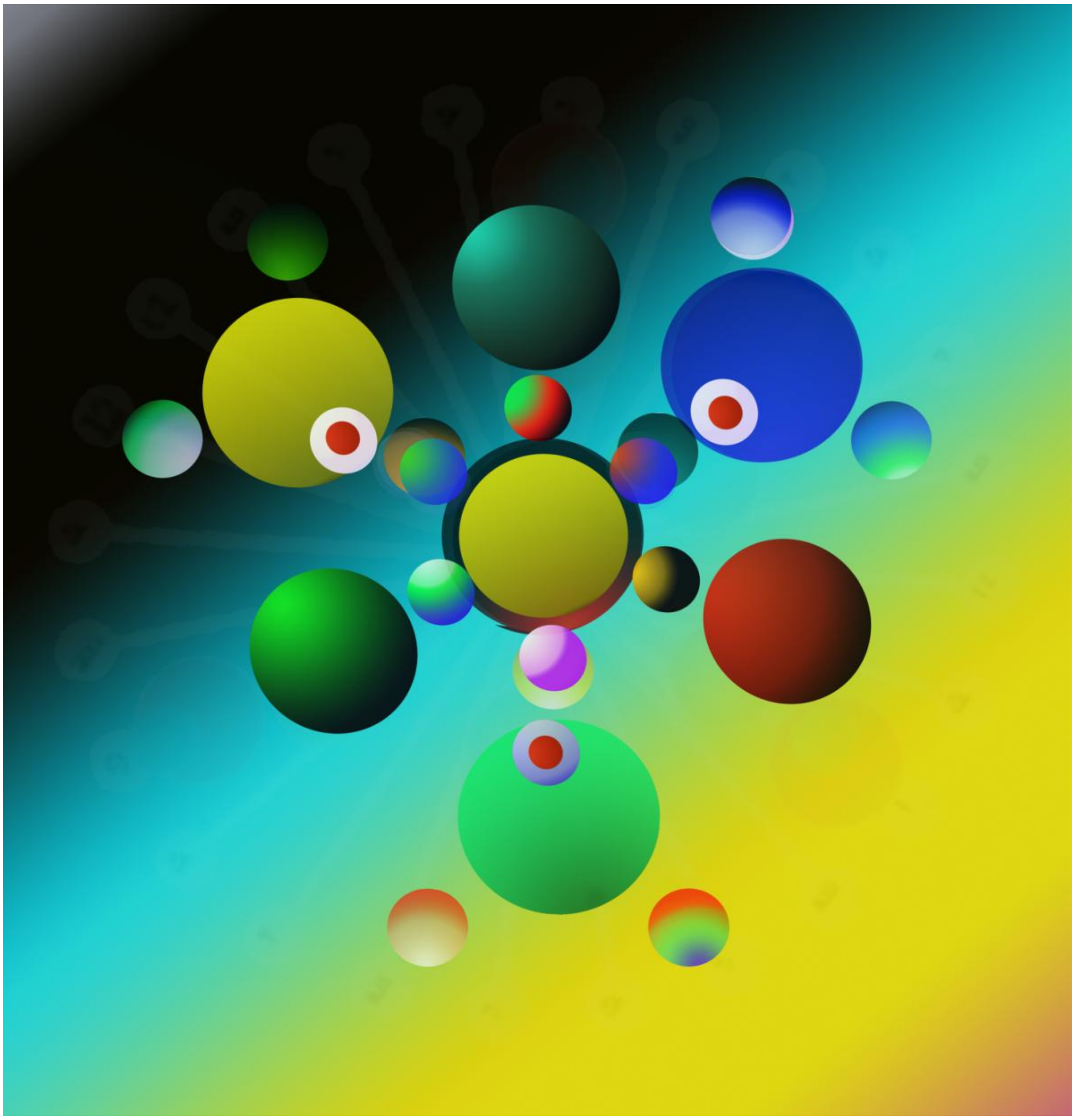
Number, space and logic

Volume II

Trees and amalgams



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The cover artwork is by Ian Dell.

