CHAPTER IV

Sequent calculus and colour logic

4.1. Introduction.

In this chapter set theory is extended to a discussion of colour sets, multivalued sets where the two values true and false are replaced by a number of values. We also cover sequent calculus and some aspects of intuitionistic logic, in which we introduce the idea that explicit implementations of intuitionistic logics can always be embedded in a colour logic where some deductive rules are declared inaccessible, but exist.

4.2. Introduction to colour sets and other set theories.

Colour or multivalued set theory occurs when the two values true and false, or in probability logic certain and impossible, are replaced in general by n separate classification types for objects, which we will call colours. For example, the values T for true and F for false could be replaced by R for red, G for green and B for blue. These are the states of the system.

This idea is inherent in the theory of generalised sets described in category theory, called toposes, where for each subobject in the topos there is a subobject classifier, in the case considered R for red, G for green and B for blue. There is also a possible application to the strong nuclear colour force in physics, where there is a noncommutative interaction described by quantum chromodynamics, based on three quark colours.

We have already met in [Ad15], chapter XIII the idea of probability logic, in which there are intermediate values between certain, τ, and impossible, υ, and we have seen that our model of probability is a set of data in which its values as a function of variables may be inconsistent with a given function. The nearness or distance away of the given function from the data can be measured, and if the set of states of the system can be determined, then a measure can be given of the probability of the given function in terms of the actual data.

We have also seen that this measure of the probability can be represented in the general case by a multipolynomial in two variables, even a matrix multipolynomial, so this system can be noncommutative, and if we wish to abandon associative algebra, can be nonassociative too. Its further generalisation is a superexponential polynomial in two variables.

We will represent colour sets in what follows by a similar idea. These can be represented by multipolynomials in n variables, so for the case n = 3 the three variables could be R, G and B.

When one column is stacked with items R, G and B without regard to their order, the number of combinations is the number of colours, three. When two columns are stacked in this way, the number of combinations of rows is 3×3, and when m columns are stacked the number of combinations is $3 \uparrow m$. Thus in general if each row has 3 possibilities, the total number of possibilities is $3 \uparrow (3 \uparrow m)$, and this means for a unary operator in the 3 colour analogue of propositional calculus (compare this with section 16.2) there are 3 types, and for a binary operation $3 \uparrow (3 \uparrow 2) = 19683$ types. If there are in general n colours, then for a unary operation there are n types and for an m-ary operation $n \uparrow (n \uparrow m)$ types, and these correspond to the number of colour tables in the generalisation of the propositional calculus.

This colour analogue of the propositional calculus is in *canonical form*, which is a unique enumeration of all its colour tables. We may transform within colour propositional calculus its variables or operators so that other states are defined and other rules apply.

If we extend the colour propositional calculus to a predicate calculus with colour ∃ and ∀, then if $\exists_G$ x meaning there exists a green variable x, the equivalent of NOT $\exists_G$ x takes on two values, $\exists_R$ y or $\exists_B$ y. Just as we defined ∀ as NOT ∃ NOT, if M and N are unary operators we can define for example M∃N as a suitable quantifier for colour predicate calculus. Note that in this context it is more natural to take a colour set theory which is a generalisation of mZFC, rather than ZFC, since mZFC allows false states with a corresponding void set, but ZFC does not.

Having achieved this generalisation, it is now possible to introduce a colour probability logic as we did, but for the propositional calculus, in chapter XIII. As for the binary colour version which represents its states by a multipolynomial in two variables, for a colour probability logic its states are representable for n colours by a variety in n variables.

We have already mentioned in [Ad15], chapter XVII that the superexponential analogue of a variety, called a supervariety, can generalise the types of objects studied as a variety.

We can then apply some of the tricks we have been using for multipolynomials and varieties to colour sets. For example, we can compact a polynomial in n variables to a polynomial in k variables, k < n, by representing say each of R, G and B as a function of say τ and υ.

We will now discuss briefly other types of generalisation of logic. It is remarkable that the advances in symbolic logic given by Gentzen have a formalisation based on a weaker logic than the Boolean logic that we have up to now been considering, the history of which is given by the theory of syllogisms developed by Aristotle and probably by others of that time. A particular type of this weaker logic is intuitionist logic, in which the statement A OR NOT A cannot be deduced in the system as a universally valid statement.

