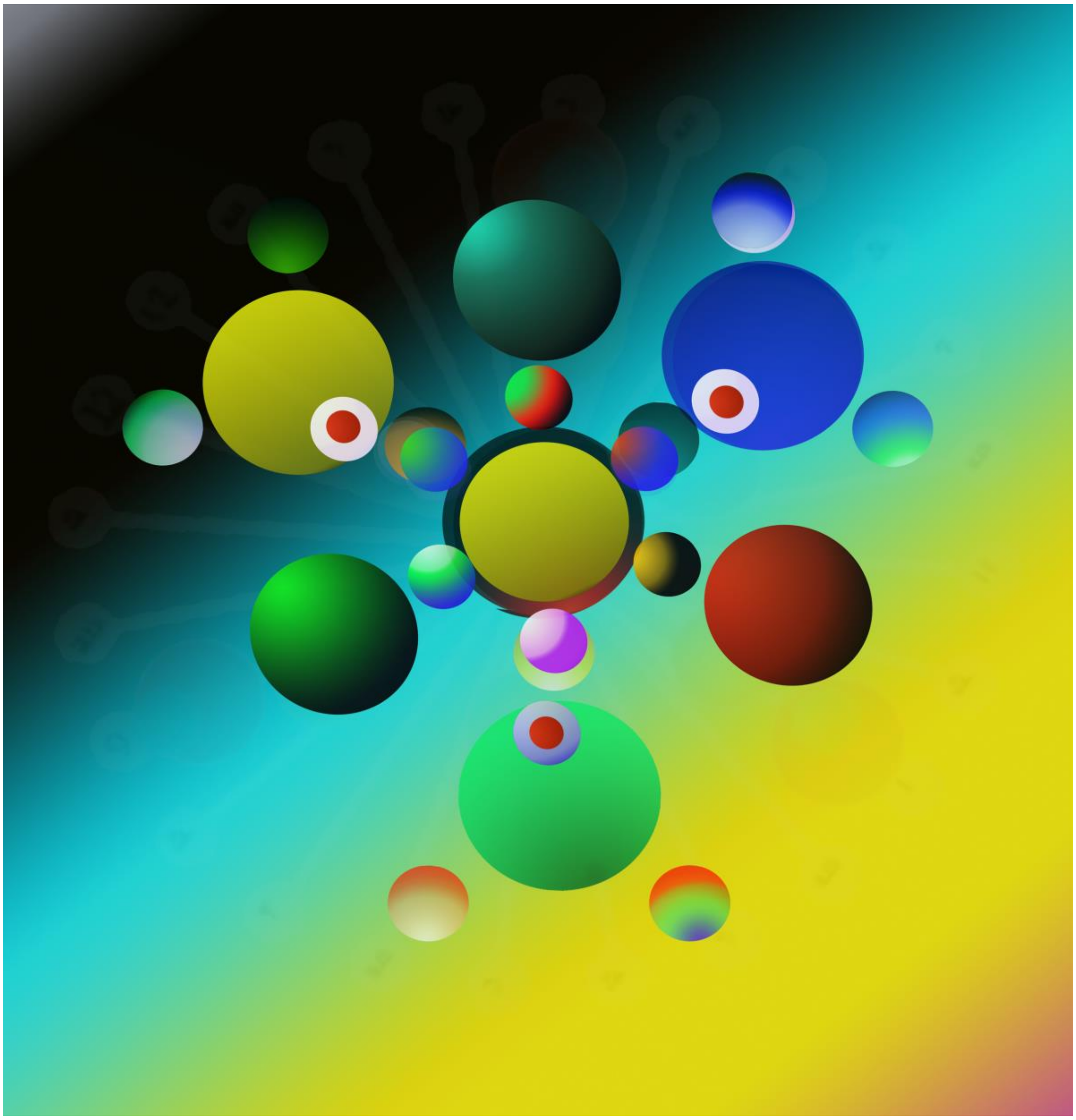
Number, space and logic

Volume III

The finite and the infinite



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The cover artwork is by Ian Dell.

Jim Hamilton Adams is a researcher in the concepts of mathematics and their explicit representation. This work is a guide to some of these ideas, and an encouragement to the graduate to develop mathematical research productively and continually.

After a career in IT, Jim joined New Music Brighton as a composer and performer of his own works. He has been extensively involved in mathematical research.

The eBooks published in 2014 were *The climate and energy emergencies* and *Innovation in mathematics*. Volumes I, II and III of *Superexponential algebra* were completed in 2017, and *Number, space and logic* is a mathematical sequel to that work.

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Foreword

Number, space and logic is a mathematical manifesto and encyclopaedia. It unifies the subject using generalised diagrams between multiobjects, which are objects in many parts, linked by multifunctions, which are transformations from something to the same type of something in many pieces, and defines and unifies three ideas, all in some ways differing from current understandings. The first concept classifies all possible universes with space and time. The second states that all consistent problems have an answer, and gives methods for solving all such problems. The third extends our new ideas of infinity, including uncountable infinity, to computable models.

We give three examples. The Irish mathematician William Rowan Hamilton discovered the quaternions representing space-time. This book extends the programme of Hamilton in novel ways which reveals that its geometrical and physical intuition, long thought to have been superseded, is still alive.

George Birch Jerrard, who studied at Trinity College Dublin 1821–1827, developed methods on the solvability of the quintic polynomial equation

$$x^5 + Qx^4 + Rx^3 + Sx^2 + Tx + U = 0$$

begun by Tschirnhaus in 1683 and Erland S. Bring in 1786, and showed that the coefficients of x^{n-1}, x^{n-2} and x^{n-3} of a polynomial of degree $n > 4$ can always be eliminated by transforming the variable x. In 1859 he wrote *An essay on the resolution of equations*. One version contains an end note by James Cockle (who became governor of New Zealand) stating that Jerrard's insistence that the quintic was solvable by radicals was incorrect. A radical is a number found from other numbers by addition, subtraction, multiplication and division, and taking nth roots. When n = 2 an nth root is a square root, for example $\sqrt{2}$, where $\sqrt{2}.\sqrt{2} = 2$. In support of Jerrard, it is well known by using the Cayley-Hamilton theorem and using companion matrices, where a matrix is an array of numbers in rows and columns with addition and a special form of multiplication, that matrix solutions of polynomial equations of arbitrary large degree n can be directly written down. Galois solvability theory, which for over 150 years has eclipsed these investigations, predicts that the solution of the sextic, which is a polynomial equation of degree n = 6, containing duplicate roots reduces to the standard Galois case of unsolvability, the theory of inseparable extensions, but it can be directly shown that this sextic can be solved by radicals. Linear transformations of variables, where x goes to x + b, which are always used to solve polynomial equations of degree $n \leq 4$ cannot be accommodated in Galois theory. Galois theory reduces the study of solvability to multiplicative group theory based on permutations of its roots, where a root is a solution of a polynomial equation, but an extension of this theory to ring automorphisms, transformations of something to itself using + and ×, predicts that the swap of two roots will not in general leave a third root intact, so a correct solvability theory cannot be based on permutations. Nevertheless, a dependency theory, based on the special case of 'killing central terms' to obtain solutions, which says we use transformations to zeroise coefficients in the middle of a polynomial, shows that no general polynomial of degree > 4 is solvable by radicals by these means. But this does not imply that other methods, say involving comparison of polynomials of different types, where we equate the coefficients of a polynomial written in one way with a polynomial written a different way, cannot yield solutions. We will show both theoretically and by direct computation, that complex solutions in radicals of complex polynomial equations of arbitrary large degree exist.

The Riemann hypothesis, concerning the zero values of a function called the zeta function, which predicts results on the distribution of primes, is proved by two methods in this book.