Jim H. Adams is a researcher in the concepts of mathematics and their explicit representation. This work is a guide to some of these ideas, and an encouragement to the graduate to develop mathematical research productively and continually.

After a career in IT, Jim joined New Music Brighton as a composer and performer of his own works. He has been extensively involved in mathematical research.

The eBooks published in 2014 were *The climate and energy emergencies* and *Innovation in mathematics*. Volumes I, II and III of *Superexponential algebra* were completed in 2017, and *Number, space and logic* is a mathematical sequel to that work.

Keywords: l-adic numbers, elliptic curves, modular forms, complex multiplication, Heegner number, Riemann hypothesis, Goldbach conjecture.

Foreword

Number, space and logic could be viewed as a mathematical manifesto and encyclopaedia. It defines and interrelates three ideas, all in some ways at variance with current understandings. The first extends our new ideas of infinity, including transcendental infinity, to computable models. The second idea classifies all possible universes with space and time. The third states that all consistent problems are decidable, and gives methods for solving all such problems.

Number, space and logic combines a practical graduate level textbook with a research project in a commentary and development of the mathematics of the late 20^{th} and the 21^{st} centuries, just as *Superexponential algebra* [Ad15] does for the 20^{th} century before that covered in this work. As a guide to this, we include the mathematics needed to prove the general Riemann hypothesis and from it the weak Goldbach conjecture. There is an exposition of the theory, to be described later in this foreword, of zargonions, polynomial wheels and branched spaces, and as detailed in volume III, a Gentzen-type proof of the consistency of analysis.

In order to present our work in a coherent way, we have tended to proceed top down, from the abstract to the specific, but so that this generality can be understood we first give an account of the meanings of these abstractions. First we describe our approach to the finite and the infinite, then how our generalisations of space fit into the theory of trees and amalgams. Finally we show how superexponential structures, which we call *superstructures*, can be used to extend these ideas in an overarching theory of mathematics.

In the chapter on *the meaning of the finite and the infinite*, after giving the rules for set theory we give an overview of standard and significant results in finite and discrete number theory, in the context of nonstandard extensions. Firstly we discuss extended sets of natural numbers, which we call transnatural numbers. We give a description of capital Ξ functions, which can be used to give a model of the real numbers. This idea is employed in Volume III to give a proof of the general Riemann hypothesis from a local field point of view. Transfinite number theory is considered for these transnatural numbers $\mathbb{M}_t$, where the theory is an extension of finite arithmetic. The essential feature of this is that, unlike our results in nonstandard set theory, these numbers are not bijective to the natural numbers, instead they have a transfinite number of members, but otherwise they share the properties of natural numbers. For instance they contain transfinite prime numbers and can be used to define transfinite rational numbers. We also deduce the inconsistency of the uncountable continuum hypothesis if it describes real numbers derived uniquely only from countably infinite Cauchy sequences. We give an account of ladder algebra which incorporates a reformulation and extension of nonstandard analysis. Then we extend results on the ladder algebra of ordinal arithmetic to $\mathbb{M}_t$.

The chapter on *the meaning of branched spaces* firstly connects ideas on vector spaces to sets and logic. It introduces us to our extension of the domain of the Euler-Poincaré characteristic used in topology, related to étale cohomology theory. In its familiar form our description appears more restricted than usual, although other formulations are equivalent to it. In its extended form it describes types of topology which have no analogue in standard topology. The model we introduce is one of branched spaces. We can put it this way: when a point is removed from the equivalent of an n-branched line, it divides into n pieces. An *explosion* is a set that contains an infinite number of branches. Indeed these are the topological analogues of other behaviour we will meet in exponential and superexponential number systems.