Since propositional calculus is an implementation of the theory of sets and vice versa, in which a statement using propositional calculus is replaced in the model by x ∊ S, where S is a set, an implementation of intuitionist logic occurs where every set, open, closed or semi-open and semi-closed, is replaced by an open set. An example of a closed set is the interval given by $0 \leq x \leq 1$, and of an open set by $0 < x < 1$, so that we can think that a closed set has a boundary, in this case {0, 1}, and an open set does not. This means that Boolean operator is first applied, and when evaluated every occurrence of the statement x ∊ S in Boolean logic is replaced by x ∊ open (S) in intuitionist logic, that is, as a *last stage* we remove the boundary of sets on evaluating that occurrence of the operator. So it is the case that NOT(NOT A) ≠ A, where A on the right is not operated on. It is now the case that A OR NOT A, called the law of the excluded middle, is not necessarily valid.

Thus there arises the possibility pointed out by S.C. Kleene [Kl52] of asserting the formula

NOT ∀x (A(x) OR NOT A(x)) . (1)

He states we should obtain an extension of the intuitionistic number theory, which has been treated as a subsystem of the classical, so that the intuitionistic and classical number theories diverge, with formula (1) holding in the intuitionistic and

∀x (A(x) OR NOT A(x)) (2)

in the classical. "Such divergences are familiar to mathematicians from the example of Euclidean and non-Euclidean geometry, and other examples".

We implement equation (1) by *firstly* applying the complement of the open set operator acting within the region of a statement $x \in S$ to be evaluated in Boolean logic, that is, we take the boundary of sets S. Then we apply the standard rules of predicate calculus. It is now the case that the set corresponding to (1) and (2) simultaneously is empty. So we have introduced as well as sets and their complements the notion of open and closed sets in the intuitionistic algebra, which increases the number of variables used to describe this logic.

Theories of intuitionist type may be described by the theory of lattices, which are sets given an order relation, provided in the theory of Heyting algebras. Given a set A with three binary operations ⇒, & and OR, and two distinguished elements, υ and τ, A is a *Heyting algebra* for these operations if and only if the following conditions hold for any element x, y and z of A.

The relation ≤ is defined by the condition that a ≤ b when a ⇒ b = τ. I suggest the reader substitutes this in (i) to (xi) below to check these rules are reasonable.

(i) If $x \leq y$ and $y \leq x$ then $x = y$
(ii) If $\tau \leq y$ then $y = \tau$
(iii) $x \leq y \Rightarrow x$
(iv) $x \Rightarrow (y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$
(v) $x \,\&\, y \leq x$
(vi) $x \,\&\, y \leq y$
(vii) $x \leq y \Rightarrow (x \,\&\, y)$
(viii) $x \leq x$ OR y
(ix) $y \leq x$ OR y
(x) $x \Rightarrow z \leq (y \Rightarrow z) \Rightarrow (x$ OR $y \Rightarrow z)$
(xi) $\upsilon \leq x$.

Finally, we define NOT x to be $x \Rightarrow \upsilon$.

We will demonstrate that there is no possibility of implementing intuitionistic logic as a 2-colour logic except trivially as a deletion of rules within the classical predicate logic or under inversion of τ and υ. So (1) cannot be implemented intuitionistically as a 2-colour logic except by swapping τ and υ. For a constant c, equation (1) asserts the invalidity of

(A(c) OR NOT A(c)), (3)

a situation which does not hold in the propositional calculus. Since we have defined NOT by $x \Rightarrow \upsilon$ above, we cannot interpret

τ OR $\upsilon = \tau$ (4)

in the intuitionalist calculus as an embedding where (4) holds, except by the means already given. ☐

It is our philosophy that the world consists of states, and that observations are a small subset of the transformations of these states. If we view mathematics as a collection of statements about possible worlds, it has axioms specifying states and rules of transformation for these. In the case of logic as conventionally formulated, the axioms define objects which are held to exist under certain choices, and the allowable transformations are rules of deduction.