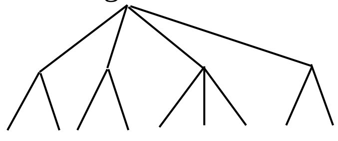
This important mathematical problem has resisted solution since it was first proposed in 1859. The earliest technique uses what we call $\mathbb{D}w$ exponential algebras. These were developed by the author, and the idea was extended by the physicist David Bohm, in conversations with me via an intermediary, Ebrahim Baravi, in the early 1980's. David thought that this had applications to the Riemann hypothesis, but at the time I did not. On reopening this research over 30 years later, I rediscovered this idea. It is correct. The second proof involves extending the idea of infinity, which was mentioned to me by a researcher on this problem, whose name I do not know, and he provided no other details. The techniques we have developed here quite independently use the idea of infinities called capital Zeta functions, $\Xi(a)$, which have the property that $1^{\Xi(a)} = a$ and are representable by standard techniques using complex numbers. These are used to incorporate the transcendental real number system within a computable model. We then apply this model to the Weil conjectures for finite fields in algebraic curves, which were developed by analogy with the Riemann hypothesis, fully proved after work by A. Grothendieck and co-workers by P. Deligne in a paper of 1974.

Number, space and logic combines a practical graduate level textbook with a research project in a commentary and development of the mathematics of the late 20th and the 21st centuries, just as *Superexponential algebra* [Ad15] does for the 20th century before that covered in this work. As a guide to this, we include the mathematics needed to prove the general Riemann hypothesis and from it the weak Goldbach conjecture, that any odd number greater than 5 is the sum of three primes. There is an exposition of the theory, to be described later in this foreword, of zargonions, polynomial wheels and branched spaces, and as detailed in volume III, a proof on the consistency of analysis, which is the study of functions which are continuous, by extending the work of Gentzen.

To present our work coherently we had intended to proceed from the abstract to the specific, and so this could be understood to give first an account of the meanings of these abstractions. Thinking these ideas could give an overarching description of mathematics, it was only in the writing of this work that I realised to my astonishment that a detailed description was available not only in theory, but it was being built in practice and could be presented in a unified way.

Having retained the majority of the work as originally conceived, after a summary of this unified mathematics, we describe as examples our approach to the finite and the infinite, then how our generalisations of space fit into the theory of trees and amalgams. Superexponentiation develops mathematical operations beyond addition, multiplication and exponentiation. We show how superexponential polynomials that we call *sunomials* fit into an abstract description of mathematics where the parentheses around expressions matters, nonassociative categories called *superstructures*.

A notion we will begin with is that of a *tree*, ordered from top to bottom in the example diagram



Each parent node at the top branches to various child nodes beneath it. We give names to the collection of a parent and its child nodes in various contexts, for example we call this collection a multiobject.

We can identify some of the nodes of the tree, so they become the same node. We call this an *amalgam*.

Suppose we have one amalgam. To transform this amalgam to another we link nodes in the first amalgam to nodes in the second. These linking transformations are described by amalgams themselves. Nodes of the two amalgams are identified with nodes of the linking transformation.

We investigate a more general situation. Instead of two amalgams linked by a transformation which is an amalgam, we allocate different *colours* to a number of amalgams. These coloured amalgams can be treated as a multiobject. We consider other coloured amalgams, not in the same colour set, as linking transformations. This is the most general mathematical object we encounter, a *xiqu*, from the Chinese 'opera', for an English speaker pronounced like 'shichu'.

All mathematics is an instance of xiqus. From one point of view, xiqus express relationships in geometry and between numbers. From another, mathematics has gone beyond these ideas. This is the overarching notion we introduce in *Number*, *space and logic*. It has originated from the construction of many examples, which at first did not use them. We investigate, extend and sometimes revolutionise mathematical thinking.

We see in history that two developments go hand in hand, the development of notions and the development of notational systems – ideas and the representation of ideas. If we restrict the way we represent things, or make this representation difficult or complicated, then we restrict how we think. An objective is to advance our notations by brazen analogies taken from history.

Pictorial ideographs were replaced in early writing by phonetic cuneiform. Although words were not separated by spaces, this drastic simplification of writing raised mass communication and allowed new extensions to languages. In China, the character representation of writing allows communication between the languages of different parts of China. The simplification of the script introduced by the communists advanced mass communication and literacy, and makes easier the understanding of Chinese culture and civilisation internationally.

In mathematics, reasoning with examples using numbers was replaced by using geometry instead as the Greeks did, where universal proofs derived from axioms describe precisely this geometry. Today algebra replaces geometry. This advanced in stages. The introduction of zero from India allowed the completion of the number system, so that the addition of two numbers could be zero. Negative numbers allow the use of complex numbers through multiplicative completion, with a number i with $i \times i = -1$, which has no analogue in the numbers hitherto considered by any human culture. This innovation took a long time to develop. Of comparable importance to the development of algebra was the book *Kitab al-jabr wa'l muqabala* [K1831], by the ninth-century Uzbek scientist in Baghdad, Mohammed ibn Mûsâ al-Khowârizmi, which is interesting socially and seems appalling to the modern reader in its easy reference to calculating the cost of concubines. Because he does not accept negative numbers, he introduces the idea of an equation. To maintain equality between two sides of an equation, so that an intermediate step does not go negative, a positive number is added to both sides. He employs a similar idea on multiplying both sides of an equation by a number. Although al-Khowârizmi introduces the name algebra, what he does is not algebra in the modern sense. Algebra, where numbers are represented by letters, is an extraordinarily late development in mathematics. Archimedes, in the cattle problem, knows how to manipulate simultaneous linear equations, but he does this using words, and his variables are white cows, black cows, yellow cows, dappled cows, white bulls, black bulls, yellow bulls and dappled bulls. Descartes replaced geometry by algebra, in that geometrical figures were replaced by coordinates. It would only take a few more lines to discuss the change from Roman numerals to the Arabic decimal system, and the introduction of differential and integral calculus. These developments of huge significance covering thousands of years we have been able to relate in a few lines.

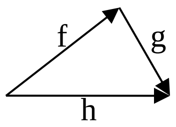
Is it possible to push mathematics further by introducing notational revolutions of comparable significance? A motto from one of my schools, translated from Latin is ‘great trees from little acorns grow’. I am a great fan of the idea ‘start small from what you know and can do, and grow big’. So to try this out, what do I know?

From my experience of music and interest in Schönberg, I know that I find his piano music not only difficult, but impossible to play. The reason is a large number of ‘accidentals’, which in Western music are sharps and flats modifying notes, as happens for a lot of serious music nowadays. I have a defect in the way I am able to think. I cannot process a forest of accidentals in music and play it in real time, although I can memorise a piece – difficult if the notes go all over the place. I discovered there is a simplified musical notation that gets rid of sharps and flats, incorporated in different shaped notes. It takes 5 minutes for a professional to learn and sing music from it, and it is easy to pick up. I think if I were given Schönberg to play in simplified notation I could do it. Why is this notation not used? Tradition! The music industry and academic teaching restrict the number of performers and performances. But it is desirable that music is not based on scarcity of these but a plentiful supply.

In conclusion, if you simplify the symbol set then you speed up and increase comprehension.

I want to describe three notational innovations I have introduced. They use simple symbols.

Functions of a variable act together to form composite functions. A function f from, say, a set A to set B may be composed with a function g from the set B to set C. An element a of A linked by an arrow in f to the element b in B then continues as an arrow taken from b in B to element c in C. These arrows, in an analogy which is not inappropriate, are then composed to give the composite arrow h, shown diagrammatically below.



The notation for these composite arrows does not normally read in the way one would expect from the above diagram, reading from left to right. Historically a function f acting on a variable x was denoted, and is still denoted today, by f(x), with x on the right. This means that when we compose functions they are written, unlike in the English language, from right to left. Two functions f and g composed together as above are written g(f(x)), or more usually since we often deal with associative objects where the brackets do not matter, gf(x). The information in the diagram above, admittedly confusingly, is usually written as

$$h = gf \text{ or } h = g\circ f.$$

This introduces cognitive difficulties in people like me who cannot process three successive functions in reverse order. To ask me to understand books on category theory written in this way is like asking a blind person ‘do you see?’, so we adopt left to right notation using an underscore

$$h = f\_g.$$

I am happy with h(x) = f_g(x).