Concerning *the meaning of superstructures*, which is the topic of the last chapter of the eBook *Superexponential algebra* [Ad15], this idea is an extension of the repetitive process which generates multiplication from addition, and exponentiation from multiplication. So we generate an (n + 1)th superexponential operation by repetition of an nth superexponential operation. In that book we discussed nonstandard exponential and superexponential algebras, which we describe here by the statement that they are unbranched compared with standard such algebras. A real number we say has a *supernorm* defining its magnitude under an exponential or superexponential operation which is similarly real. Thus superexponentiation on numbers has two features that are interesting to investigate: supernorms and branching. We give a basic description of the meaning of terms in category theory, which describes mathematics from the point of view of transformations, called morphisms, and develop some technical results in it.

Having built up the meaning of these abstractions, from their axiomatic formulation we then provide a detailed exposition of these ideas in reverse order: of superstructures, trees and amalgams, and the finite and the infinite, and look at some of their consequences.

It had been my belief that these ideas could be developed to give an overarching description of mathematics, but it has only been in the construction of this work that I realised to my astonishment that a detailed overarching description of mathematics was available not only in theory, but that it was being built in practice.

Two essential leading ideas which are further components in this supermathematics is that of the nonassociative zargonions and the deconstruction of the current theory of polynomial equations and its replacement, now completed, by polynomial wheel methods, so called because I ask whether their implications are as important as the invention of the wheel, giving access to computable structures for the whole of mathematics. The difference between our approach and the standard one is that we say that there is no accommodation of nonabelian group theory within the theory of abelian varieties, although there is always a mapping from noncommutative structures to commutative ones. Our methods reveal a connection between the solvability of abelian varieties, elliptic curves and modular forms. The structural problems we solve can be generalised to all mathematical problems. Mathematics hitherto has stated that there exist undecidable problems. But the statement that there are consistent undecidable problems is false; we have proved that all consistent problems are decidable. When these computable structures combine with zargonion superstructures, mathematics describes number, space and logic in full and coherent generality.

Volume II in more detail

An example of a 2-branched space is the set of rational or algebraic numbers in a line. When a point is removed there are two sets remaining: that to the left of the point, and that to the right. This also holds when the number of points is transinfinite. An example of a point which is 3-branched is a cusp in a plane. Spaces which are n-branched may be partitioned into n sets when a point is removed from them. When a set contains an infinite number of n-branches at every point in it, we call it an explosion.

Homology, which defines holes, and cohomology refer to 2-branched spaces. The general theory we develop refers to n-branched spaces, which may be derived from superexponential structures, or it may be considered independently of a superexponential structure, as an extension of models of topology.

Homotopy theory describes paths. We also extend these ideas to n-branched spaces.

On inviting the reader to mathematics

What is mathematics? Possible worlds! These worlds have strict definitions. They have states which describe what is in these possible worlds, and specified transformations of these states, which describe how these possible worlds may change.

It is my philosophy – a leap from some thinking about physics – that what is possible and consistent in mathematics corresponds in some way with what exists. Since mathematics can describe physics, consistent mathematics can describe the real world. If a symbol represents 1 as zero, this is inconsistent. There is a relationship between consistency and the stability of solutions of mathematical problems represented by symbols. In this way of thinking, states that evolve stably correspond with what now exists in the physical universe, and other states have physically self-annihilated in an inconsistency.

It is then an interesting programme to document what mathematical worlds are possible. Some of these worlds appear to be hallucinations – they have no implementation as far as we can see in the physical world. Others may have implementations which may only be used by future generations on the discovery of new physics, or otherwise have seemingly obvious practicality but may be generalised in ways which have no apparent immediate physical significance.

The objective of this work is to introduce you the reader to these possible worlds and to ask you to become creatively involved in their construction and the analysis of their behaviour, so that giving new definitions and working out their consequences comes naturally to you. This programme is both abstract and potentially useful, to manipulate the physical world in ways which may be at the frontiers of human knowledge and experience, or may be mundane applications of direct and practical use.

Mathematics proceeds outside the social group by the analysis of abstract systems whose meaning is derived from features of the world, and also within this group, as culture, history, revolution, extension and rewriting of its basis. In order to connect with its modern modes of reasoning, we need also to be aware of the human features which have led to the development of mathematics.

