CHAPTER V

Zeta functions

5.1. Introduction.

In this chapter we prove the global Riemann hypothesis and also its generalisation for L-series, the proof of which implies a new proof of the weak Goldbach theorem – every odd number > 5 is the sum of three primes. The method uses independence of zeta functions for $\mathbb{D}$w exponential algebras where the w are imaginary numbers $\pm$i. This technique carries over to the generalised Riemann hypothesis defined for L-series, which uses Dirichlet characters to generalise the zeta function. To present these ideas we begin by discussing gamma functions.

5.2. Γ-functions. [Ar64]

The theory of the gamma function was developed to generalise the factorial function. Euler extended the factorial function n! = n(n – 1) ... 3.2.1 from the natural numbers, n, to all Eudoxus (real, which are countable) numbers greater than -1 by noting that on integration by parts, for ordinal infinity $\boldsymbol{\Omega}$

$$n! = \int_0^{\Omega} e^{-y}y^n dy, \tag{1}$$

where on the right the integral converges for noninteger values of n whenever n > -1.

This was also discussed by Gauss. Later Legendre introduced the notation which has now become standard, for a Eudoxus number u

$$\Gamma(u) = \int_0^{\Omega} e^{-y}y^{u-1}dy, \ (u > 0), \tag{2}$$

where Γ(u) is the *gamma function*, with values (u – 1)! for u an integer > 0, in what is known as *Euler's second integral* on the right side of (2), and this is a *meromorphic function* on the whole complex plane (by definition a meromorphic function is complex differentiable about each neighbourhood of a point except a set of isolated poles of the function, which coincide with dividing the function there by zero). Then by analytic continuation Γ(s) is defined for complex numbers, s.

It is possible to show that on holding y fixed, decrementing ε and increasing δ, the following integrals exist

$$\int_{\varepsilon}^{\Omega} e^{-y}y^{u-1}dy, \ (u > 0), \qquad \text{and} \qquad \int_0^{\delta} e^{-y}y^{u-1}dy, \ (u > 0),$$

and therefore so does $\Gamma(u)$ for all Eudoxus u.

In order to prove some of the theorems which follow, we will include a description of the terms convex and log convex for functions. The discussion does not extend to ladder algebra, where there can be infinite series of infinitesimals, each an infinitesimal of a previous one, and derivatives can exist in each infinitesimal region.

Let g(u) be a Eudoxus valued function on an open interval t < u < w of the Eudoxus line. For each pair (u_a, u_b) of distinct numbers in the interval we can form the difference quotient

$$\eta(u_a, u_b) = \frac{g(u_a) - g(u_b)}{u_a - u_b} = \eta(u_b, u_a), \tag{3}$$

and for each triple of distinct numbers (u_a, u_b, u_c) the quotient

$$\Phi(u_a, u_b, u_z) = \frac{\eta(u_a,u_c) - \eta(u_b,u_c)}{u_a - u_b} = \frac{(u_c - u_b)g(u_a) + (u_a - u_c)g(u_b) + (u_b - u_a)g(u_c)}{(u_a - u_b)(u_b - u_c)(u_c - u_a)}. \qquad (4)$$

The value of the function $\Phi(u_a, u_b, u_c)$ does not alter when its parameters u_a, u_b, u_c are permuted.

An open interval (t, w) has values > t and < w. A closed interval [t, w] is $\geq$ t and $\leq$ w. A function g(u) is called *convex* on the open interval (t, w) if, for every number u_c of this interval, $\eta(u_a, u_c)$ increases whenever u_a does. This steadily increasing growth is called *monotonic*. This implies that for any pair of numbers $u_b > u_a$ that are distinct from u_c the relation $\eta(u_b, u_c) \geq \eta(u_a, u_c)$ holds, so that $\Phi(u_b, u_a, u_c) \geq 0$, and because this value is not changed on permuting its parameters, the convexity of g(u) is equivalent to the relation

$$\Phi(u_a, u_b, u_c) \geq 0. \qquad (5)$$

If h(u) is a convex function defined on the same interval t < u < w, on adding (5) for g(u) to the corresponding inequality for h(u), we see that g(u) + h(u) is also convex. To generalise, this also holds when the two functions are replaced by a sequence of functions, and even when this sequence tends to a limit, so that (5) holds for this limit. To restate this

Theorem 5.2.1. *The sum of convex functions is convex, likewise the limit function of a convergent sequence of convex functions is convex. A convergent infinite series whose terms are all convex has a convex sum*. ☐

