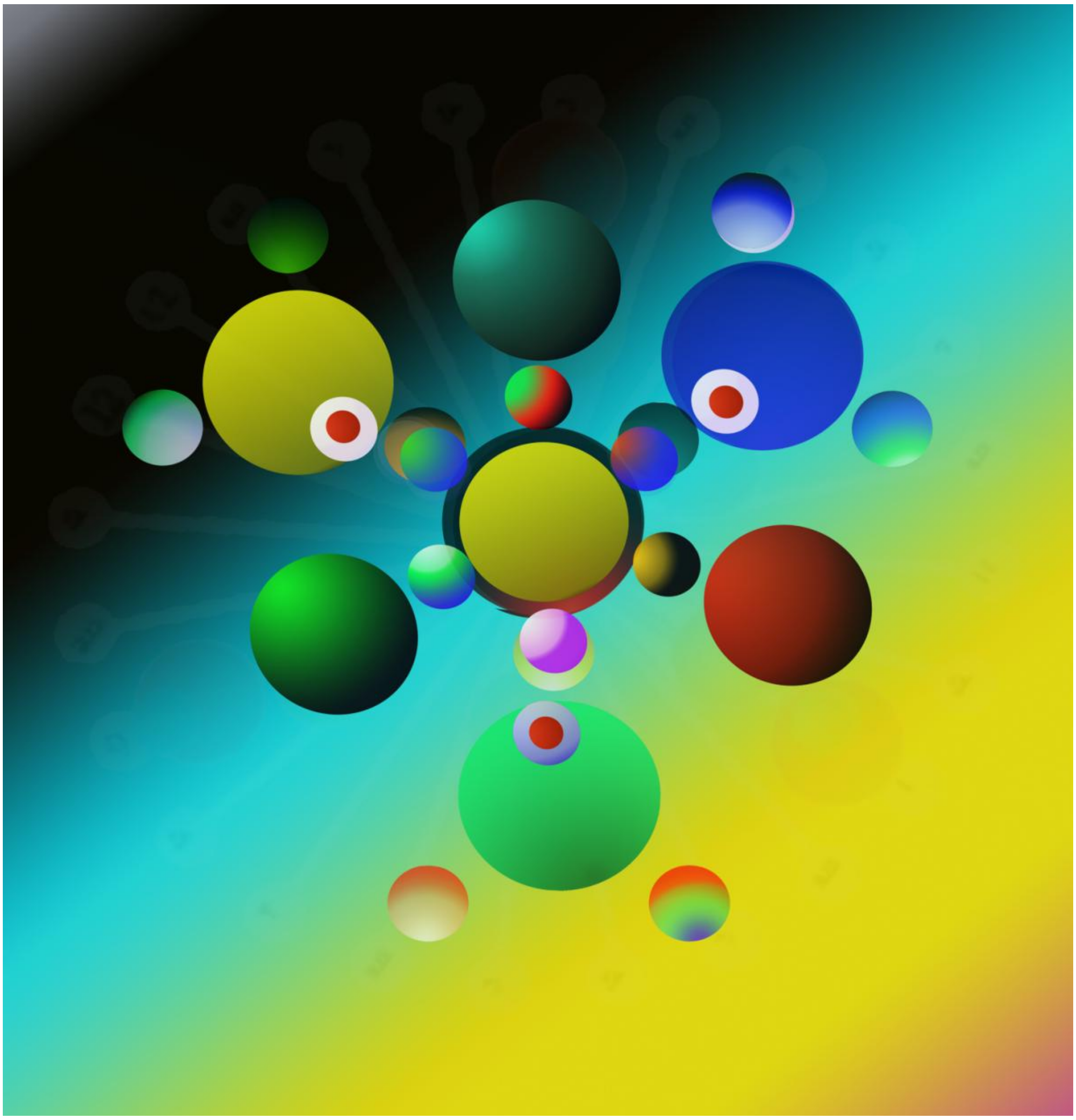
Number, space and logic

Volume III

The finite and the infinite



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The cover artwork is by Ian Dell.