Mathematics is usually confined to three operations and their inverses, addition, multiplication and exponentiation. Multiplication is generated from repeated addition

$$a + a + \ldots + a \text{ (m times)} = a \times m.$$

The notation for exponentiation, a good one, was introduced by Euler

$$a \times a \times \ldots \times a \text{ (m times)} = a^m.$$

We will also introduce

$$a \times a \times \ldots \times a \text{ (m times)} = a \uparrow m.$$

It is now easy to see that where you put the parentheses around exponentiation matters. It is nonassociative

$$a \uparrow (b \uparrow c) \neq (a \uparrow b) \uparrow c,$$

as well, of course, being noncommutative

$$a \uparrow b \neq b \uparrow a.$$

I asked at the age of nine what comes after exponentiation. It is called tetration. By and large mathematics does not use it, or higher order operations. It should.

We will extend the operations +, which we will write as ${}^{1}|$ or as a word “onesu” and speak as “onesoo”, × written as ${}^{2}|$ or “twosu”, exponentiation ↑ as in $a \uparrow b$ more usually written as a^b, and written with ${}^{3}|$, and a general nth suoperator ${}^{n}|$.

Usual notation	Suoperator notation
$a + b$	$a\,{}^{1}|b$
$ab = a \times b$	$a\,{}^{2}|b$
$a^b = a \uparrow b$	$a\,{}^{3}|b$
area of sphere = $4\pi r^2$	area of sphere = $4\,{}^{2}|\pi\,{}^{2}|(r\,{}^{3}|2)$

We note the following points. An nth suoperator generates an (n + 1)th suoperator by induction. Then

$$a + a + a \ldots + a \text{ (m terms)} = am$$

$$(\ldots((a \uparrow a) \uparrow a) \ldots) \text{ (m terms)} = a\,{}^{4}|m,$$

where all the brackets are collected together from the left, or as we say, are nested on the left, so that, for instance for +, given by ${}^{1}|$

$$(\ldots((a\,{}^{1}|a)\,{}^{1}|a) \ldots) \text{ (m terms)} = a\,{}^{2}|m,$$

a general case being

$$(\ldots((a\,{}^{n}|a)\,{}^{n}|a) \ldots) \text{ (m terms)} = a\,{}^{n+1}|m.$$

So we have introduced ${}^{n}|$ to indicate nesting on the left, for example

$$(((a\,{}^{n}|\,b)\,{}^{n}|\,c) \ldots {}^{n}|\,d) = a\,{}^{n}|\,b\,{}^{n}|\,c \ldots {}^{n}|\,d.$$

For nesting on the right, we introduce a completely analogous notation, including suone, etc.

$$(a\,|^{n} \ldots (b\,|^{n}\,(c\,|^{n}\,d))) = a\,|^{n} \ldots b\,|^{n}\,c\,|^{n}\,d.$$

What trends are discernible in mathematical notation today? Algebra has been replaced by diagrams. Transformations from algebraic objects to algebraic objects are represented by arrows. They can be commutative diagrams shown below.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

A functor is an arrow of an arrow. The commutative diagram for it is better represented as

$$\begin{tikzcd} A \arrow[rr] & {} \arrow[dd] & B \\ & & \\ A' \arrow[rr] & {} & B' \end{tikzcd}$$

If this is the embryonic form of a change of representation from algebra to diagrams, like the change from geometry to algebra, how should this mathematics develop into infancy? I think one of the problems with commutative diagrams is that they can get multidimensional. Here is a functor of a functor of a functor (a functor cubed)

$$\begin{tikzcd}
A \arrow[r] & B & C \arrow[r] & D \\
A' \arrow[r] & B' & C' \arrow[r] & D' \\
E \arrow[r] & F & G \arrow[r] & H \\
E' \arrow[r] & F' & G' \arrow[r] & H'
\end{tikzcd}$$

A bit bulky, but it gets the idea. Do you really want to represent functors in the traditional way for this level of complexity?

We have holograms, though they are not often available yet, which can represent objects three dimensionally. Irrespective of that, we need a notation that deals with diagrams in dimensions greater than three. Maybe we have just given it. I think notational ways need to be developed to label these components algebraically, and the arrows between them.

In the chapter on *the meaning of the finite and the infinite*, after giving new rules for set theory we give an overview of standard and significant results in finite and discrete number theory for nonstandard extensions. Firstly we discuss extended sets of positive whole numbers, which we call transnatural numbers. We give a description of capital Ξ functions, which can be used to give a model of the real numbers. This idea is employed in Volume III to give a proof of the general Riemann hypothesis from a local field point of view. Transfinite number theory is considered for the transnatural numbers $\mathbb{M}_t$, where the theory is an extension of finite arithmetic. Unlike our results in nonstandard set theory, these numbers are not two-way, or bijective, to the natural numbers, instead they have a transfinite number of members, but they share the other properties of natural numbers. They contain transfinite prime numbers and can be used to define transfinite rational numbers. We also deduce the inconsistency of the uncountable continuum, if we describe real numbers derived uniquely only from countably infinite sequences, where elements become arbitrarily close to one another as the sequence progresses. We give the standard axioms for a field, which defines the rules for addition and multiplication of numbers, and their inverses. Division by zero is not consistent for a field, or rather all numbers in it then become the same. We introduce zero algebras, which are not fields and contain multizeros which do have division. A detailed working out of the properties of algebraic systems using zero algebras is developed in volume II. We describe ladder algebra which reformulates and extends nonstandard analysis. Then we extend the ladder algebra of infinite ordinal number arithmetic to $\mathbb{M}_t$ and its analogue for real numbers. We consider the irreducible ordinal infinities of ladder algebra as ordered colour sets.

The chapter on *the meaning of branched spaces* connects ideas about vector spaces, their generalisation to modules and what we call boxes, which are n-dimensional arrays like tensors, to sets and logic. It considers dependent and independent probability as a logic derived from set theory. We look at partially ordered sets, lattices, which are logics with an order structure, and topology. An extension of the idea of a lattice, which for two objects a and b has one join a ∨ b and one meet a ∧ b, is to consider multiple meets and joins. The algebraic properties of these multilattices is discussed in the section on Dedekind-MacNeille completion, where new nodes are introduced into a multilattice to complete it back to a lattice. This can be related to notions of closed and open sets, where we use the ideas of respectively finished and unfinished sets. A set may be simultaneously closed and open. We prefer to discuss sets which are finished or unfinished but not both. We discuss topology, which is a generalisation of geometry, and relate our discussion on lattices to Hausdorff spaces used in topology. As an example of the relation of sets to logic, we describe twisted and untwisted logic. We introduce our extension of the Euler-Poincaré characteristic used in topology (for a graph this is the number of vertices, minus the number of edges plus the number of faces), related to étale cohomology theory introduced by Grothedieck to prove the Weil conjectures. In its familiar form this appears restricted, although other formulations are equivalent. In its extended form it describes types of topology which have no analogue in standard topology. We introduce branched spaces, putting it this way: when a point is removed from an n-branched line, it divides into n pieces. An *explosion* is a set that contains an infinite number of branches. These are the topological analogues of behaviour we meet in exponential and superexponential number systems. The Euler-Poincaré characteristic is an instance of a superexponential polynomial, a sunomial.