We are now going to investigate some important properties of a convex function g(u) defined on the interval t < u < w. For a fixed u_f in this interval, let u_a range over all numbers < u_f and u_b range over all numbers > u_f. We have

$$\eta(u_b, u_f) \geq \eta(u_a, u_f). \qquad (6)$$

If u_a is kept fixed and u_b decreases approaching u_f, the left hand side of equation (6) will decrease but always stay greater than the right hand side. This implies that the *right hand derivative* of g(u) exists, so that the limit is

$$\lim_{\substack{u_b > u_f \\ u_b \to u_f}} \eta(u_b, u_f) = \lim_{\substack{u_b > u_f \\ u_b \to u_f}} \frac{g(u_b) - g(u_f)}{u_b - u_f},$$

which we denote by $g'(u_f + 0)$, where according to equation (6)

$$g'(u_f + 0) \geq \eta(u_a, u_f).$$

Theorem 5.2.2. *For a convex function* g(u) *defined on the interval* t < u < w, *the one-sided derivatives increase monotonically*.

Proof. If we let u_a increase to approach u_f, we see the left hand derivative $g'(u_f - 0)$ also exists and

$$g'(u_f + 0) \geq g'(u_f - 0). \qquad (7)$$

Whenever $u_a < u_c$ in the interval, we can select u_b, u_d so that $u_a < u_b < u_c < u_d$, giving

$$\eta(u_c, u_a) \leq \eta(u_d, u_a) = \eta(u_a, u_d) \leq \eta(u_b, u_d) = \eta(u_d, u_b), \qquad (8)$$

and on letting u_c tend to u_a and u_d to u_b, we get

$$g'(u_a + 0) \leq g'(u_b - 0) \text{ for } u_a < u_b, \qquad (9)$$

proving one-sided derivatives of a convex function satisfy (8) and (9) and are always present. The equations (8) and (9) describe the monotonically increasing nature of these one-sided derivatives. ☐

We will now demonstrate the converse, by generalising the mean-value theorem to functions with one-sided derivatives. Here is the analogue of Rolle's theorem.

Theorem 5.2.3. *For a function* g(u) *defined and continuous on the interval* $t \leq u \leq w$, *with one-sided derivatives in the open interval* $t < u < w$, *if* $g(t) = g(w)$ *then there is a value* z *with* $t < z < w$ *where one of the values* $g'(z+0)$ *or* $g'(z-0) \geq 0$ *and the other* ≤ 0.

Proof. There are three cases.
(i) If g(u) is at its maximum in the interior of the interval [t, w], then

$$\frac{g(z+h) - g(z)}{h}$$

has a value ≤ 0 for positive h and ≥ 0 for negative h, so that in the limit $g'(z+0) \leq 0$ whereas $g'(z-0) \geq 0$.
(ii) If z is a minimum in the interior of the interval [t, w], then correspondingly $g'(z+0) \geq 0$ and $g'(z-0) \leq 0$.
(iii) If the maximum and minimum are both on the boundary of [t, w], then g(u) is constant and z can be allocated anywhere in the interior. ☐

The mean-value theorem is expressed as follows.
Theorem 5.2.4. *For a function* g(u) *defined and continuous on the interval* [t, w], *with one-sided derivatives in the open interval* $t < u < w$, *there exists a value* z *in the interior of* [t, w], *with* $[g(w) - g(t)]/(w - t)$ *between* $g'(z-0)$ *and* $g'(z+0)$.

Proof. The function

$$G(u) = g(u) - \frac{g(w) - g(t)}{w - t}(u - t)$$

is continuous, has one-sided derivatives

$$G'(y \pm 0) = g'(y \pm 0) - \frac{g(w) - g(t)}{w - t},$$

and $G(t) = g(t)$, $G(w) = g(w)$, so that according to Rolle's theorem there is a z in the interior of [t, w] with one of the values

$$g'(z+0) - \frac{g(w) - g(t)}{w - t} \quad \text{or} \quad g'(z-0) - \frac{g(w) - g(t)}{w - t}$$

≥ 0 and the other ≤ 0. ☐

Theorem 5.2.5. *Let* g(u) *be a function in the open interval* $t < u < w$, *with one-sided derivatives that are monotonically increasing*. *Then* g(u) *is convex.*

Proof. Let u_a, u_b and u_c be distinct numbers in the interval, where we will assume $u_a < u_b < u_c$. According to the mean value theorem, we can find a z and x with $u_a < x < u_b < z < u_c$ so that $\eta(u_c, u_b)$ lies between $\leq g'(z-0)$ and $g'(z+0)$ and a $\eta(u_a, u_b)$ which lies between $g'(x-0)$ and $g'(x+0)$. Thus equation (7) gives

$$\eta(u_c, u_b) \geq g'(z-0) \text{ and } \eta(u_a, u_b) \geq g'(x-0).$$

From (4) we get

$$\Phi(u_a, u_b, u_c) \geq \frac{g'(z-0) - g'(x-0)}{u_c - u_a},$$

so that

$$\Phi(u_a, u_b, u_c) \geq 0. \ ☐$$

On combining theorems 5.2.2 and 5.2.5 we obtain
Corollary 5.2.6. *A function* g(u) *is convex if and only if* g(u) *has monotonically increasing one-sided derivatives*. ☐

Corollary 5.2.7. *A twice differentiable function* g(u) *is convex if and only if* $g''(u) \geq 0$ *for all* u *in its interval.*

Proof. The function g(u) is monotonically increasing if and only if $g''(u) \geq 0$ for all u in its domain. □

Definition 5.2.8. A function g(u) is *log convex* if and only if the function log g(u) is convex.