If we view intuitionistic logic as axioms with a set of rules, some of which have been deleted, then if the axiom system corresponds to the usual mode of predicate calculus, we have chosen a specific type of intuitionistic logic which is not the most general. Under this selection no rules exist which are in contradiction with the predicate calculus, although some theorems may be deduced in predicate calculus for which there is no corresponding intuitionistic proof.

For colour intuitionistic logic, we proceed in the manner of the construction of intuitionistic logic by the choice of a subset of the classical propositional calculus, that is, by eliminating a subset of rules generated by propositional truth tables. Since for an m-ary operator on n colours we can enumerate the n ↑ (n ↑ m) rules from colour tables, it is possible to select a subset of these rules as those for a colour propositional fragment. Intuitionistic colour formulas can now be expressed using colour quantifiers $M\exists_i N$, i = 1, ... n, acting on this fragment. As for intuitionistic logic, other rules operating on the fragment can be introduced.

However, intuitionistic logic may be a logic not embedded within 2-colour predicate logic. If an intuitionistic logic does not contain a propositional logic derived from its underlying states, we know from the canonical form of propositional logic, as the enumeration of all of its states in a full set of truth tables, that either the intuitionistic logic can be converted to canonical form, or it is inconsistent.

The intuitionistic logic can be embedded in n-colour logic with an appropriate deletion of rules. We then come to the situation where the underlying logic may contain states which are inconsistent, but for which the undeleted rules of deduction do not allow this to be detected (this is obtained in the trivial case where all rules are deleted). The question then arises whether we say the fundamental objects of the logical system are its states, or whether we can dispense with its states and deal only with its rules of deduction.

In the case of consistent states we will need to prove a theorem that all intuitionistic logics can be embedded in n-colour logics with appropriate deletion of rules. In the standard intuitionistic system these deletions occur wherever colour variables other than τ and υ would otherwise be displayed. This embedding is obtained by definition on adjoining to standard predicate calculus with its propositional variables τ for true and υ for false, the mappings to O(τ) and O(υ) as two new variables, where in the interpretation of this model

$x \in S \rightarrow \tau$ (or, unusually, υ),

$x \notin S \rightarrow \upsilon$ (or respectively τ),

$x \in$ open(S) → O(τ) (or O(υ))

$x \in$ closed(S) (which means $x \notin$ open(S)) → O(υ) (or respectively O(τ)).

We are using the axioms for open and closed sets given in chapter II.

It is clear that any n-colour logic can be embedded in an (n + 1)-colour logic. ☐

4.3. Supervarieties and colour logic.

Proofs in colour logic may be subsumed under the following idea. The propositional fragment of colour logic is obtained from multivalued tables in n colours, which in the case n = 2 reduces to the truth tables of Boolean propositional logic.

Consider the case n = 3, with three colours, R, G and B. We can extend the propositional fragment to a 3-colour probability logic, which we introduced in chapter XIII, by allocating as a basic building block probabilities of the form

$$\gamma R + \delta G + (1 - \gamma - \delta)B, \qquad (1)$$

where γ and δ are real numbers or other types of number we have considered.

To include the predicate n-colour logic, we will find it convenient to give a model of $\exists_R$, $\exists_G$ and $\exists_B$ section 2. ∃ means *for some*, so we are dealing in some cases with collections of objects. If we have symbols a, b, c, ... then we can enumerate the situations when a, b, c, ... have the value R. In the notation we introduce in section 4, this is the case for formulas A[a/R], etc. So as a model using collections of equations we can describe this as

$$R = a$$
$$R = b$$
$$R = c, \ldots \tag{2}$$

or alternatively express this as the polynomial equation in R

$$(R - a)(R - b)(R - c) \ldots = 0. \tag{3}$$

The formula (3) now expresses the fact that we have enumerated states of the colour variable R, and the collection of all such states can be represented by (3), where the polynomial equation is finite, countably infinite, or one of the transfinite systems we have been considering. Further, equation (1) for probabilities may be used to express a multiplicative variety

$$(\gamma R + \delta G + (1 - \gamma - \delta)B)(\gamma' R + \delta' G + (1 - \gamma' - \delta')B) \ldots = 0, \tag{4}$$

and we can also consider varieties in additive format.