Concerning *the meaning of superstructures*, a big topic in the last chapter of the eBook *Superexponential algebra* [Ad15], this idea extends the repetitive process which generates multiplication from addition, and exponentiation from multiplication. We generate an (n + 1)th suoperator, short for superexponential operator, by repeating an nth suoperator. We give a representation of matrices called the hyperintricate representation which is used in examples. Features of this representation describe nonassociative, or parenthesis dependent, mathematics. In [Ad15] we discussed nonstandard exponential and suoperator algebras that are single valued, or unbranched, unlike standard such algebras. We describe general $\mathbb{D}$w suoperator algebras for all types. A new idea is that as well as describing suoperators starting from n = 1 for addition of two objects, n = 2 for multiplication, and so on, we can introduce suoperators for negative n. These are examples of crude suoperators which do not reduce to suoperators acting on binary objects. We give a description of a standard, or canonical, way of representing sunomials, the equivalent for suoperators of polynomials which are expressed using + and ×. A real number has a *sunorm* which is similarly real defining its size under exponentiation or a suoperator. So a suoperator on numbers has two items interesting to investigate: sunorms and branching. We introduce for sunomials the ideas of sudifferentiation and suintegration. These are suoperator analogues of differentiation and integration. Then we describe the meaning of category theory as objects and transformations, and our extension to nonassociative transformations, which we call superstructure theory. These transformations, called morphisms in category theory, can be depicted by arrows. This theory includes a description of functors, which are arrows acting on arrows. Duality occurs in category theory when we reverse the direction of all arrows. Category theory introduces in a general and abstract way a description of an example that is so typical it represents all cases. This is called a universal. We come back to lattices and graphs which we introduced in the chapter on branched spaces, to look at them in a categorical way. We develop notions often used in category theory, hom-sets, adjoints, initial objects, terminal objects, pullbacks, equalisers and limits, their dual notions, and give concrete examples. The concept of a comma category and the case for superstructures is given. A categorical description of a set is called a topos. We introduce sutoposes, which are their superstructural analogues. Finally we describe Kan extensions, which are useful for describing sudifferentiation and suintegration in superstructures.

Having built up the meaning of these abstractions, from their axioms we provide a detailed exposition of these ideas in reverse order: of superstructures, trees and amalgams, and the finite and the infinite, and look at some of their consequences.

Two leading ideas which are further components in this supermathematics are those of the nonassociative zargonions and the deconstruction of the current theory of polynomial equations and its replacement by polynomial wheel methods, so called because I ask whether their implications are as important as the invention of the wheel, giving access to computable structures for the whole of mathematics. Our approach cannot accommodate the standard one. Nonabelian group theory with $ab \neq ba$ does not model the solution sets of abelian varieties as is supposed to happen in Galois theory, although there is a mapping from noncommutative structures to commutative ones. Our methods reveal a link between the solvability of abelian varieties, elliptic curves and modular forms. The problems we solve can be generalised to all of mathematics. Mathematics hitherto has stated that there exist undecidable problems. The proposition that there are consistent undecidable problems is false; we have proved that all consistent problems are decidable. When computable structures combine with zargonion superstructures, mathematics describes results in number, space and logic in full and coherent generality.

Volume III in more detail

In this volume we look more deeply into an exposition of our nonstandard outlook on the continuum hypothesis, and an extension of the idea of natural and rational numbers to the transfinite case, called respectively transnatural and transrational numbers. In this context we discuss class field theory and provide a Gentzen-type proof of the consistency of analysis.

We prove Fermat's last theorem by elementary methods, but also using standard techniques of the modularity theorem derived in volume I.

We then prove the general Riemann hypothesis by two distinct methods. The first uses an extension of exponential algebra to the $\mathbb{D}$w exponential algebras, developed by myself and David Bohm, the latter in an attempt to prove the Riemann hypothesis. In [Ad15] these are given in the case of w an integer. In this volume we introduce imaginary $\mathbb{D}$w exponential algebras. These reduce to two types, the classical exponential algebra and the nonclassical. Both types give the same result for zeta functions, and this gives sufficient information to prove the theorem. This is the most direct method. The second method uses an extension of the case of local fields, where we prove the general Riemann hypothesis conjecture, with related ideas described by theorems developed by Weil, Grothendieck and Deligne. For the viewpoint developed here, we need not only these theorems extended to the transnatural and transalgebraic case, but also a discussion of those transcendental numbers independent of these two previous types. Thus we have three types of counting to do: for transnatural numbers, for transalgebraic numbers which are not transnatural, and then for transcendental numbers which are not of the other two types. For this we need a theory of transcendental independence extended to complex numbers, and a method of counting solutions for 3-branched spaces, in fact for 3-explosions. The latter is provided by an extension of the infinite superexponential methods of Gentzen.

Having proved the generalised Riemann conjecture by these means, we are able to give a proof of the weak Goldbach conjecture, both by the direct method of Harald Helfgott proved in 2013, and using the our result on the Riemann hypothesis combined with the work of J-M. Deshouilles, G.W. Effinger, H.I.J. te Riele and D. Zinoviev in 1997.

On inviting the reader to mathematics

What is mathematics? Possible worlds! These worlds have strict definitions. They have states which describe what is in these possible worlds, and specified transformations of these states, which describe how these possible worlds may change.

It is my philosophy – a leap from some thinking about physics – that what is possible and consistent in mathematics matches what exists. Since mathematics is used to describe physics, consistent mathematics can describe the real world. If a symbol represents 1 as 0, in some systems this is inconsistent. Consistency and the stability of solutions of equations represented by symbols are related. In this way of thinking, states that evolve stably are what now exists in the physical universe, and other states have physically self-annihilated in an inconsistency. Now I think of true, false, 1 and 0 as aspects of many valued logics with consistent algebras. These algebras describe consciousness.

It is then an interesting programme to document what mathematical worlds are possible. Some of these worlds appear to be hallucinations – they have no implementation as far as we can see in the physical world. Others may have implementations which may only be used by future generations on the discovery of new physics, or otherwise have seemingly obvious practicality but may be generalised in ways which have no apparent immediate physical significance.

The objective of this work is to introduce you the reader to these possible worlds and to ask you to become creatively involved in their construction and the analysis of their behaviour, so that giving new definitions and working out their consequences comes naturally to you. This programme is both abstract and potentially useful, to manipulate the physical world in ways which may be at the frontiers of human knowledge and experience, or may be everyday applications of direct and practical use.

Mathematics develops outside the social group by the analysis of abstract systems whose meaning is derived from features of the world, and also within this group, as culture, history, revolution, extension and rewriting of its basis. In order to connect with its modern modes of reasoning, we need also to be aware of the human features which have led to the development of mathematics.

The objective of creative mathematics is the production of ideas. These need caring support, and part of this objective is not just to replicate results, but to reconstruct mathematics from new principles, to have the resilience to deconstruct what has gone before if it is needed, and to extend results by an analysis of the core ideas of the subject as it currently stands. Thus it becomes feasible to develop generalised new methods to tackle outstanding problems.

We need also to be aware that as a human endeavour we are not at a unique point in history where all aspects have been decided with certainty, and that human systems have triumphs and failings. We must have the courage of our convictions, and the sure analysis of our own and others thinking to develop this civilisation, and where necessary to have the resilience to upturn what has gone before, because even in mathematics we are not in utopia, and some things, promoted in the system from one generation to another, are wrong.

Mathematics, once constructed, is about the truth. It may be that this truth has no respect for high-ranking authority. I feel it is good to be aware of this possibility when we conduct our research.

Processes of mathematical thinking

An objective in writing this work has been to make progress and to explain to the reader. To make progress in mathematics I have relied on what is to hand, my own understanding or misunderstanding, and my intuition.

My approach to understanding the subject is reliant on a feature of my personality, that I am not good at memorising a large number of facts, and therefore I must proceed by isolating a clear, distinct and small number of basic principles, and by applying rules of deduction to them which are recorded on some external medium, I can then reach finally a full set of results.

To isolate these principles, I often find they cannot be derived from studying only the latest mathematics texts, but to make sense of these basic building blocks requires backtracking through the mathematical literature, until the principles on which a sector of mathematics is based is simple and its future form of development is clear. This history is often a long one, because each new mathematical generation feels compelled to revolutionise the ideas of its predecessors, and indeed eventually we find in Western culture the language of discourse is no longer English, but mainly German and also French, with many other languages included, and before this the language of scientific communication is Latin, or if we want to go further back than this, it is Greek. It is my assertion that to follow this history is extremely important in isolating key concepts and the processes inherent in current thinking. This history is not, as is often presented by historians with standard mathematical culture, a history of personalities. In essence, it is the history of the development of ideas.