Theorem 5.2.9. *A product of log convex functions is log convex. A convergent sequence of log convex functions has a log convex limit function provided the limit is positive, or alternatively that the sequence of logarithms of the individual terms is convergent.* □

Theorem 5.2.10. *If* g(u) *is a twice differentiable function for which*

$$g(u) > 0, \qquad g(u)g''(u) - [g'(u)]^2 \geq 0,$$

then g(u) *is log convex.*

Proof. This follows from the fact that the second derivative of log g(u) has the value

$$\frac{g(u)g''(u) - [g'(u)]^2}{[g(u)]^2}. \ □$$

Theorem 5.2.11. *Suppose that* g(u) *and* h(u) *are functions defined on a common interval. If both are log convex, then* g(u) + h(u) *is log convex.*

Proof. If $g(u) > 0, \quad g(u)g''(u) - [g'(u)]^2 \geq 0,$

$h(u) > 0$ and $h(u)h''(u) - [h'(u)]^2 \geq 0,$

then the following equality holds

$$\begin{aligned} g(u)h(u)[(g(u) + h(u))(g''(u) + h''(u)) - (g'(u) + h'(u))^2] = \\ h(u)[(g(u) + h(u)][\ g(u)g''(u) - [g'(u)]^2] + \\ g(u)[(g(u) + h(u)][\ h(u)h''(u) - [h'(u)]^2] + \\ [g(u)\ h'(u) - h(u)\ y'(u)]^2, \end{aligned}$$

where the term after the equals sign is ≥ 0, so the result follows. □

Theorem 5.2.12. *The integral* $\Gamma(u) = \int_0^{\Omega} e^{-y}y^{u-1}dy$, $(u > 0)$, *is log convex.*

Proof. e^{-y} is positive and continuous for $y \in [0, \mathbf{\Omega}]$. If we take the logarithm of the integrand and differentiate twice with respect to y, we get 0. □

On integrating (2) by parts,

$$\Gamma(u+1) = \int_0^{\Omega} e^{-y}y^{u}dy = [-e^{-y}y^{u}]_0^{\Omega} + u\int_0^{\Omega} e^{-y}y^{u}dy = u\int_0^{\Omega} e^{-y}y^{u}dy,$$

where for $y \to 0$, $e^{-y} \to 1$, and for $y \to \mathbf{\Omega}$, $e^{y} > y^{u+1}$ giving $e^{-y}y^{u} < y^{-1}$, so the $e^{-y}y^{u}$ terms drop out, and we have the formulas

$$\Gamma(1) = 1,$$

$$\Gamma(u + 1) = u\Gamma(u), \tag{10}$$

which represent the generalisation of the identity

$$n! = n(n - 1)!$$

By repeated use of equation (10)

$$\Gamma(u + n) = (u + n - 1)(u + n - 2) \ldots (u + 1)u\Gamma(u), \tag{11}$$

for every positive integer n.

If we extend the definition (2) to include negative Eudoxus numbers, so that u lies in the interval $-n < u < -n + 1$, we define the gamma function by

$$\Gamma(u) = \{1/[u(u + 1) \ldots (u + n - 1)]\}\Gamma(u + n), \tag{12}$$

but if u is a negative integer or zero, the right hand side of (12) has meaning only if we use the zero algebra. Otherwise the right hand side of (12) is always well defined. □

Let n be an integer > 1. The inequality

$$\frac{\log\Gamma(-1+n)-\log\Gamma(n)}{(-1+n)-n} \le \frac{\log\Gamma(u+n)-\log\Gamma(n)}{(u+n)-n} \le \frac{\log\Gamma(1+n)-\log\Gamma(n)}{(1+n)-n}$$

expresses the monotonic increase of the difference quotient. Since Γ(n) = (n – 1)!, we have

$$\log(n-1) \le \frac{\log\Gamma(u+n)-\log(n-1)!}{u} \le \log n,$$

or

$$\log(n-1)^u(n-1)! \le \log\Gamma(u+n) \le \log n^u(n-1)!,$$

and since the logarithm is a monotonically increasing function

$$(n-1)^u(n-1)! \le \Gamma(u+n) \le n^u(n-1)!$$

With the help of equation (10), we get the following inequality for $0 < u \le 1$