More generally we can describe supervarieties in this format.

To define proofs for logics of this kind, say in a sequent calculus, we need to ask whether a theorem is *correct* or *incorrect*. We will allocate correct proofs to the Nullstellensatz variety case of [Ad15], chapter XII, section 5, namely the weak Nullstellensatz case of varieties falling in the situation of a set of multipolynomials

$$T_1(R, G, B) = T_2(R, G, B) = \ldots = 0, \tag{5}$$

having a solution in a field, which may correspond to the existence of solutions like (2), or otherwise that there exist multipolynomials such that

$$T_1U_1 + T_2U_2 + \ldots = 1, \tag{6}$$

which corresponds to the case of no solution like (2), so the state the theorem describes does not exist as (5). In other words, equations (5) and (6) are mutually exclusive.

Generally speaking when we have collections of solutions like (3), varieties of type (3) can be interpreted as

$$(R = a) \text{ OR } (R = b) \text{ OR } (R = c) \ldots, \tag{7}$$

but this is not exclusive OR, XOR, so we need an interpretation of (R = a) & (R = b) which can happen for inclusive OR. If the variety is in multiplicative format, we now say that this occurs for a = b, so the interpretation for a series of varieties of the form (2) is that we are dealing with varieties under &, and of the form (3) under this new interpretation under OR. We can also consider the case of the strong Nullstellensatz. Colour sequent calculus now has the task of mapping proofs to colour sets for which at least one allocation of equation (5) or (6) is possible.

An interesting case is when (6) is a quadratic form, so that if we represent R, Y, G, B, etc. by R_j, j = 1, ... n, then

$$\sum_{j=1}^{n} \sum_{k=1}^{j} \delta_{jk} R_j R_k$$

with n(n + 1)/2 terms can be represented under the transformation

$$R_j = \sum_{k=1}^{n} \varepsilon_{jk} R'_k$$

with n terms, so that R_jR_k has n(n – 1)/2 distinct terms $\neq (R_j)^2$, so that we can choose

$$\sum_{j=1}^{n} \sum_{k=1}^{j} \delta'_{jk} R'_j R'_k$$

with $\delta'_{jk} = 0$ if $j \neq k$ and $\delta'_{jk} = 1$ if $j = k$, that is, n(n + 1)/2 conditions in all.

Because there are quadratic $R'_j R'_k$ terms, the R'_j solutions may be complex numbers. Then these solutions satisfy (6), so (6) becomes a type of n-dimensional space satisfying a complex Pythagoras theorem. Thus we have a distance *norm* on the space. In [Ad15], chapter XI, section 2 we introduced Sylvester's law of inertia, in which the R'_j solutions are real and we obtain $\delta'_{jk} = \pm 1$ if $j = k$.

Further, any such space may contain a curved submanifold of arbitrary dimension $\leq$ n. This submanifold has a local coordinate system, so it is a Riemannian manifold, and we have introduced Riemannian geometry into the study of colour logics. ☐

4.4. Classical sequent calculus and Gentzen normal form. [Sz69], [1Wa90]

As well as the states of a logical system, rules of deduction may be attached to them.

We wish to describe a method first developed by Gentzen to generate a proof or disproof of any theorem. In this section we will discuss proofs in two forms of classical logic we have introduced, propositional calculus and its extension to predicate calculus.

[Ac67] An argument may be taken as a sequence $S_1, S_2, ..., S_{n-1}, S_n$ of $n \geq 1$ assertions, where $S_1, S_2, ..., S_{n-1}$ are the *premises* of the argument and S_n is the *conclusion*. An argument is considered correct if and only if the conclusion S_n follows from the premises $S_1, S_2, ..., S_{n-1}$. The process of building a formal proof is one where the sentences $S_1, S_2, ..., S_{n-1}, S_n$ are converted to well-formed formulas (wff's) $L_1, L_2, ..., L_{n-1}, L_n$ of section 2.