Jim H. Adams is a researcher in the concepts of mathematics and their explicit representation. This work is a guide to some of these ideas, and an encouragement to the graduate to develop mathematical research productively and continually.

After a career in IT, Jim joined New Music Brighton as a composer and performer of his own works. He has been extensively involved in mathematical research.

The eBooks published in 2014 were *The climate and energy emergencies* and *Innovation in mathematics*. Volumes I, II and III of *Superexponential algebra* were completed in 2017, and *Number, space and logic* is a mathematical sequel to that work.

Keywords: l-adic numbers, elliptic curves, modular forms, complex multiplication, Heegner number, Riemann hypothesis, Goldbach conjecture.

Foreword

Number, space and logic could be viewed as a mathematical manifesto and encyclopaedia. It combines a graduate level textbook with a research project in a commentary and development of the mathematics of the late 20th and the 21st centuries, just as *Superexponential algebra* [Ad15] does for the period in the 20th century before that covered in this work. As a practical guide to this, we include the mathematical machinery necessary for a proof of the general Riemann hypothesis and from it the weak Goldbach conjecture. There is an exposition of the theory of branched spaces and a Gentzen-type proof of the consistency of analysis.

In order to present our work in a coherent way, it has been necessary to proceed top down, from the abstract to the specific, but so that this generality can be understood we first give an account of the meanings of these abstractions. First we describe our approach to the finite and the infinite, then how our generalisations of space fit into the theory of trees and amalgams. Finally we show how superexponential structures can be used to encapsulate these ideas and extend them to an overarching theory of mathematics.

In the chapter on *the meaning of the finite and the infinite* we include a description of standard and significant results in finite and discrete number theory, including on p-adic numbers. We also discuss nonstandard results. Firstly we discuss extended sets of natural numbers, which we call transnatural numbers. The essential feature of these is that, unlike our results in nonstandard set theory, these numbers are not bijective to the natural numbers, instead they have a transfinite number of members, but otherwise they share the properties of natural numbers. For instance they contain transfinite prime numbers and can be used to define transfinite rational numbers. We also deduce the inconsistency of the uncountable continuum hypothesis compared with the countable rational numbers. We give an account of ladder algebra which incorporates a reformulation and extension of nonstandard analysis.

The chapter on *the meaning of branched spaces* introduces us to our extension of the domain of the Euler-Poincaré characteristic used in topology. In its familiar form our description appears more restricted than usual, although other formulations are equivalent to it. In its extended form it describes types of topology which have no analogue in standard topology. The model we introduce is one of branched spaces. We can put it this way: when a point is removed from the equivalent of an n-branched line, it divides into n pieces. An *explosion* is a set that contains an infinite number of branches. Indeed these are the topological analogues of other behaviour we will meet in exponential and superexponential number systems.

Concerning *the meaning of superexponentiation*, which is the topic of the last chapter of the eBook *Superexponential algebra* [Ad15], this idea is an extension of the repetitive process which generates multiplication from addition, and exponentiation from multiplication. So we generate an (n + 1)th superexponential operation by repetition of an nth superexponential operation. In that book we discussed nonstandard exponential and superexponential algebras, which we describe here by the statement that they are unbranched compared with standard such algebras. A real number we say has a *supernorm* defining its magnitude under an exponential or superexponential operation which is similarly real. Thus superexponentiation on numbers has two features that are interesting to investigate: supernorms and branching.

Having built up the meaning of these abstractions, from their axiomatic formulation we then provide a detailed exposition of these ideas in reverse order: of superexponential structures, trees and amalgams, and the finite and the infinite, and look at some of their consequences.

Volume III in more detail

In this volume we look more deeply into an exposition of our nonstandard outlook on the continuum hypothesis, and an extension of the idea of natural and rational numbers to the transfinite case, called respectively transnatural and transrational numbers. In this context we discuss class field theory and provide a Gentzen-type proof of the consistency of analysis.

We prove Fermat's last theorem by standard techniques on the modularity theorem derived in volume I.