$$\frac{(n-1)^u(n-1)!}{u(u+1)\cdots(u+n-1)} \le \Gamma(u) \le \frac{n^u(n-1)!}{u(u+1)\cdots(u+n-1)},$$

so that as a corollary derived by Gauss

$$\Gamma(u) = \lim_{n\to\infty}\frac{n^u n!}{u(u+1)\cdots(u+n)}, \tag{13}$$

and to expand its validity beyond $0 < u \le 1$

$$\Gamma(u+1) = u\Gamma(u)\frac{n}{n+u+1},\ \Gamma(u) = \Gamma(u+1)\frac{u+n+1}{nu}. \ \square \tag{14}$$

Γ(u) in equation (10) does not specify a unique function. Indeed, for an arbitrary polynomial g(u) in the interval $0 < u \le 1$ we can specify the properties

$$g(u + n) = [(u + n - 1) \ldots (u + 1)u]g(u), \tag{15}$$

and for $-n < u < -n + 1$

$$g(u) = \{1/[u(u + 1) \ldots (u + n - 1)]\}g(u + n),$$

so that g(u) is defined for all Eudoxus numbers, with the exception of 0 and the negative integers.

Theorem 5.2.13 (Bohr-Mollerup). *A function* g(u) *with the following properties is identical throughout its domain of definition to the gamma function.*
(i) g(1) = 1.
(ii) g(u + 1) = ug(u)
(iii) g(u) *is defined for* u > 0 *and is log convex*.

Proof. The existence of a function with these properties, the gamma function, has already been proved.

With g(u) replacing Γ(u), equation (11) is valid because of condition (ii), and g(n) = (n – 1)! for all integers > 0 because of condition (i), so it will be sufficient to show that g(u) = Γ(u) on the interval $0 < u \le 1$. If this is the case, then g(u) agrees with Γ(u) everywhere since (ii) holds. But the argument leading to equation (13) holds with g(u) replacing Γ(u), and this expression is identical to the gamma function. □

We note other formulas in [Ar64] using Γ(u), namely Gauss's multiplication formula

$$\Gamma\left(\frac{u}{p}\right)\Gamma\left(\frac{u+1}{p}\right)\cdots\Gamma\left(\frac{u+p-1}{p}\right) = \frac{(2\pi)^{(p-1)/2}}{p^{u-1/2}}\Gamma(u), \tag{16}$$

from which on putting p = 2 we get Legendre's duplication formula, otherwise known as Legendre's relation

$$\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{u+1}{2}\right) = \frac{\sqrt{\pi}}{2^{u-1}}\Gamma(u), \tag{17}$$

then with u = 1 we find

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{18}$$

and the formula known as Euler's functional equation

$$\Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin \pi u}. \tag{19}$$

We will prove (18), (17) and (19) in that order. Formula (16) is not used in what follows.

As well as Euler's second integral, Euler discovered another function related to the gamma function, the *beta function*, containing on the right *Euler's first integral*

$$B(u,v) = \int_0^1 y^{u-1}(1-y)^{v-1}dy, \tag{20}$$

which is in two variables, u and v.

To prove this integral exists for positive u and v, define it as the sum of two integrals

$$B(u,v) = \int_0^{\frac{1}{2}} y^{u-1}(1-y)^{v-1}dy + \int_{\frac{1}{2}}^1 y^{u-1}(1-y)^{v-1}dy,$$

for which the integral for the first is smaller than $y^{u-1}(1-y)^{-1}$, and thus smaller than $2y^{u-1}$, and the second is smaller than $y^{-1}(1-y)^{v-1}$, and hence smaller than $2y^{v-1}$. The method in the exercises to prove the existence of Euler's second integral can now be used to show the existence of Euler's first integral.

If we replace u by u + 1 in (20) and write the integral in the form

$$B(u+1,v) = \int_0^1 (1-y)^{u+v-1}\left(\frac{y}{1-y}\right)^u dy,$$

we can integrate by parts to obtain under more general limits of integration

$$\int_\varepsilon^{1-\delta}(1-y)^{u+v-1}\left(\frac{y}{1-y}\right)^u dy$$

$$= \left[-\frac{(1-y)^{u+v}}{u+v}\left(\frac{y}{1-y}\right)^u\right]_\varepsilon^{1-\delta} + \int_\varepsilon^{1-\delta}\frac{u}{u+v}(1-y)^{u+v}\left(\frac{y}{1-y}\right)^{u-1}\frac{1}{(1-y)^2}dy$$

$$= \left[-\frac{(1-\varepsilon)^v\varepsilon^u - \delta^v(1-\delta)^u}{u+v}\right]_\varepsilon^{1-\delta} + \frac{u}{u+v}\int_\varepsilon^{1-\delta} y^{u-1}(1-y)^{v-1}dy,$$