In this context, let us look at the rules for implication. The theorems which are often cited as an objection to the use of the rules for implication from these logics are

$$\text{NOT } p \Rightarrow (p \Rightarrow q) \tag{1}$$

$$p \Rightarrow (p \Rightarrow q). \tag{2}$$

The existence of these theorems in the propositional calculus seems to sanction as valid any argument from a false premise to any conclusion whatsoever, and any argument to a true conclusion from any premise.

The critics maintain there is a confusion in the role of ⇒, since it is sometimes used to articulate the truth function structure of a sentence, and sometimes to represent the relation between the premises and conclusion of an argument in a validity test. They have attempted to find a calculus with an *entailment* operator → so that if $S_1, S_2, ..., S_{n-1}, S_n$ is an argument sequence and $L_1, L_2, ..., L_{n-1}, L_n$ are wff's derived from it, then the argument is correct if and only if the formula

$$(L_1 \,\&\, L_2 \,\&\, ..., \&\, L_{n-1}) \rightarrow L_n \tag{3}$$

is a theorem of the calculus.

It follows that we have separated out the use of the implication operator ⇒ in propositional calculus and that of the entailment operator →. We do not abandon the use of ⇒, but by rules we will eventually construct employ both ⇒ and → together. It will then turn out that for example $p \rightarrow (q \Rightarrow p)$ is a theorem of the calculus but $p \rightarrow (q \rightarrow p)$ is not.

In a proof we may have in intermediate stages formulas

$$(L_1 \,\&\, L_2 \,\&\, ..., \&\, L_{n-1}) \rightarrow L_n,$$

$$(L'_1 \,\&\, L'_2 \,\&\, ..., \&\, L'_{n'-1}) \rightarrow L_{n'}, \text{ etc.} \tag{4}$$

The proof of a formula proceeds by the use of logical rules from a set of axioms to the concluding formula. An objective of sequent calculus is to derive proofs of formulas in the above entailment form using a tree structure for premises and conclusions. We have defined trees [Se00] in chapter I.

For predicate calculus described in [Ad15], chapter XIV, we have defined in 14.2.1 a free or bound variable for wff's according as the variable is not or is in the scope of a corresponding ∃ or ∀ quantifier.

Notation 4.4.7. The *substitution* of a constant c for a free variable x in the wff A is denoted by A[c/x].

Subformulas are defined in [Ad15], 14.2. It is sometimes useful to display all the subformulas of a given formula, given by

Definition 4.4.8. The *formation tree* for a formula A is defined inductively on the structure of A as follows.

(a) the root is labelled by A.
(b) If a node is labelled by NOT B, it has one child labelled by B.
(c) If a node is labelled by B & C, B OR C or B ⇒ C, it has two children labelled respectively by B and C.
(d) If a node is labelled by ∃x A or ∀x A, it has one child labelled with A.

For example, consider the formation tree for the formula in propositional calculus [2Sm68]

[(p & q) ⇒ (NOT p OR NOT NOT q)] OR (q ⇒ NOT p)

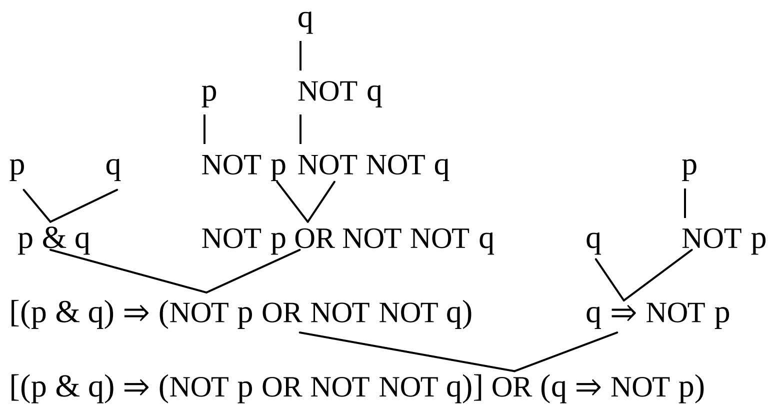

Figure 4.4.9. A formation tree in propositional calculus.

The most general form of the sequence of formulas (4) is known as a sequent.

Definition 4.4.10. A *sequent* is an ordered pair <Γ, Δ>, alternatively written Γ → Δ, of sets of formulas Γ and Δ.