We then prove the general Riemann hypothesis by two distinct methods. The first uses an extension of exponential algebra to the $\mathbb{D}$w exponential algebras, developed by myself and David Bohm, the latter in an attempt to prove the Riemann hypothesis. In [Ad15] these are given in the case of w an integer. In this volume we introduce imaginary $\mathbb{D}$w exponential algebras. These reduce to two types, the classical exponential algebra and the nonclassical. Both types give the same result for zeta functions, and this gives sufficient information to prove the theorem. This is the most direct method. The second method uses an extension of the case of local fields, where we prove the general Riemann hypothesis conjecture, with related ideas described by theorems developed by Weil, Grothendieck and Deligne. For the viewpoint developed here, we need not only these theorems extended to the transnatural and transalgebraic case, but also a discussion of those transcendental numbers independent of these two previous types. Thus we have three types of counting to do: for transnatural numbers, for transalgebraic numbers which are not transnatural, and then for transcendental numbers which are not of the other two types. For this we need a theory of transcendental independence extended to complex numbers, and a method of counting solutions for 3-branched spaces, in fact for 3-explosions. The latter is provided by an extension of the infinite superexponential methods of Gentzen.

Having proved the generalised Riemann conjecture by these means, we are able to give a proof of the weak Goldbach conjecture, both by the direct method of Harald Helfgott proved in 2013, and using the our result on the Riemann hypothesis combined with the work of J-M. Deshouilles, G.W. Effinger, H.I.J. te Riele and D. Zinoviev in 1997.

The author and the reader

A central objective of this work is to encourage the graduate to produce mathematical ideas, and also to give assistance by providing a minimally fussy exposition of the research into number theory of the last fifty years. We are seeking to foster insight so that the reader can pursue further investigations into the technical literature in a spirit of an understanding of its background.

Except for marketing, our inclination is that a proof is only the final stage in the presentation of a mathematical idea. Proofs are necessary, but not sufficient.

The objective of creative mathematics is the production of ideas. These need caring support, and part of this objective is not just to replicate results, but to reconstruct mathematics from new principles, to have the resilience to deconstruct what has gone before if it is needed, and to extend results by an analysis of the core ideas of the subject as it currently stands. Thus it becomes feasible to develop generalised new methods to tackle outstanding problems.

I am aware that the first language of the reader may not be English, so I have looked at the text and removed high literary style. Technical terms are I hope well explained, and examples are given.

I have provided no exercises to work through the text, the reason being that these are no more needed for the development of creative skills than the memorisation of words and standard literary works is necessary for an active participation in becoming an author. My hope is that the diligent reader will provide what is necessary for constructive work, and that it is no longer necessary to give direction.

The contents of this work are my own, as is the responsibility for any errors.

Jim H. Adams

Brighton & Hove

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Mathematical terms

The following terms, symbols, ideas and definitions are used in the text. The arrangement is by ideas rather than alphabetic. This may be scanned as a more technical alternative to the details of the contents in the Foreword, or as a further introduction to the contents which follow in the eBook. The notation for example (SA III) refers to *Superexponential algebra* chapter III.

1. *Sets* S, T (SA III).
$\varnothing$ (*the empty set*). The set with no members.
$\odot$ (*the void set*). The set satisfying a false condition.
$\in$ (*belongs to*). If x is a member of a set S then $x \in S$.
$\subset$ (*properly included in*). If a set S is included in a set T and S does not equal T.
$\subseteq$ (*included in*). Inclusion, when S = T is possible.
$\complement S_T$ (*complement of* S *in* T). Those x not belonging to S but that belong to T, and $S \subset T$.
$\cup$ (*union of sets*). If x belongs to S or x belongs to T then x belongs to $S \cup T$.
$\cap$ (*intersection of sets*). If $x \in S$ and $x \in T$ then $x \in S \cap T$.