Once this subject is understood as a way of thinking, it is necessary to analyse it. Basic to the way I respond to this culture is the *insight* that I need. Perhaps some mathematicians have no intuition, just as some people are blind or deaf. Whatever the case, intuition is available to me, and I wish to apply it. Intuition may be thought of as an approach which grasps the whole of a subject at once, in an immediate understanding of all of its parts. It is my contention that in order to connect my intuition, which is what I feel about a result, to mathematical culture, I have to bring to the surface of my consciousness what is immediately grasped subconsciously, in order to analyse the contents of what I am thinking so that it is externally and explicitly expressed. Once this intuition has been transferred to written form, it is in a form in which it can be analysed. It is then possible to subject it to the mathematical culture which has arisen in describing logic, by devising methods of calculation in which ideas can be tested out. This initial stage is usually one in which the total system is not immediately expressed in an axiom system and regularised rules of deduction, but it allows further development in which these features are adopted.

The result of this process is that intuitive ideas are collected together and are compared with current mathematical culture. By these methods I have seen an explosion of new ideas. This intensively creative process gives rise to significant mathematics. Riemann and Grothendieck advanced mathematics by 50 years, Euler by 100 and Archimedes by 200. This work pushes forward mathematics as if Archimedes had developed algebra, differentiation and integration and then intended to apply cohomology to the classification of topological groups. But had he been able to do so, to be understood he would have had to communicate in the notational encumbrances of the age in which he lived.

We need to continue the process and explain this thinking, for two reasons. The first reason is that creative ideas are eventually subject to systemisation, so that they can be analysed using standard techniques adopted by the mathematical community. The second is so that jumps in reasoning which the author thinks are justified are often not obvious enough to a reader new to this thinking. Thus I must be forced to describe all stages in my thinking. Proper mathematical argument is about explicit reasoning, so that all processes are described and a clear argument can be followed without dispute.

The existence of disputes in mathematics is interesting. Thus, having introduced disputes in the first place by possibly controversial intuitive reasoning, an objective becomes to minimise or remove them by the introduction of explicit rules of deduction and symbolic manipulations together with a set of concrete meanings of these symbols which point to the world. This is the method I have chosen to develop new symbolic generalisations and create new mathematics.

The most significant directions of my life have been determined by the labour I wanted to do. Had it been that I were sent to a pig farm in the cultural revolution in China, on adjustment to the collapse of my excessive ambition, I would probably have become an exemplary worker exceeding all quotas, without a thought to my personal promotion. But that would have been a waste. I have received an excellent education that I did not request, and did not find myself succeeding particularly well within it, but having adjusted to the culture of science, I became fascinated by its ideas and content. The wonderful mathematician Ramanujan did not succeed fully at university because he was too interested in his investigations of the subject to follow properly the course work, and he was not interested in social promotion through the system. In the pursuit of my obsessive interests in science, in my life this has befallen me too.

Aesop's fable of the tortoise and the hare ends with the tortoise winning the race. In research a certain level of intelligence is useful, but what is important is persistent curiosity. I am the tortoise. In my research I have exceeded in duration, but despite my wishes, not in collective effort, many Long Marches. The result of this work is no longer mine, it belongs to human culture, but social promotion through the system as a consequence of this is of no interest to me, except where it might promote general social welfare.

At the start of my life my attitude was to ignore the rules and do the right thing. But now I have come to the conclusion that it is very useful to know what the rules are. They can be used to define acceptable and unacceptable behaviour, and to navigate through the social system to attain objectives.

I started my scientific career with an innate conservatism towards scientific conventional wisdom and an aspiration to reach the summit of its understandings. In this respect I treated science as a religion. But at university I began to have doubts as to the correctness of some theories, although I did not have the means or resources to question them properly.

In order to succeed well in the academic system, it is necessary to give the right answer to questions. Many mathematicians now teaching the subject freely admit that they crammed for examinations, and did not properly understand results, although they were concerned to memorise and replicate them. I believe some mathematicians writing on the subject today do so from the point of view that a proof has to be memorised, and if it is viewed abstractly it is not important to understand properly what is going on, provided the conclusion is reached rigorously.

In pursuit of these interests, time and again in mathematics I am confronted with the fact that I do not understand a proof. When I inspect my own proofs and come to the same conclusion, the answer is easy. However much I am attached to an idea behind a proof, which sometimes comes from experience of the material concerning it, if on reviewing what I am trying to say, the result is no nearer after all the effort, the theorem has to be jettisoned.

There are standard proofs in mathematics which I have not understood after considerable effort at coming to grips with what they are saying. These proofs are sometimes technically erudite and sometimes extremely long. They can be surprising or counterintuitive. I do not believe the majority of mathematicians spend time on inspecting them, but rely on the peer review process to check their validity. Doubts are allayed if the result is well-established and a considerable confirmatory literature surrounds it.

I have learnt that if, after considerable investigations, such a result is no clearer, that initial doubts are confirmed, and that the theorem is counterintuitive, a not unreasonable strategy is to assume that this does not arise from my own innate stupidity, but that the result is wrong. This at least allows an entry into other approaches to investigate what really might be the situation, since there is a logical mode of deduction which states that to prove a result, first assume the opposite and then prove a contradiction. Very often when I try this approach, the result is not a contradiction but an escape route, and if after much research an escape route cannot be closed off, persistent investigation has led to a refutation of current findings.

To begin with, this astonished me, and an implication going beyond the proclaimed rigorous methods of mathematics to subjects of weaker intent is that much human reasoning of the current day is suspect. But now in new work I have taken this conclusion not as an end view in my investigations, but its starting point.

As a word of encouragement, to make progress in mathematics it is best to assume that the truth can be arrived at by a process of successive corrections of theory. If I look at the work of Aristotle, whose deep thinking and wisdom ranged over history, logic, physics, biology and philosophy in a way that seems impossible because of intense specialisation today, we see someone who is concerned to find the truth when others of his generation were not. There is an erudite and difficult passage in his work where he deals with the motion of the moon. It is apparent 2200 years later that, not to put too fine a point on it, it is a load of codswallop. I feel that some intimidated students might have been examined on it, and heaven help them if they gave unapproved answers. It has been said that although Aristotle had many false ideas, that the history of much of scientific thought has consisted in correcting them. But it is not possible to make much progress when no prior thinking has been made of a topic. That progress has been made in mathematics has been contingent on the fact that those studying the subject had some wrong ideas but did not stay silent.

I wish to end this section with a remark on learning and unlearning. In general, I have found no problems in presenting my ideas to undergraduates in their twenties who are studying for degrees in a science subject, particularly mathematics and physics. These ideas are absorbed and accepted without trauma. The reason that revolution has its meaning is that Copernicus's book *On the revolutions of the heavenly spheres* could not be cognitively accepted by those who had been trained in a different outlook and had accepted it in their earlier years. I have found that those who have been taught the subject of mathematics in their youth, have reached maturity and maybe have been teaching this too, and have accepted this teaching cannot accommodate to the double burden of learning this new material which is at variance with what they have accepted, and at the same time unlearning what has been embedded in their way of thinking for very many years. A response from some younger people already trained is to find at every opportunity ways of undermining the ideas, always framed from the point of view of total acceptance of what they have been taught, and for such people who are older, of refusing to engage in any way with these ideas, and even the refusal to read anything about them. The task I face is a difficult one. I do not know enough about psychology to understand the proper techniques which people can use to unlearn a subject. I know that when learning a piece of music, if you introduce an error into the performance of it, it becomes necessary to unlearn the error, and this is difficult, because the error becomes embedded in the way the mind operates in producing automatic responses. Have I then been able to escape from this situation myself? Perhaps I have an immediate readiness, frequently applied, of being able to change my conclusions in the face of evidence, together with the determination to find the facts and long term to maintain my stance or suspend my judgement when I believe the evidence is unproven. The only other way I can think that the mind uses to unlearn responses is to cry, but this is a speculation. I think this situation is about natural behaviour and needs to be viewed with compassion and understanding. In the long term, the logic of the viewpoint I am presenting will win out. That it may take a long time is unfortunate, but given the human condition this is understandable.

