)^{id},$

where $(a + ib)^{id}$ may also be expressed as $a^{id}(1 + i(b/a))^{id}$ or $(ib)^{id}(1 - i(a/b))^{id}$. Consequently there exist convergent solutions.

There are at least two ways of expanding out $(a + ib)^{id}$. The first is

$$(a + ib)^{id} = \{a^{id} + id(a^{id-1})ib + [id(id - 1)/2](a^{id-2})i^2b^2 + ...\},$$

and the second is

$$(ib + a)^{id} = \{(ib)^{id} + id((ib)^{id-1})a + [id(id - 1)/2]((ib)^{id-2})a^2 + ...\},$$

the first when $b \leq a$ and the second when $a \leq b$.

The binomial theorem may be converted to exponential algebra form under the allocation, for r real, of $r^i = (e^{\lambda})^i = e^{i\lambda}$. For -r to an ith power this is $[(e^{\pi i})(e^{\lambda})]^i = -e^{i\lambda}$ with the caveat, for example, that $[(e^{\pi i})(e^{\lambda\alpha})] \neq e^{\pi i + \lambda\alpha}$. Thus where terms are multiplied as e to powers, the result is not in general e to the sum, and multiplication and the multiplicative order have to be retained.

For 2-hyperintricates $(A + B)_{(C + D)} = A_C + A_D + B_C + B_D$, but $A_C + B_D$ is not of the form G_H in general, and thus we are reduced to considering general terms $E_F + G_H$ which cannot be further reduced. To apply the binomial theorem in the J-abelian case, apply it to $E_F + G_H$. Since the hyperintricate algebra we have introduced can evaluate $E_F{}^L$, this is possible

$$\begin{aligned}(E_F + G_H)^L &= \{E_F(1 + E^{-1}{}_{F^{-1}}G_H)\}^L \\ &= E_F{}^L(1 + E^{-1}{}_{F^{-1}}G_H)^L,\end{aligned}$$

but in general the commutative property does not hold. □

CHAPTER XI

Superexponentiation

Notes

Except that intricate basis elements for SL(2, R) are given on page 25 of [Ge75], a partial literature search, e.g. both historical [Fr73], [Sch73] and contemporary [Se03a] has failed to find reference to the explicit idea of the hyperintricate representation. The matrix representation of complex numbers is mentioned by R. Remmert on page 69 of [Eb91], from which we developed this notion in 2008. The representation of quaternions by matrices is given in [He68] and [No83].

There are many references to the upper triangular representation of matrices in the literature [JLa70], [La80], [ST92]. A. Borel is often quoted in this context, but the idea without the technology goes back to Gauss [Gau1863].

We have derived the equivalence of algebras within an abelian subgroup of $GL^+(2, R)$ corresponding to the bijection between i and $J = bi + c\alpha + d\phi$, with J constant and $J^2 = -1$, as in equation (6.2.5). A consequence is that complex analysis may be extended, for which the Cauchy-Riemann equations and the Cauchy integral formula can be re-expressed in terms of J instead of i, whilst variable J can be tackled using sheaf theory [MM00] or more mundanely under the $J_1 \to J_2$ homotopy $[L, 0] \to [0, M]$ of (1.2) with u and x of the same sign, as in the complex case, and similarly we can introduce Laplace transforms. In like manner, polynomial theory carries over for such algebras [Ad12k]. The same can be said for a theory of modular forms [Ad12d], [BN97], [Ne97]. There is also the classification of Lie groups, which can be extended [Ad12h], [Dy00], [Hu90], [MT00], [Ros02]. Our representation is an extension of all these items.

References

AB95 J.L. Alperin and R.B. Bell, *Groups and representations*, Springer, 1995.

AdJF60 J.F. Adams, *On the non-existence of elements of Hopf invariant one,* Ann. Math., **72**, No. 1, 1960.

Ad12a J.H. Adams, *An elementary proof of a formula on quadratic residues* I, to appear.

Ad12b J.H. Adams, *Division algebras,* to appear.

Ad12c J.H. Adams, *Fermat's little theorem for matrices*, to appear.

Ad12d J.H. Adams, *Hyperanalysis*, to appear.

Ad12e J.H. Adams, *Hyperintricate exponential algebras* I, to appear.

Ad12f J.H. Adams, *Hyperintricate exponential algebras* II, to appear.

Ad12g J.H. Adams, *Hyperintricate number theory*, to appear.

Ad12h J.H. Adams, *Hyperintricate rings*, to appear.

Ad12i J.H. Adams, *Introduction to intricate and hyperintricate numbers* I, to appear.

Ad12j J.H. Adams, *Introduction to intricate and hyperintricate numbers* II, to appear.