2. $\mathbb{N}$ is the set {1, 2, 3 ...} of *positive whole numbers*, also called *natural numbers*. If this set contains the element 0, we denote it in this eBook by $\mathbb{N}_{\cup 0}$. If we wish to emphasise that it does not contain zero, we use $\mathbb{N}_{\neq 0}$.
$\mathbb{Z}$ (from the German Zahl for number) is the set {..., -3, -2, -1, 0, 1, 2, ...} of negative, zero or positive *integers*.
$\mathbb{Q}$ is the set of *rational numbers* m/n, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, for example ½.
$\mathbb{A}$ is used in this eBook as the set of *algebraic numbers*, sums and differences of numbers of the form p^q, where $p, q \in \mathbb{Q}$, but p and q together are not both zero, for example $1 + 2\sqrt[3]{\frac{-1}{5}}$.
$\mathbb{M}_t$ is the set of *transnatural numbers* for index t, satisfying the rules for $\mathbb{N}$. $\mathbb{M}_1 = \mathbb{N}$. There are no surjections $\mathbb{M}_t \to \mathbb{M}_{t+1}$ or injections $\mathbb{M}_t \to \mathbb{M}' \to \mathbb{M}_{t+1}$ for distinct $\mathbb{M}'$.
$\mathbb{Z}_t$ *Transintegers* are positive, zero or negative transnatural numbers.
$\mathbb{Q}_t$ *Transrational numbers* are of the form m/n where $m \in \mathbb{Z}_t$ and $n \in \mathbb{M}_t$.
$\mathbb{A}_t$ *Transalgebraic numbers* are sums and differences of the form p^q, where $p, q \in \mathbb{Q}_t$.

3. *Congruence arithmetic* (*mod n*). Finite or 'clock' arithmetic where transnatural numbers come back to themselves, so its set is {0, 1, ... (n – 1)} and n = 0.
Prime number. A transnatural number which only when divided by 1 and itself gives a transnatural number. Example: 7.
Totient ($\varphi(s)$). For a transnatural number s as a product of primes p, q, ... r to powers j, k, ... m, if $s = (p^j)(q^k)...(r^m)$ then $\varphi(s) = s\left[\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\cdots\left(1-\frac{1}{r}\right)\right]$. Example: $\varphi(20) = 8$.

4. *Abelian*. Occurs for a set with a general operation + (not necessarily addition as usually understood) when a + b = b + a always.
Commutative. Abelian, but generally written for × rather than +.
Associative. Satisfying a + (b + c) = (a + b) + c, or a(bc) = (ab)c, etc.

5. *Real* (*or Eudoxus*) *numbers*, $\mathbb{U}$. (chapter I and *Discussion on ladder numbers and zero algebras* in the eBook Innovation in mathematics, [Ad14]).
Complex numbers, $\mathbb{C}$. Numbers of the form a + bi, where a, b $\in \mathbb{U}$ and i = $\sqrt{-1}$.
Gaussian integers. Complex numbers where a and b above are integers.
$\mathbb{F}$ is a *field* (SA III). It contains axioms (rules) for addition and multiplication. Examples could be the real (or Eudoxus) numbers $\mathbb{U}$ and complex numbers $\mathbb{C}$.
$\mathbb{Y}$ is a *zero algebra* (SA III). This is similar to a field except for the existence of multizeros.
Exponential algebra, (SA XII and XIII). Contains axioms for exponentiation.
$\mathbb{D}$w *exponential algebra*, (SA XIII). A nonstandard exponential algebra.
Superexponentiation (chapter III and SA XV). An operation, of which the first three are addition, multiplication and exponentiation, where the nth is found by repeating the (n – 1)th.
Superexponential algebra. Contains axioms connecting superexponential nth operations for various n.

6. *Implies*. (exercise, SA XI). For statements A and B, A implies B is only false when A is true and B is false.
Sufficient. A is sufficient for B means A implies B.
Necessary. A is necessary for B means A is implied by B (the same as B implies A).

7. *Function* (SA III). A set of pairs, {x, f(x)}, all x of which have a value f(x).
Injection. A mapping from all the sets {a, b} to {f(a), f(b)}, where f(a) $\neq$ f(b) if a $\neq$ b.
Surjection. A mapping where every f(x) in the set {f(x)} has a value from an x.
Bijection. A mapping which is simultaneously injective and surjective.