Definition 4.4.11. In the sequent Γ → Δ, the set of formulas Γ is called the *antecedent* and the set of formulas Δ the *succedent* of the sequent <Γ, Δ>.

Definition 4.4.12. The sequents of the calculus obey *rules* which transform one set of sequents to another.

Notation 4.4.13. The first, or initial, sequent is also called the *axioms* of the sequent.

Gentzen took instances of the sequent

$$A \to A$$

for initial sequents.

Notation 4.4.14. The rules will be displayed as *lines*.

Definition 4.4.15. The sequents above the line in a rule are called the *premises* of the rule. The sequent below the line is called the *conclusion* of the rule.

Definition 4.4.16. An *inference* is an instance of a logical rule read from premises to conclusion.

Definition 4.4.17. A *reduction* is an instance of a logical rule read in an inverted way from the conclusion to its premises.

Definition 4.4.18. A *derivation* in the calculus is a tree regulated by the logical rules.

Definition 4.4.19. The *endsequent* of a derivation is a sequent at the root of its derivation tree.

Definition 4.4.16. The *leaves* of a derivation are the sequents at the leaves of its derivation tree.

We now give the rules for the sequent calculus in both its propositional form, which contains the logical operators NOT, &, OR and ⇒ defined by the truth tables of chapter XVI section 2, and its form in predicate calculus with two operators, ∃, for some, and ∀, for every. We will find it is necessary to introduce one set of rules for antecedents and another for succeedents. The rules for antecedents are labelled by lNOT, l&, lOR, l⇒, l∃ and l∀, where l stands for left, and the rules for succedents by rNOT, r&, rOR, r⇒, r∃ and r∀, where r stands for right.

$$A \to A$$

$$\frac{\Gamma \to A, \Delta \qquad \Gamma, B \to A}{\Gamma, A \Rightarrow B \to \Delta}\ \mathrm{l}\Rightarrow \qquad\qquad \frac{\Gamma, A \to B, \Delta}{\Gamma \to A \Rightarrow B, \Delta}\ \mathrm{r}\Rightarrow$$

$$\frac{\Gamma, A, B \to A}{\Gamma, A \,\&\, B \to \Delta}\ \mathrm{l}\& \qquad\qquad \frac{\Gamma \to A, \Delta \qquad \Gamma \to B, \Delta}{\Gamma \to A \,\&\, B, \Delta}\ \mathrm{r}\&$$

$$\frac{\Gamma, A \to \Delta \qquad \Gamma, B \to \Delta}{\Gamma, A \text{ OR } B \to \Delta}\ \mathrm{lOR} \qquad\qquad \frac{\Gamma \to A, B, \Delta}{\Gamma \to A \text{ OR } B, \Delta}\ \mathrm{rOR}$$

$$\frac{\Gamma \to A, \Delta}{\Gamma, \text{NOT } A \to \Delta}\ \mathrm{lNOT} \qquad\qquad \frac{\Gamma, A \to \Delta}{\Gamma \to A \,\&\, B, \Delta}\ \mathrm{rNOT}$$

$$\frac{\Gamma, A[c/x] \to \Delta}{\Gamma, \forall x\, A \to \Delta}\ \mathrm{l}\forall \qquad\qquad \frac{\Gamma, A[a/x] \to \Delta}{\Gamma \to \forall x\, A, \Delta}\ \mathrm{r}\forall$$

$$\frac{\Gamma, A[a/x] \to \Delta}{\Gamma, \exists x\, A \to \Delta}\ \mathrm{l}\exists \qquad\qquad \frac{\Gamma, A[c/x] \to \Delta}{\Gamma \to \exists x\, A, \Delta}\ \mathrm{r}\exists$$

Note: For r∀ and l∃, a must not occur in the conclusion.

For further discussion of formula trees for sequent calculus in classical and intuitionistic logic, their non-permutability when predicate quantifiers are present, and their matrices, paths and connections, see Wallend [1Wa90]. For a discussion of trees and amalgams, and their relation to groups and their generators, see Serre [Se00] and chapter I.

4.5. A colour model for modal logics.

4.6. Colour sequents.