CHAPTER I

Trees and amalgams

1.1. Introduction.

1.2. Undirected graphs.

1.3. Cubic graphs.

1.4. Frucht's theorem.

1.5. Directed graphs.

Definition 1.5.1. A *directed graph* Γ consists of a set X of *vertices* and a set Y of *edges*, with two maps

$Y \to X \times X$: $y \to (\text{origin}(y), \text{terminus}(y))$

$Y \to Y$: $y \to \text{reverse}(y)$

which for every $y \in Y$ satisfy the conditions

$\text{reverse } y \neq y$,

$\text{reverse}(\text{reverse}(y)) = y$

and

$\text{origin}(y) = \text{terminus}(\text{reverse}(y))$.

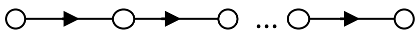
y reverse(y)

origin(y) terminus(y)

1.6. Trees.

We first define trees [Se00].

Definition 1.6.1. A *path* is a finite sequence of edges $y_1, y_2, ..., y_n$ with $\text{terminus}(y_i) = \text{origin}(y_{i+1})$, $i < n$. The path may be said to have an origin, terminus pair (y_1, y_n).



Definition 1.6.2. A *circuit* is a path with $\text{terminus}(y_n) = \text{origin}(y_1)$.

A circuit remains a circuit under a cyclic permutation of the y_i, since the circuit may be defined from a new path with origin y_k for some k.

Definition 1.6.3. A directed graph is said to be *connected* if any pair of vertices is the origin, terminus pair of at least one path.

Definition 1.6.4. A *tree* is a connected nonempty directed graph without circuits.

Definition 1.6.5. A *node* is an origin or a terminus in a tree. A *parent node* or *parent* is an origin in a tree. A *child node* or *child* is a terminus in a tree. A *root* of a tree is a child with no parent, a *leaf* of a tree is a parent with no child.

1.7. Free groups. [Ar88]

Definition 1.7.1. A subset X of a group G is a *free set of generators* for G if every g ∈ G not equal to the identity can be uniquely expressed as a finite product

$$g = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

where the x_i are distinct and $m_i \in \mathbb{N}_{\neq 0}$.

Definition 1.7.2. A *word* in the alphabet X is a finite associative product

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

where an *empty word* is the word x^0 with no symbols.

Definition 1.7.3. A *reduced word* satisfies the property that the x_i are distinct and $m_i \in \mathbb{Z}_{\neq 0}$.

The sequence of words x_1x_2 may have a common factor between the last word of x_1 and the first word of x_2, but may be expressed as the reduced word $\overline{x_1x_2}$.

Theorem 1.7.4. *Reduced words form a group*.

Proof. It is associative

$$\overline{\overline{(x_1x_2)}x_3} = \overline{x_1\overline{(x_2x_3)}} = \overline{x_1x_2x_3},$$

it has an identity, the empty word $\overline{x^0} = x^0$, and reduced words in the group have an inverse

$$\overline{(x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n})} \times \overline{(x_1^{-m_n} x_2^{-m_{n-1}} \cdots x_n^{-m_1})} = x^0. \square$$

Corollary 1.7.5. *Reduced words are uniquely expressed*.

Proof. This follows from the fact that elements of a group are unique. □

Definition 1.7.6. The group of reduced words over the alphabet X is called the *free group generated by the elements of* X and is denoted by F(X).

Theorem 1.7.7. Let X be a subset of the group G.

1.8. The Nielsen-Schreier theorem. [Ar88]

1.9. Branched retracts.

1.10. Amalgams.