8. *Magma*, M (SA III). A set with one binary operation, with no other properties specified.
Polymagma (SA XV). Maps a number of copies of a set to itself.
Group, G (SA III). This satisfies the multiplicative axioms for a field, except multiplication may be noncommutative: ab $\neq$ ba.
Subgroup. A set of elements in a group which satisfies within itself all the properties of the containing group.
Order of a group. The number of elements (or members) in a group.
Homomorphism of groups is a surjective map h: G $\to$ G′ of groups with h(ab) = h(a)h(b).
Isomorphism of groups. A bijective homomorphism.
Automorphism of a group is an isomorphism of a group to itself.
Inner automorphism of a group is an automorphism of the form x $\leftrightarrow$ a^{-1}xa.
Outer automorphism of a group. An automorphism which is not inner.
Normal subgroup is invariant under all inner automorphisms of the containing group G.
Simple group has no normal subgroups other than itself and 1.

9. *Ring*, A. Satisfies the additive and multiplicative axioms of a field, except there is no general division and multiplication may be noncommutative. Example: matrices.
Unital ring. A ring with a multiplicative identity, 1. We assume rings are unital.
Automorphism of a ring. (SA X). A bijective map of a ring A, H: A $\leftrightarrow$ A, where H(ab) = H(a)H(b) and H(a + b) = H(a) + H(b).

10. *Matrix* (*plural matrices*). An array of numbers B = b_{jk}, where the element b_{ij} exists in the i^{th} row and j^{th} column. (SA I and II).
Symmetric matrix. U = u_{jk} = u_{kj}.
Antisymmetric matrix. V = v_{jk} = $-v_{kj}$.
Matrix transpose. If W = w_{jk} , then the transpose W^T = w_{kj}.
Unit diagonal matrix. Denoted by I = b_{jk}, where b_{jk} = 1 when j = k, otherwise b_{jk} = 0.
Trace of a matrix. The sum of all (main) diagonal entries b_{jk}, where j = k.
Determinant (*or hypervolume*) *of a matrix*, (*det* B). (SA I and II).
Singular matrix, D. Satisfies det D = 0.

11. *Intricate number*. A representation of 2 × 2 matrices, that is, with two rows and two columns, given by a1 + bi + cα + dϕ. (SA I).
Intricate basis element. One of the vectors 1, i, α or ϕ above.
Real basis element. The number 1 in its intricate representation.
Imaginary basis element. The number i in its intricate representation.
Actual basis element. The number α in its intricate representation.
Phantom basis element. The number ϕ in its intricate representation.
Intricate conjugate. The number a1 – bi – cα – dϕ.
J. J = bi + cα + dϕ in which J^2 = 0 or ±1.
JAF. A changed basis for i, α and ϕ.

12. *Hyperintricate number*. A representation of $2^n \times 2^n$ matrices. (SA II).
Layer. For example, a hyperintricate number with a component in 3 layers is $A_{B,C}$ where A, B and C are intricate numbers, possibly intricate basis elements.
n-hyperintricate number. A hyperintricate number representable by sums of components in n layers. Sometimes denoted by $Я_n$.
n-hyperintricate conjugate, $Я^*_n$. Satisfies $Я^*_n Я_n$ = det $Я_n$.
J-abelian hyperintricate number. A number giving the example $A_B + ... + D_E$, where A = p1 + qJ, B = p′1 + q′J′, ..., D = t1 + uJ, E = t′1 + u′J′. Two such hyperintricate numbers with identical J and J′ commute.
Compression. The map κ: $A_B \to AB$.
Expansion. A map $κ^{op}$: $AB \to A_B$.