The author's responsibility

This work is provided in the Mathematics section of the website www.jimhadams.org, where a brief synopsis of each chapter is provided. It is the sequel to *Superexponential algebra* [Ad15], displayed after *Number, space and logic* in the Mathematics section. Nevertheless, I have made *Number, space and logic* depend less on [Ad15] by taking some material from it.

A central objective of this work is to encourage the graduate to produce mathematical ideas, and also to give assistance by providing a minimally fussy exposition of the research into number theory of the last fifty years. The creation of new mathematics can start from humble beginnings. Its process is too long delayed in the present academic system. We are seeking to foster insight so that the reader can pursue further investigations into the technical literature in a spirit of an understanding of its background. The reader will face a vast panorama of unified but simple new mathematics in this book. Its invention and discovery can be your work too.

Exercises are not provided to work through the text, the reason being that these are no more needed for the development of creative skills than the memorisation of words and standard literary works is necessary for an active participation in becoming an author. My hope is that the diligent reader will provide what is necessary for constructive work, and that it is no longer necessary to give direction.

Since I am aware that the first language of the reader may not be English, I have looked at the text and removed high literary style. Technical terms are I hope well explained, and examples are given.

I am surrounded by people who definitely dislike and do not relate to abstraction, particularly mathematical abstraction. Some writers in our subject use abstract methods of reasoning to obscure simple ideas and clothe them in a system of thought which is remote. But abstraction can also reveal, and even attain ends which cannot be achieved by other means. Mathematics is best written in a style appropriate to its audience. For a graduate text I am confronted with the need and indulgent necessity, if given enough time, of writing in a rigorous mathematical style that will appeal to a mature mathematical audience. I have attempted, after much effort in attaining results, to rewrite this text in a clear classical style that reveals the subject as well as developing it.

Except for marketing, our inclination is that a proof is only the final stage in the presentation of a mathematical idea. Proofs are necessary, but not sufficient.

I would advise that the academic system can refer without proof to topics containing accepted well-known but false results, or demand the application of wrong reasoning to reach standard and erroneous conclusions. Errors may propagate through the examination system from one generation to another by selection of people who accommodate to them, with the authority structure too rigid to allow change. Much stress can be avoided on acknowledging this fact.

Reflecting on my practical experience of the idea that major new understandings are never encountered in history without considerable resistance, I wish to issue a caution. Some such ideas have eventually been adopted by systems of authority not only because they represent the truth, but because their originator said so. I wish to repudiate such a notion. The authority is the truth. Period. Should it occur in the future that these ideas become the conventional wisdom, and if any one of them were in error, be aware that had I been there, my primary mathematical objective would be to deconstruct it.

Faced with my own condition, I love to promote an attitude: getting it wrong is the first stage in getting it right.

I thank Graham Ennis, who has sustained and inspired me with his encouragement. I have found John Baez's website beautifully presented and a treasure-trove of links to interesting and accessible accounts of the research literature. Very useful to my mathematical development have been the books *A first course in modular forms* by Fred Diamond and Jerry Shurman [DS05], and *Sphere packings, lattices and groups* by John H. Conway and Neil Sloane [CN98]. An essential first task for a writer on the Riemann hypothesis is to investigate the work of Terry Tao and explore the insights he reveals. Research needs ideas, and also the people to pull them back to Earth. As well as the usual fulsome acknowledgement to Doly García in her criticism of this work, who cannot disagree with everything I say because sometimes I quite easily adopt her suggestions, I wish to express especially my gratitude to the stimulating interest of James Hamilton whose calculational checks amounted to an interactive collaboration on the chapter on polynomial wheel methods and comparison techniques. The contents of this work are my own, as is the responsibility for any errors.

As this work continued to develop, the conclusions have surprised and delighted me. I hope *Number, space and logic* will delight and surprise you too.

Jim Hamilton Adams
Dublin
September 2017

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Definitions of mathematical terms

The following terms, symbols, ideas and definitions are used in the text. The arrangement is by ideas rather than alphabetic. This may be scanned as a more technical alternative to the details of the contents in the Foreword, or as a further introduction to the contents which follow in the eBook. The notation, for example (SA III), refers to *Superexponential algebra* chapter III, and (NSL I, IV) refers to this work, volume I, chapter IV.

1. *Sets* S, T (SA III).
$\varnothing$ (*the empty set*). The set with no members.
$\odot$ (*the void set*). The set satisfying a false condition.
$\in$ (*belongs to*). If x is a member of a set S then $x \in S$.
$\subset$ (*properly included in*). If a set S is included in a set T and S does not equal T.
$\subseteq$ (*included in*). Inclusion, when S = T is possible.
CS_T (*complement of* S *in* T). Those x not belonging to S but that belong to T, and $S \subset T$.
$\bigcup$ (*union of sets*). If x belongs to S or x belongs to T then x belongs to $S \bigcup T$.
$\bigcap$ (*intersection of sets*). If $x \in S$ and $x \in T$ then $x \in S \bigcap T$.

2. $\mathbb{N}$ is the set {1, 2, 3 ...} of *positive whole numbers*, also called *natural numbers*. If this set contains the element 0, we denote it in this eBook by $\mathbb{N}_{\cup 0}$. If we wish to emphasise that it does not contain zero, we use $\mathbb{N}_{\neq 0}$.
$\mathbb{Z}$ (from the German Zahl for number) is the set {..., -3, -2, -1, 0, 1, 2, ...} of negative, zero or positive *integers*.
$\mathbb{Q}$ is the set of *rational numbers* m/n, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, for example ½.
$\mathbb{A}$ is used in this eBook as the set of *algebraic numbers*, sums and differences of numbers of the form p^q, where p, q $\in \mathbb{Q}$, but p and q together are not both zero, for example $1 + 2\sqrt[3]{\frac{-1}{5}}$.
$\mathbb{M}_t$ is the set of *transnatural numbers* for index t, satisfying the rules for $\mathbb{N}$. $\mathbb{M}_1 = \mathbb{N}$, with proper injections $\mathbb{M}_t \to \mathbb{M}_{t+1}$, $\mathbb{M}_t \to \mathbb{M}'$ and surjections $\mathbb{M}' \to \mathbb{M}_{t+1}$ for distinct $\mathbb{M}'$.
$\mathbb{Z}_t$ *Transintegers* are positive, zero or negative transnatural numbers.
$\mathbb{Q}_t$ *Transrational numbers* are of the form m/n where $m \in \mathbb{Z}_t$ and $n \in \mathbb{M}_t$.
$\mathbb{A}_t$ *Transalgebraic numbers* are sums and differences of the form p^q, where p, q $\in \mathbb{Q}_t$, or alternatively formed in a similar way from superexponential operations.
$\mathbb{R}$ *Real numbers* have no imaginary component but are possibly not transalgebraic.
p-adic number. Arithmetic with a nonstandard idea of closeness between numbers.

3. *Peano axioms*. The rules for natural number arithmetic.
UCH. The false *uncountable continuum hypothesis* that $\mathbb{R}$ is bijective to $2^{\mathbb{N}}$.
ZFC. *Zermelo-Fraenkel* standard set theory with the axiom of choice.
mZFC. *Modified ZFC*, allowing a set satisfying a false condition (but not invalid choice).
Propositional calculus. The logic of truth tables for true and false.
Predicate calculus. Propositional calculus allowing the statement ‘there exists’.
Untwisted logic. Logic for a set embedded in an oriented manifold.
Twisted logic. Logic for a set embedded in a twisted manifold.