and if we let ε and δ converge to zero we get

$$B(u+1,v) = \frac{u}{u+v}B(u,v). \tag{21}$$

We now wish to obtain an equation that satisfies equation (10). To do this we put

$$g(u) = B(u, v)\,\Gamma(u + v), \tag{22}$$

which satisfies condition (ii) of the Bohr-Mollerup theorem 5.2.13. Moreover, since B(u, v) and Γ(u + v) as functions of u are log convex, g(u) is the product of two log convex functions and thus is also log convex. This implies that condition (iii) of the Bohr-Mollerup theorem is satisfied. However, condition (i) does not hold, because

$$B(1,v) = \int_0^1 (1-y)^{v-1}dy = \frac{1}{v}, \tag{23}$$

so that

$$g(1) = \frac{1}{v}\Gamma(1+v) = \Gamma(v), \tag{24}$$

but this is not a big problem. We can define from (22) the equation

$$h(u) = g(1)\Gamma(v), \tag{25}$$

which now satisfies (i), and still (ii) and (iii). This gives

$$h(u) = \Gamma(u)\Gamma(v),$$

and hence using equations (25), (24), (22), (21) and (20) supplies the solution

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \int_0^1 y^{u-1}(1-y)^{v-1}dy. \tag{26}$$

On setting u = $\frac{1}{2}$ and v = $\frac{1}{2}$, we can evaluate the above integral using the substitution

$$y = \sin^2\theta$$

to obtain

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = 2\int_0^{\frac{\pi}{2}} d\theta = \pi,$$

in other words the required equation (18)

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \ \square$$

Putting y = x^2 in (20), so dy = 2xdx, gives

$$B(u,v) = 2\int_0^1 x^{2u-1}(1-x^2)^{v-1}dx. \tag{27}$$

Now put w = u = v, but this time y = (1 + x)/2, which gives dy = dx/2, to obtain from (26)

$$\begin{aligned}\frac{\Gamma(w)^2}{\Gamma(2w)} &= \int_{-1}^{1}\left(\frac{1+x}{2}\right)^{w-1}\left(1-\frac{1+x}{2}\right)^{w-1}\frac{1}{2}dx\\ &= \frac{1}{2}\int_{-1}^{1}\left(\frac{1+x}{2}\right)^{w-1}\left(\frac{1-x}{2}\right)^{w-1}dx\\ &= \frac{1}{2^{1+2(w-1)}}\int_{-1}^{1}(1-x^2)^{w-1}dx,\end{aligned}$$

and since $x^2 = (-x)^2$, the limits of the above integral can be changed to give

$$\frac{\Gamma(w)^2}{\Gamma(2w)} = 2^{1-2w}\int_0^1 2(1-x^2)^{w-1}dx, \tag{28}$$

so that on putting (26), (27) and (28) together gives

$$\frac{\Gamma(w)^2}{\Gamma(2w)} = 2^{1-2w}B\left(\tfrac{1}{2}, w\right) = 2^{1-2w}\frac{\Gamma(\frac{1}{2})\Gamma(w)}{\Gamma(\frac{1}{2}+w)}, \tag{29}$$

which means, using equation (18) for $\Gamma\left(\frac{1}{2}\right)$ and with the substitution w = u/2, we obtain equation (17)

$$\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{u+1}{2}\right) = \frac{\sqrt{\pi}}{2^{u-1}}\Gamma(u). \ \square$$

In order to prove equation (19), set

$$\psi(u) = \Gamma(u)\Gamma(1-u)\sin\pi u, \tag{30}$$

which is only defined for noninteger parameters. On replacing u by (u + 1), $\Gamma(u)$ becomes $u\Gamma(u)$ and the function $\Gamma(1-u)$ is transformed to

$$\Gamma(-u) = \frac{\Gamma(1-u)}{-u},$$

and $\sin\pi u$ changes its sign. This means $\psi(u)$ is left fixed, and is thus periodic with period 1.

$$\psi(u) = \psi(u+1). \tag{31}$$

On using Legendre's duplication formula (17) and replacing u by (1 – u)

$$\Gamma\left(\frac{1-u}{2}\right)\Gamma\left(1-\frac{u}{2}\right) = 2^u\sqrt{\pi}\,\Gamma(1-u).$$

We now consider

$$\begin{aligned}\psi\left(\tfrac{u}{2}\right)\psi\left(\tfrac{u+1}{2}\right) &= \Gamma\left(\frac{u}{2}\right)\Gamma\left(1-\frac{u}{2}\right)\sin\frac{\pi u}{2}\,\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{1-u}{2}\right)\cos\frac{\pi u}{2}\\ &= \pi\,\Gamma(u)\Gamma(1-u)\sin\pi u,\end{aligned}$$

and we obtain the equation

$$\psi\left(\tfrac{u}{2}\right)\psi\left(\tfrac{u+1}{2}\right) = \pi\psi(u). \tag{32}$$