13. *Vector*, **v** (in bold). A matrix as one row (a row vector), or as one column (a column vector). Example: the row vector (x, y, z).
Scalar product of two vectors. The matrix product of multiplying each element of a row vector in turn with the corresponding elements of a column vector. Example: $x^2 + y^2 + z^2$.
Eigenvector. A vector **x** satisfying for matrix B, B**x** = λ**x**.
Eigenvalue. A value λ for the eigenvector **x** above. Example: λ is a complex root value.
Vector space. Contains vectors with magnitude and direction, which can be added together and multiplied by scalars in a field.
Module. A module over a ring is a generalisation of a vector space over a field, being an additive abelian group like a vector space, where the scalars are the elements of a ring.

14. *Division algebra*. (SA III and V). A ring with division where multiplication might be nonassociative. Two elements of a division algebra cannot be multiplied giving zero unless one of them is zero.
Quaternions. (SA III and V). A type of associative division algebra.
Exquaternions. (SA V). A type of algebra obtained from quaternions, neither associative nor with complete division.
Octonions, $\mathbb{O}$. (SA V). A nonassociative division algebra.
Exoctonions. (SA V) A type of algebra derived from the octonions, neither associative nor with complete division.
n-novanions. (SA V). An n dimensional nonassociative division algebra, but not when both the real parts in a multiplication are zero.

15. *Norm*. Applied to complex numbers $a + bi$, the norm is $\sqrt{}(a^2 + b^2)$. For intricate numbers $a1 + bi + c\alpha + d\phi$ the norm squared is $a^2 + b^2 - c^2 - d^2$. Applied to a $n \times n$ matrix B, the norm is the positive nth root of det B. Applied to n-novanions $a1 + bi + c\alpha + d\phi + b'i' + c'\alpha' + d'\phi' + ...$, the norm is $\sqrt{}(a^2 + b^2 + c^2 + d^2 + b'^2 + c'^2 + d'^2 + ...)$.
Interlayer operator $\underline{\vee}_P$. (SA IV).
Diamond operator $\lozenge$. SA IV).
Left roll operator s°. (SA IV).
Right roll operator $^\circ s$. (SA IV).
Split product. (Sa IV).

16. *Standard protocol*. (chapter I). The ordinal infinity $\mathbf{\Omega}_{\mathbb{N}} = \sum_{\text{all } \mathbb{N}} 1$. This is not a natural number, and is treated as being irreducible.
Ladder number. A superexponential expression in $\mathbf{\Omega}_{\mathbb{N}}$, with Eudoxus coefficients.
Strict transfer principle. The axioms for variables in a superexponential algebra also hold for the variable $\mathbf{\Omega}_{\mathbb{N}}$.
Winding number. The number of times a loop winds round a point.

17. *Additive format of a polynomial equation*. The form $ax^n + bx^{n-1} + ... + d = 0$. (SA VII and VIII).
Monic polynomial. Example in the case of a polynomial equation: when a above = 1.
Fundamental theorem of algebra. The complex polynomial in additive format given by $ax^n + bx^{n-1} + ... + d$ always has some values which are zero.
Multiplicative format of a polynomial equation. The form $(x - p)(x - q) ... (x - t) = 0$.
Zero of a polynomial. A value of a polynomial $f(x) = ax^n + bx^{n-1} + ... + d$ so that $f(x) = 0$.
Root of a polynomial equation. The roots of a polynomial equation $f(x) = 0$ are the values of x satisfying this.
Degree of a polynomial. The value of n for f(x).
Duplicate root. A root of the equation $(x + a)^2 = 0$.
Antiduplicate root. A root of the equation $(x + a)(x - a) = 0$.
Independent roots. Occur when no known dependency relation is used in the solution of a polynomial equation.

Dependent roots. Occur when a known dependency relation is used in the solution of a polynomial equation.
Polynomial entity. A polynomial equation with dependent roots.
Multivariate polynomial. A polynomial in a number of variables.
Variety. A polynomial equation in a number of variables. Example: $3x^2y + xyz + 4x^2z^2 = 0$.