4. *Congruence arithmetic* (*mod n*). Finite or 'clock' arithmetic where transnatural numbers come back to themselves, so its set is {0, 1, ... (n – 1)} and n = 0.

Prime number. A transnatural number which only when divided by 1 and itself gives a transnatural number. Example: 7.

Totient ($\varphi(s)$). For a transnatural number s as a product of primes p, q, ... r to powers j, k, ... m, if s = $(p^j)(q^k)...(r^m)$ then $\varphi(s) = s\left[\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\cdots\left(1 - \frac{1}{r}\right)\right]$. Example: $\varphi(20) = 8$.

5. *Abelian*. Occurs for a set with a general operation + (not necessarily addition as usually understood) when a + b = b + a always.

Commutative. Abelian, but generally written for × rather than +.

Associative. Satisfying a + (b + c) = (a + b) + c, or a(bc) = (ab)c, etc.

6. *Eudoxus numbers*, $\mathbb{U}$. Any number which multiplied by some integer can have a size within a range of finite positive natural numbers.

Complex numbers, $\mathbb{C}$. Numbers of the form a + bi, where a, b $\in \mathbb{U}$ and i = √-1.

Gaussian integers. Complex numbers where a and b above are integers.

$\mathbb{F}$ is a *field* (SA III). It contains axioms (rules) for addition and multiplication. Examples could be the Eudoxus numbers $\mathbb{U}$ and complex numbers $\mathbb{C}$.

$\mathbb{Y}$ is a *zero algebra* (SA III). This is similar to a field except for the existence of multizeros.

Exponential algebra, (SA XII and XIII). Contains axioms for exponentiation.

$\mathbb{D}$w *exponential algebra*, (SA XIII). A nonstandard exponential algebra.

Suoperator (chapter III and SA XV). An operation, of which the first three are addition, multiplication and exponentiation, where the nth is found by repeating the (n – 1)th.

Superstructure. Contains axioms connecting suoperator nth operations for various n.

7. *Standard protocol.* (chapter I). The ordinal infinity $\boldsymbol{\Omega}_{\mathbb{N}} = \sum_{\text{all } \mathbb{N}} 1$. This is not a natural number, and is treated as being irreducible.

Ladder number. A suoperator expression in $\boldsymbol{\Omega}_{\mathbb{N}}$, with Eudoxus coefficients.

Strict transfer principle. The axioms for variables in a suoperator algebra also hold for the variable $\boldsymbol{\Omega}_{\mathbb{N}}$.

Capital Ξ function. Satisfies $1^{\Xi(a)} = a$.

8. *Implies*. (exercise, SA XI). For statements A and B, A implies B is only false when A is true and B is false.

Sufficient. A is sufficient for B means A implies B.

Necessary. A is necessary for B means A is implied by B (the same as B implies A).

9. *Function* (SA III). A set of pairs, {x, f(x)}, all x of which have a value f(x).

Injection. A mapping from all the sets {a, b} to {f(a), f(b)}, where f(a) ≠ f(b) if a ≠ b.

Surjection. A mapping where every f(x) in the set {f(x)} has a value from an x.

Bijection. A mapping which is simultaneously injective and surjective.

10. *Magma*, M (SA III). A set with one binary operation, with no other properties specified.

Polymagma (SA XV). Maps a number of copies of a set to itself.

Group, G (SA III). This satisfies the multiplicative axioms for a field, except multiplication may be noncommutative: $ab \neq ba$.
Subgroup. A set of elements in a group which satisfies within itself all the properties of the containing group.
Order of a group. The number of elements (or members) in a group.
Homomorphism of groups is a surjective map $h: G \to G'$ of groups with $h(ab) = h(a)h(b)$.
Isomorphism of groups. A bijective homomorphism.
Automorphism of a group is an isomorphism of a group to itself.
Inner automorphism of a group is an automorphism of the form $x \leftrightarrow a^{-1}xa$.
Outer automorphism of a group. An automorphism which is not inner.
Normal subgroup is invariant under all inner automorphisms of the containing group G.
Simple group has no normal subgroups other than itself and 1.

11. *Ring*, A. Satisfies the additive and multiplicative axioms of a field, except there is no general division and multiplication may be noncommutative. Example: matrices.
Unital ring. A ring with a multiplicative identity, 1. We assume rings are unital.
Automorphism of a ring. (SA X). A bijective map of a ring A, $H: A \leftrightarrow A$, where $H(ab) = H(a)H(b)$ and $H(a + b) = H(a) + H(b)$.

12. *Vector*, **v** (in bold). A matrix as one row (a row vector), or as one column (a column vector). Example: the row vector (x, y, z).
Vector space. Contains vectors with magnitude and direction, which can be added together and multiplied by scalars in a field.
Base point. The origin for a vector space.
Module. A module over a ring is a generalisation of a vector space over a field, being an additive abelian group like a vector space, where the scalars are the elements of a ring.
Scalar product of two vectors. The matrix product of multiplying each element of a row vector in turn with the corresponding elements of a column vector. Example: $x^2 + y^2 + z^2$.
Eigenvector. A vector **x** satisfying for matrix B, $B\mathbf{x} = \lambda\mathbf{x}$.
Eigenvalue. A value λ for the eigenvector **x** above. Example: λ is a complex root value.

13. *Matrix* (*plural matrices*). An array of numbers $B = b_{jk}$, where the element b_{ij} exists in the i^{th} row and j^{th} column. (SA I and II).
Symmetric matrix. $U = u_{jk} = u_{kj}$.
Antisymmetric matrix. $V = v_{jk} = -v_{kj}$.
Matrix transpose. If $W = w_{jk}$, then the transpose $W^T = w_{kj}$.
Unit diagonal matrix. Denoted by $I = b_{jk}$, where $b_{jk} = 1$ when $j = k$, otherwise $b_{jk} = 0$.
Trace of a matrix. The sum of all (main) diagonal entries b_{jk}, where $j = k$.
Determinant (*or hypervolume*) *of a matrix*, (*det* B). (SA I and II).
Singular matrix, D. Satisfies det D = 0.
Units, K*. The invertible elements of a ring, for example giving det $\neq 0$.
Box. An n-dimensional array of numbers.
Box scalar product of two boxes. A scalar value obtained from boxes using scalar products of vectors and determinants of matrices.

14. *Intricate number*. A representation of 2×2 matrices, that is, with two rows and two columns, given by $a1 + bi + c\alpha + d\phi$. (SA I).
Intricate basis element. One of the vectors 1, i, α or ϕ above.
Real basis element. The number 1 in its intricate representation.
Imaginary basis element. The number i in its intricate representation.
Actual basis element. The number α in its intricate representation.
Phantom basis element. The number ϕ in its intricate representation.
Intricate conjugate. The number $a1 - bi - c\alpha - d\phi$.
J. $J = bi + c\alpha + d\phi$ in which $J^2 = 0$ or ± 1.
JAF. A changed basis for i, α and ϕ.

15. *Hyperintricate number*. A representation of $2^n \times 2^n$ matrices. (SA II).
Layer. For example, a hyperintricate number with a component in 3 layers is $A_{B,C}$ where A, B and C are intricate numbers, possibly intricate basis elements.
n-hyperintricate number. A hyperintricate number representable by sums of components in n layers. Sometimes denoted by $Я_n$.
n-hyperintricate conjugate, $Я^*_n$. Satisfies $Я^*_n Я_n = \det Я_n$.
J-abelian hyperintricate number. A number giving the example $A_B + ... + D_E$, where $A = p1 + qJ$, $B = p'1 + q'J'$, ..., $D = t1 + uJ$, $E = t'1 + u'J'$. Two such hyperintricate numbers with identical J and J′ commute.