Ad12k J.H. Adams, *Polynomial equations for non-commutative algebras*, to appear.

Ad12l J.H. Adams, *Rescaling linear and polynomial probabilities*, to appear.

Ad13a J.H. Adams *Discussion on ladder numbers*, to appear.

Ad13b J.H. Adams, *Some simple proofs of general reciprocity*, to appear.

Ar59 E. Artin, *Galois theory*, Notre Dame, Indiana, 1959.

Be05 D. Bernstein, *Matrix mathematics*, Princeton University Press, 2005.

Bo73 N. Bourbaki, *Algebra* I, Hermann, 1973.

BN97 J. Bak and D.J. Newman, *Complex analysis*, 2nd edn., Springer, 1997.

Bu89 D.M. Burton, *Elementary number theory*, 2nd edn., Wm C. Brown, 1989.

CLS97 D. Cox, J. Little and D. O'Shea, *Ideals, varieties and algorithms*, (2nd edn), Wiley, 1997.

Co89 D. Cox, *Primes of the Form* $x^2 + ny^2$. Wiley, 1989.

Co12 D. Cox, *Galois theory*, (2nd edn), Wiley, 2012.

Da80 H. Davenport, *Multiplicative number theory*, 2nd edition, Springer, 1980.

De60 E. Dehn, *Algebraic equations*, Dover, 1960.

Dy00 E.B. Dynkin, *Selected papers*, A.M.S. International Press, 2000.

Eb91 Ebbinghaus et al., *Numbers*, Springer, 1991.

Ed84 H. Edwards, *Galois theory*, Springer, 1984.

Fri26 R. Fricke, *Lehrbuch der Algebra II*, 1926.

Fr73 F.G. Frobenius, *Gesammelte Abhandlungen*, vols. 1, 2 and 3, Springer, 1973.

Gal62 É. Galois, *Écrits et mémoires mathématiques*, Gauthier-Villars, 1962.

Gau1863 C.F. Gauss, *Werke*, vols. 1 – 6, Dietrich, Göttingen, 1863 – 1874, vols. 7 – 12, B.G. Teubner, Leipzig, 1906 – 1917.

Ge75 S.S. Gelbart, *Automorphic forms on adele groups*, Ann. Math Studies 83, Princeton University Press, 1975.

Ha69 W.R. Hamilton, *Elements of quaternions,* Vol I, Chelsea, 1969.

HB82 B. Huppert and N. Blackburn, *Finite groups II*, Springer, 1982.
Hec33 E. Hecke, *Algebraische zahlen*, 1933.
Her1858 C. Hermite, *Sur la résolution de l'équation du cinquiéme degré,* Comptes Rendus de l'Acad. des Sciences XLVI, 508-513, 1858.
He68 I.N. Herstein, *Noncommutative rings*, Wiley, 1968.
Hu90 J.E. Humphreys, *Reflection and Coxeter groups*, Cambridge University Press, 1990.
JLa70 H. Jacquet and R.P. Langlands, *Automorphic forms on* GL(2), Springer, 1970.
JLi09 G. James and M Liebeck, *Representations and characters of groups*, Cambridge U. P., 2009.
Jo1870 M.C. Jordan, *Traité des substitutions*, Gautier-Villars, 1957.
Kl56 F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*, Dover, 1956.
La80 R.P. Langlands, *Base change for* GL(2), Princeton U. P., 1980.
Ma1886 G.B. Mathews, *Theory of numbers*, 1886, reprint Chelsea Publishing.
MM00 S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic*, Springer, 2000.
MT00 M. Mimura and H. Toda, *Topology of Lie groups*, (Tr.), A.M.S., 2000.
Ne97 T. Needham, *Visual complex analysis*, Clarendon Press, 1997.
No83 E. Noether, *Gesammelte Abhandlungen* (Collected Papers), Springer, 1983: with E. Brauer, *Über minimale Zerfällungskörper irreduzible Darstellungen*.
Ros02 W. Rossmann, *Lie groups*, Oxford University Press, 2002.
Ro90 J. Rotman, *Galois theory*, Springer, 1990.
Ro95 J. Rotman, *An introduction to the theory of groups*, Springer, 1995.
Ro10 J. Rotman, *Advanced modern algebra* (*2nd edn*), AMS, 2010.
Sch73 I. Schur, *Gesammelte Abhandlungen*, vols. 1, 2 and 3, Springer, 1973.
Se03a J-P Serre, *Oeuvres*, vols. 1, 2, 3 and 4, Springer, 2003.
Se03b J-P. Serre, *Extensions icosaédriques*, Oeuvres, Vol III, **123**, Springer, 2003.
ST92 J.H. Silverman and J. Tate, *Rational points on elliptic curves*, Springer, 1992.