18. *Equivalence relation* ≡ *in a set* S. Satisfies a ≡ a (*reflexive*), if a ≡ b then b ≡ a (*symmetric*) and if a ≡ b and b ≡ c then a ≡ c (*transitive*), for a, b, c ∈ S.
Equivalence class. A partition of a set where an equivalence relation between elements defines membership of the partition.
Partial order ≤ *of a set* S. Satisfies a ≤ a, if a ≤ b and b ≤ a then a = b (*antisymmetric*) and if a ≤ b and b ≤ c then a ≤ c, for a, b, c ∈ S.
Total order ≤ *of a set* S is a partial order existing for all a, b, c ∈ S.
Well-ordering ≤ *of a set* S. A total order where every nonempty subset has a least element.

19. *Left* (*or right*) *coset* of a subgroup S of G is the set of elements aS (or respectively Sa), with s ∈ S and a ∈ G.
Quotient group G/S *of* G mod S. The family of left cosets of the group G with subgroup S, sG, s ∈ S.

20. *Ideal*, C. (SA III and XI). A subset of a ring, A, with the rule that {c, d} ∈ C and a ∈ A implies (c – d) ∈ C and both ac and ca ∈ C.
Principal ideal, (a). The ideal generated by one element, a, of the ring A. For every r ∈ A, (a) is ra. Example: for a ≠ 0 belonging to the integers $\mathbb{Z}$, (3a) ⊂ (a) ⊂ $\mathbb{Z}$.
Prime ideal, P. If a and b are two elements of A such that their product ab is an element of P, then a or b is in P, and P is not equal to the whole ring A. Example: integers containing all the multiples of a given prime number, together with zero. Example: the zero ideal (0).
Maximal ideal, M. In any ring A, this is an ideal M contained in just two ideals of A, M itself and the entire ring A. Every maximal ideal is prime. Nonexistence: the zero ideal (0) is not a maximal ideal of $\mathbb{Z}$ because (0) ⊂ (2) ⊂ $\mathbb{Z}$, nor is the ideal (6), since (6) ⊂ (2) ⊂ $\mathbb{Z}$.
Nilradical, N(A). The intersection of all prime ideals of a ring.
Jacobson radical, J(A). The intersection of all maximal ideals of a ring.

21. *Monomial* (SA XI) in a variety is a term without the coefficient. Example: x^4y^2z.
Monomial order of monomials in a variety. For example: $1 < x^4y^2z$ and if $x^4y^2z < x^4y^3z^2$, then $(x^4y^2z)(x^ay^bz^c) < (x^4y^3z^2)(x^ay^bz^c)$.
Lexicographic order $<_{\text{lex}}$ of monomials in a variety. The monomial order where the first powers g, h that differ satisfy g < h. Example: $x^4y^gz < x^4y^hz^2$, with g < h.
Degree lexicographic order $<_{\text{deglex}}$ of monomials in a variety. For example: $x^ay^bz^c <_{\text{deglex}} x^dy^ez^f$ if a + b + c < d + e + f, and if they are equal, then revert to lexicographic order.
Degree reverse lexicographic order $<_{\text{degrevlex}}$ of monomials in a variety is defined by $x^ay^bz^c <_{\text{degrevlex}} x^dy^ez^f$ if a + b + c < d + e + f, and if they are equal, then revert to right to left lexicographic order with the degrees interchanged. Example: $x^2y^2z^3 <_{\text{degrevlex}} x^4yz^2$ so the total degrees are equal, but then order with 3 and 2 in z interchanged.
Gröbner basis. A set of multivariate polynomial divisors of a multivariate polynomial with unique remainder.

22. *Open set*. Example: the interval $a < x < b$ with the end points a and b removed.
Closed set. Example: the interval $a \leq x \leq b$ with the end points a and b present.
Topology. A theory of space using open and closed sets.
Homotopy. A theory of paths through a topological space.

23. *Exact sequence*. (SA III).
Homology. A theory of holes. The dimension of the nth homology is the number of holes in a space for dimension n.

24. *Explanation*. A theory or theorem using matrices or other combinatorial means.
Charade. The image of an explanation in homological algebra.
Deconstruction. The mathematical refutation of a generally accepted result.