16. *Division algebra*. (SA III and V). A division ring where multiplication might be nonassociative. Multiplying two elements of such an algebra cannot give zero unless one of them is zero.
Zargon algebra. A division algebra except possibly for zero scalar components.
Quaternions. (SA III and V). A type of associative division algebra.
Octonions, $\mathbb{O}$. (SA V). A nonassociative division algebra.
n-vulcannions. General nonassociative division algebras of dimension $n = 6k + 2$.
Vulcan number, v. The dimension of a vulcannion minus one – the number of its space components.
T-junction. A diagram used to classify vulcannions.
n-novanions. (SA V). An n dimensional nonassociative division algebra, but not when both the real parts in a multiplication are zero.
Zargonion. A combination of algebras obtained from vulcannions and novanions.
Tribble. A zargonion except space components have square 0. It may not have a divisor.
Tharlonion. A zargonion except space components have square 1. It may not have a divisor.

17. *Norm*. Applied to complex numbers $a + bi$, the norm is $\sqrt{(a^2 + b^2)}$. For intricate numbers $a1 + bi + c\alpha + d\phi$ the norm squared is $a^2 + b^2 - c^2 - d^2$. Applied to a $n \times n$ matrix B, the norm is the positive nth root of det B. Applied to n-zargonions $a1 + bi + c\alpha + d\phi + b'i' + c'\alpha' + d'\phi' + ...$, the norm is $\sqrt{(a^2 + b^2 + c^2 + d^2 + b'^2 + c'^2 + d'^2 + ...)}$.

18. *Additive format of a polynomial equation*. The form $ax^n + bx^{n-1} + ... + d = 0$. (SA VII and VIII).

Monic polynomial. Example in the case of a polynomial equation: when a above = 1.
Fundamental theorem of algebra. The complex polynomial in additive format given by $ax^n + bx^{n-1} + ... + d$ always has some values which are zero.
Multiplicative format of a polynomial equation. The form $(x - p)(x - q) ... (x - t) = 0$.
Zero of a polynomial. A value of a polynomial $f(x) = ax^n + bx^{n-1} + ... + d$ so that $f(x) = 0$.
Root of a polynomial equation. The roots of a polynomial equation $f(x) = 0$ are the values of x satisfying this.
Equaliser of two polynomials. The intersection of their values.
Degree of a polynomial. The value of n for f(x).
Duplicate root. A root of the equation $(x + a)^2 = 0$.
Antiduplicate root. A root of the equation $(x + a)(x - a) = 0$.
Independent roots. Occur when no known dependency relation is used in the solution of a polynomial equation.
Dependent roots. Occur when a known dependency relation is used in the solution of a polynomial equation.
Multivariate polynomial. A polynomial in a number of variables.
Variety. A polynomial equation in a number of variables. Example: $3x^2y + xyz + 4x^2z^2 = 0$.

19. *Equivalence relation ≡ in a set S*. Satisfies $a \equiv a$ (*reflexive*), if $a \equiv b$ then $b \equiv a$ (*symmetric*) and if $a \equiv b$ and $b \equiv c$ then $a \equiv c$ (*transitive*), for $a, b, c \in S$.
Equivalence class. A partition of a set where an equivalence relation between elements defines membership of the partition.
Partial order ≤ of a set S. Satisfies $a \leq a$, if $a \leq b$ and $b \leq a$ then $a = b$ (*antisymmetric*) and if $a \leq b$ and $b \leq c$ then $a \leq c$, for $a, b, c \in S$.
Poset. A partially ordered set.
Total order ≤ of a set S is a partial order existing for all $a, b, c \in S$.
Well-ordering ≤ of a set S. A total order where every nonempty subset has a least element.

20. *Left (or right) coset* of a subgroup S of G is the set of elements aS (or respectively Sa), with $s \in S$ and $a \in G$.
Quotient group G/S *of* G mod S. The family of left cosets of the group G with subgroup S, sG, $s \in S$.

21. *Ideal*, C. (SA III and XI). A subset of a ring, A, with the rule that $\{c, d\} \in C$ and $a \in A$ implies $(c - d) \in C$ and both ac and ca $\in C$.
Principal ideal, (a). The ideal generated by one element, a, of the ring A. For every $r \in A$, (a) is ra. Example: for $a \neq 0$ belonging to the integers $\mathbb{Z}$, $(3a) \subset (a) \subset \mathbb{Z}$.
Prime ideal, P. If a and b are two elements of A such that their product ab is an element of P, then a or b is in P, and P is not equal to the whole ring A. Example: integers containing all the multiples of a given prime number, together with zero. Example: the zero ideal (0).
Maximal ideal, M. In any ring A, this is an ideal M contained in just two ideals of A, M itself and the entire ring A. Every maximal ideal is prime. Nonexistence: the zero ideal (0) is not a maximal ideal of $\mathbb{Z}$ because $(0) \subset (2) \subset \mathbb{Z}$, nor is the ideal (6), since $(6) \subset (2) \subset \mathbb{Z}$.
Nilradical, N(A). The intersection of all prime ideals of a ring.

22. *Unfinished set*. Example: the interval a < x < b with the end points a and b removed.
Finished set. Example: the interval a $\leq$ x $\leq$ b with the end points a and b present.
Topology. A theory of space using finished and unfinished sets.

23. *Exact sequence*. (SA III).
Homology. A theory of holes. The dimension of the nth homology is the number of holes in a space for dimension n.

24. *Euler-Poincaré characteristic*. In 3-space the number of vertex points – edges + faces of a space divided into n-dimensional polygons.
Möbius strip. A reconnected rectangle with a twist.
Handle. Obtained on a surface by cutting out two holes and gluing in a cylinder.
Crosscap. Obtained on a surface by cutting out a hole and gluing in a Möbius strip.

25. *Graph*. A set of vertices and arrows (or edges) with a mapping from origin to terminus, and provided with a reverse mapping changing orientation.
Path. A finite sequence of edges with the terminus of each edge connecting to an origin of the next edge.
Circuit. A path with its end vertices connected together (start origin = end terminus vertex).
A *graph is connected* if all vertices are contained in a path.
Tree. A connected nonempty graph without circuits.
Node. An origin or terminus in a tree.
Parent node. An origin in a tree.
Child node. A terminus in a tree.
Root of a tree. A child with no parent.
Leaf of a tree. A parent with no child.

26. *Homotopy*. A theory of paths through a topological space.
Winding number. The number of times a loop winds round a point.

27. *n-branched space*. A space where the removal of a point disconnects the space into n pieces.
Explosion. An n-branched space with an infinite or transinfinite number of points.
Supernorm. An evaluation of the magnitude of an explosion.
Branch number. An evaluation of the number of branches in an explosion.

28. *Deformation retract*. The set of all occurrences of a vector transported along another vector.
Branched vector. A directed tree, with arrows proceeding from its root and splitting to its branches.
Branched deformation retract. A branched vector transported and split into copies along another branched vector.
Amalgam. A branched retract with some of its nodes connected.

29. *Morphism*. An associative mapping with identity in category theory.
Category. A description of mathematics in terms of morphisms.
Object. The element where a morphism comes from or goes to.
Arrow. A morphism considered as a directed mapping between objects.
Hom-set. The description of categories as a collection of arrows.
Functor (*covariant*). Describes states and transformations as if they were on the same footing. If f and g are morphisms, and T is a functor, it is covariant if T(f_g) = T(f)_T(g).
Contravariant functor. A functor reversing the order of composition: T(f_g) = T(g)_T(f).
Morphism of functors (*natural transformation*). An example is the determinant, as a transformation from commutative rings to groups.
Topos. A categorical description of a set.
Adjoint functor. A specific type of interrelationship between functors, of general use in mathematics.
Kan extension. The combinations of mappings between two sets are described in terms of hom-sets by an exponential, and this can be differentiated or integrated. Kan extensions implement this idea.

30. *Explanation*. A theory or theorem using matrices or other combinatorial means.
Charade. The image of an explanation in homological algebra.
Deconstruction. The mathematical refutation of a generally accepted result.