Since both $\Gamma(u)$ and $\sin \pi u$ can be differentiated indefinitely, $\psi(u)$ can be as well. Then from equation (10) we can write

$$\psi(u) = \frac{\Gamma(1+u)}{u}\Gamma(1-u)\sin \pi u,$$
$$= \Gamma(1+u)\Gamma(1-u)\left(\pi - \frac{\pi^3 u^2}{3!} + \frac{\pi^5 u^4}{5!} - \cdots\right),$$

where $\psi(u)$ is not defined at u = 0, but as $u \to 0$, $\psi(u)$ can be extended analytically to u = 0. $\psi(u)$ is now continuous for all u.

To show that $\psi(u)$ is a constant, define h(u) as the second derivative of log $\psi(u)$, so that h(u) is also periodic with period 1, and by equation (32)

$$\frac{1}{4}\left(h\left(\frac{u}{2}\right) + h\left(\frac{u+1}{2}\right)\right) = h(u). \tag{33}$$

Because h(u) is continuous in the interval [0, 1], it is bounded in this interval, say $|h(u)| \leq N$, and this inequality holds for all u, since h(u) is periodic.

We can now show that h(u) = 0. Equation (33) implies

$$|h(u)| \leq \frac{1}{4}\left(\left|h\left(\frac{u}{2}\right)\right| + \left|h\left(\frac{u+1}{2}\right)\right|\right) \leq \frac{N}{4} + \frac{N}{4} = \frac{N}{2},$$

which means the upper bound can be decreased from N to N/2. Repeating this procedure halves the bound at each stage, so its limit is zero. Since h(u) is continuous, h(u) = 0. But h(u) is defined as the second derivative of log $\psi(u)$, so that log $\psi(u)$ is linear in $\psi(u)$ and because $\psi(u)$ is periodic, log $\psi(u)$ is a constant, which implies $\psi(u)$ is a constant for all u. This means $\psi(u)$ maintains its value from the previously derived value of π, and hence equation (19) holds

$$\Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin \pi u}. \square$$

5.3. The basic problem. [Ed74]

Riemann's work of 1859 *On the number of primes less than a given magnitude* begins with the Euler product formula

$$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)}, \tag{1}$$

where n is a positive natural number, and p ranges over all primes (p = 2, 3, ...), which is obtained by expanding the factors on the right

$$\frac{1}{\left(1 - \frac{1}{p^s}\right)} = 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \cdots$$

and observing that the product of these factors is a sum of terms of the form

$$\frac{1}{(p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r})^s},$$

where p_1, p_2, ... p_r are distinct primes and n_1, n_2, ... n_r are natural numbers, and thus by the fundamental theorem of arithmetic of chapter III, section 9 of *Superexponential algebra*, that every natural number up to order of factors can be written in only one way as a product of primes, we infer that their sum is the left hand side of (1).

The difference from previous authors on this formula is that Riemann considers s to be a complex number, whereas Dirichlet, one of Riemann's former teachers, used a real variable. The function of a complex variable s which the two sides of (1) define is the Riemann zeta function ζ(s).

In what follows in this section the complex exponential algebras we will consider are of standard and not $\mathbb{D}$w type.

Substitution of ny, n a natural number, for y in 2.(2) gives

$$\frac{\Gamma(s)}{n^s} = \int_0^{\Omega} e^{-ny} y^{s-1} dy, \ (s > 0), \tag{2}$$

where Riemann sums this over n and uses

$$\sum_{n=1}^{\Omega} r^{-n} = (r - 1)^{-1}$$

to get

$$\int_0^{\Omega} \frac{y^{s-1}}{e^y - 1} dx = \Gamma(s) \sum_{n=1}^{\Omega} \frac{1}{n^s}, \ (s > 1). \tag{3}$$

Next he considers the contour integral

$$\int_{+\Omega}^{+\Omega} \frac{(-y)^s}{e^y - 1} dy = \left(e^{i\pi s} - e^{-i\pi s}\right) \int_0^{\Omega} \frac{y^{s-1}}{e^y - 1} dy,$$

where the limits of integration indicate a contour which starts at $+\boldsymbol{\Omega}$ moves on the left down the positive Eudoxus axis, circles the origin once in a counterclockwise direction, and returns up the positive Eudoxus axis to $+\boldsymbol{\Omega}$, so that combined with equation (3) we obtain

$$\int_{+\Omega}^{+\Omega} \frac{(-y)^s}{e^y - 1} \frac{dy}{y} = 2i \sin \pi s \, \Gamma(s) \sum_{n=1}^{\Omega} \frac{1}{n^s}, \ (s > 1), \tag{4}$$

or on using equation 2.(12)

$$\zeta(s) = \frac{\Gamma(1 - s)}{2\pi i} \int_{+\Omega}^{+\Omega} \frac{(-y)^s}{e^y - 1} \frac{dy}{y}, \ (s > 1), \tag{5}$$

which remains valid for all s (the integral (5) converges because e^y grows much faster than y^s as $y \to \Omega$, and the function it defines is complex analytic, where the left hand side of (1) is defined for n = 2, 3, ..., except it is not analytic at s = 1, where it has a simple pole).

Let D consist of the domain in the complex plane of all points lying within a distance δ of one of the singularities $x = \pm 2\pi n$ in (1). Cauchy's integral formula gives over the boundary of D oriented clockwise

$$\oint \frac{(-y)^s}{e^y - 1} \frac{dy}{y} = 0. \tag{6}$$

One component of (6) is the integral (5) with counterclockwise orientation, and the others are integrals over circles $\delta = |y \pm 2\pi ni|$ with clockwise orientation. Thus, integrating in the counterclockwise sense around δ

$$-\zeta(s) = \sum \frac{\Gamma(1 - s)}{2\pi i} \oint \frac{(-y)^s}{e^y - 1} \frac{dy}{y}. \tag{7}$$

These contour integrals around $|z| = \delta$ can be obtained by setting $y = z + 2\pi n$ to find

$$\frac{\Gamma(1 - s)}{2\pi i} \oint \frac{(-2\pi ni - z)^s}{e^{2\pi ni} - 1} \frac{dz}{2\pi ni + z} = \frac{\Gamma(1 - s)}{2\pi i} \oint (-2\pi ni - z)^{s-1} \frac{z}{e^z - 1} \frac{dz}{z} \tag{8}$$

which gives

$$\zeta(s) = \sum_{n=1}^{\Omega} \Gamma(1 - s)[(-2\pi ni)^{s-1} + (2\pi ni)^{s-1}], \tag{9}$$

or

$$\zeta(s) = \Gamma(1 - s)(2\pi)^{s-1}[(i)^{s-1} + (-i)^{s-1}] \sum_{n=1}^{\Omega} n^{s-1}. \tag{10}$$

5.4. Some observations on the Riemann hypothesis.

From equation 5.3.(10) we are presented with the equation

$$\zeta(s) = (2\pi)^{s-1} [i^{s-1} + (-i)^{s-1}] \, \Gamma(1 - s) \, \zeta(1 - s), \tag{1}$$

so in the conventional circumstances where

$$[i^{s-1} + (-i)^{s-1}] = [i^{s-1} + (i^{-1})^{s-1}]$$

$= -i[i^s - i^{-s}]$
$= -i[e^{s\pi i/2} - e^{-s\pi i/2}]$ (2)
$= 2\sin(\pi s/2)$

we obtain Riemann's functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin(\pi s/2)\,\Gamma(1-s)\,\zeta(1-s). \quad (3)$$

The expression (2) is identical to

$$[i^{s-1} + i^{-(s-1)}] = 2\cos[(s-1)\pi/2]. \quad (4)$$

Using 5.2.(19) the functional form of the zeta function in conventional circumstances can be written

$$\Gamma(s/2)\,\pi^{-s/2}\zeta(s) = \Gamma((1-s)/2)\pi^{-(1-s)/2}\zeta(1-s), \quad (5)$$

and thus is invariant under the substitution $s \to 1 - s$.

In particular, since for noninteger parameters we have shown

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$

so here $\Gamma(s) \neq 0$, we find from (5) that if $\zeta(s)$ is zero, $\zeta(1-s)$ is zero. It is clear if $\zeta(s) = 0$, that the imaginary parts of s and $(1-s)$ are of opposite sign and if the real part of s is ½ then $\zeta(1-s)$ gives to $(1-s)$ a real part of ½. ☐

Let $s = a + ib$, and s^* be its complex conjugate $a - ib$. Consider the $\mathbb{D}$w exponential algebra with w the imaginary number i. From volume I, chapter III, section 2, this satisfies

$$(e^c)^d = e^{cd} \quad (6)$$
$$(e^{ic})^d = e^{icd} \quad (7)$$
$$(e^c)^{id} = e^{idc} \quad (8)$$
$$(e^{ic})^{id} = e^{widc}. \quad (9)$$

This is clearly the classical exponential algebra. There also exists a nonclassical algebra with $w = -i$.

Now the zeta function series 5.3.(1) being a sum of terms

$$(e^{-c})^{a+ib} = (e^{-ca})(e^{-icb})$$

is not dependent on the allocation (9), only on (6) and (8). But the variable

$$i^s = (e^{i\pi/2})^{a+ib}$$

is dependent on equation (9). Then in equation (2) the nonclassical algebra term

$$-i[(e^{i\pi/2})^{a+ib} - (e^{i\pi/2})^{-a-ib}]$$

maps the classical value $2\sin(\pi s/2) \to 2\sin(\pi s^*/2)$ as measured in classical algebra.

We can allocate by definition a value of $\Gamma(1-s)$ which is stable under transformation from the classical to the nonclassical algebra, since definition 5.2.(2) can be made independent of (9) above.

We note that, as discussed from 5.2.(30) onwards, equation 5.2.(19) is dependent on $\sin \pi u$ and we have seen that $\sin \pi u$ depends on its classical or nonclassical implementation. This equation now becomes, writing $\Gamma'(s)$ instead of $\Gamma(s)$

$$\psi(s) = \Gamma'(s)\Gamma(1-s)\sin \pi s. \quad (10)$$

This means the allocation and (3) is consistent for $\Gamma(1-s)$ by restricting the real part of s to between zero and less than ½ or one minus this. Then the log convex definition of $\Gamma(1-s)$ holds between $0 < s < ½$ and this is invariant under (9), and the $\Gamma'(s)$ part will be defined as compliant with equation (10), which was formerly described as $\Gamma(s)$ and is now dependent on

(9), whereas a new definition of $\Gamma(s)$ is chosen which is invariant under (9) for 0 < s < ½ and (11) below holds for $\Gamma'(1 - s)$.

$$\psi(s) = \Gamma(s)\Gamma'(1-s)\sin \pi s. \tag{11}$$

So $\Gamma(s)$ is defined by two methods which exclude s = ½. Let us assume this.

Then when ζ(s) → 0 the real part remains the same in equation (3), but the ζ(1 – s) imaginary part changes sign from the classical value. This means the imaginary parts with parameters s and (1 – s) in the analogue of equation (3) are now of the same sign, and thus correspond in the limit to the same root. This is the same as setting s = (1 – s*) in the limit, so the real part of s = ½, in contradiction with the removable assumption that s ≠ ½. But we have already seen that s = ½ has good credentials. Thus s = ½. ☐

5.5. L-series.

We have seen in chapter III section 5 of *Superexponential algebra* that if a finite commutative group has order, or number of elements, m, it is cyclic and may be represented as a product of commutative groups, each with prime order. We know this cyclic group may also be represented by an Argand diagram of complex roots of unity, so that n (mod m) maps to $e^{2\pi i m/n} = \omega^n$.

A *Dirichlet character* of such a group is a function char(n) which maps the elements n of the group to the unit circle in the complex plane for it, so that (mod m) it satisfies for n ∈ ℕ

(i) char(n + m) = char(n)

(ii) char(n.n′) = char(n). char(n′)

(iii) char(n) ≠ 0 if and only if the greatest common divisor of n and m is 1.

If n is mapped to any mth root of unity, this directly represents char, so that the characters form a multiplicative group with

$$\text{char}(n.n') = \text{char}(n).\ \text{char}(n'), \tag{1}$$

for which

$$\text{char}(1) = 1. \tag{2}$$

For the inverse n^{-1} of n, which by (2) is the complex conjugate in char,

$$\text{char}(n^{-1}) = \text{char}^{-1}(n). \tag{3}$$

Because the sum of roots of unity

$$1 + \omega + \omega^2 + ... + \omega^{m-1} = m \text{ or } 0,$$

has a value m for this sum when ω = 1 and 0 otherwise, another important relation is that the sum, or trace, of all char is m for n = 1, and 0 otherwise. This can be rephrased as saying that the characters char_k, k = 1, ..., m, satisfy the orthogonality relations

$$\frac{1}{m}\sum_n \text{char}_i^{-1}(n)\,\text{char}_j(n)$$
$$= 1 \text{ for } i = j$$
$$= 0 \text{ for } i \neq j. \tag{4}$$

If such a character is given, we define the corresponding Dirichlet L-function by

$$L(\text{char}, s) = \sum_{n=1}^{\Omega} \frac{\text{char}(n)}{n^s}. \tag{5}$$

By analytic continuation this function can be extended about each neighbourhood of a point to be complex differentiable except at a set of isolated poles of the function, so it is a meromorphic function, defined on the whole complex plane. The generalised Riemann hypothesis asserts that for every Dirichlet character char and every complex number s with L(char, s) = 0, if the real part of s is between 0 and 1, then this real part is actually ½.

The case char(n) = 1 for all n yields the ordinary Riemann hypothesis.

Now $\mathrm{L}(\mathrm{char}, s)$ can also be represented in a form independent of equation 5.4.(9). Thus the argument of section 4 carries over to this case, which means that the generalised Riemann hypothesis holds. □