The objective of creative mathematics is the production of ideas. These need caring support, and part of this objective is not just to replicate results, but to reconstruct mathematics from new principles, to have the resilience to deconstruct what has gone before if it is needed, and to extend results by an analysis of the core ideas of the subject as it currently stands. Thus it becomes feasible to develop generalised new methods to tackle outstanding problems.

We need also to be aware that as a human endeavour we are not at a unique point in history where all aspects have been decided with certainty, and that human systems have triumphs and failings. We must have the courage of our convictions, and the sure analysis of our own and others thinking to develop this civilisation, and where necessary to have the resilience to deconstruct what has gone before, because even in mathematics we are not in utopia, and some things, promoted in the system from one generation to another, are wrong.

Mathematics, once constructed, is about the truth. It may be that this truth has no respect for high-ranking authority. I feel it is good to be aware of this possibility when we conduct our research.

Processes of mathematical thinking

An objective in writing this work has been to make progress and to explain to the reader. To make progress in mathematics I have relied on what is to hand, my own understanding or misunderstanding, and my intuition.

My approach to understanding the subject is reliant on a feature of my personality, that I am not good at memorising a large number of facts, and therefore I must proceed by isolating a clear, distinct and small number of basic principles, and by applying rules of deduction to them which are recorded on some external medium, to reach finally a full set of results.

I often find that to isolate these principles cannot be derived from studying only the latest mathematics texts, but to make sense of these basic building blocks requires backtracking through the mathematical literature, until the principles on which a sector of mathematics is based is simple and its future form of development is clear. This history is often a long one, because each new mathematical generation feels compelled to revolutionise the ideas of its predecessors, and indeed eventually we find in Western culture the language of discourse is no longer English, but mainly German and also French, with many other languages included, and before this the language of scientific communication is Latin, or if we want to go further back than this, it is Greek. It is my assertion that to follow this history is extremely important in isolating key concepts and the processes inherent in current thinking. This history is not, as is often presented by historians with standard mathematical culture, a history of personalities. In essence, it is the history of the development of ideas.

Once this subject is understood as a way of thinking, it is necessary to apply analysis to its form. Basic in the way I respond to this culture is the *insight* that I need. It may be the case that some mathematicians have no intuition, just as some people are blind or deaf. Whatever that case, intuition is available to me, and I wish to apply it. Intuition may be thought of as an approach which grasps the whole of a subject at once, in an immediate understanding of all of its parts. It is my contention that in order to connect my intuition, which is what I feel about a result, to mathematical culture, I have to bring to the surface of my consciousness what is immediately grasped subconsciously, in order to analyse the contents of what I am thinking so that it is externally and explicitly expressed. Once this intuition has been transferred to written form, it is in a form in which it can be analysed. It is then possible to subject it to the mathematical culture which has arisen in describing logic, by devising methods of calculation in which ideas can be tested out. This initial stage is usually one in which the total system is not immediately expressed in an axiom system and regularised rules of deduction, but it allows further development in which these features are adopted.

The result of the application of this process is that intuitive ideas are collected together and are compared with current mathematical culture. Apologising at this moment for ignoring necessary modesty, in my opinion I have seen an explosion of new ideas and feel that this intensively creative process has given rise to significant mathematics.

We need to continue the process and explain this thinking, for two reasons. The first is that creative ideas are eventually subject to systemisation, so that their contents can be analysed using standard techniques adopted by the mathematical community. The second reason is so that jumps in reasoning which the author thinks are reasonable are often not obvious enough to the reader introduced to this thinking. Thus I must be subject to being forced to describe all stages in my thinking process. Proper mathematical argument is about explicit reasoning, so that all processes are described and a clear argument can be followed without dispute.

The existence of disputes in mathematics is interesting. Thus, having introduced disputes in the first place by possibly controversial intuitive reasoning, an objective becomes to minimise or remove them by the introduction of explicit rules of deduction and symbolic manipulations together with a set of concrete meanings of these symbols which point to the world. This is the method I have chosen by which new symbolic generalisations are developed and new mathematics is created.

The most significant directions of my life have been determined by the labour I wanted to do. Had it been that I were sent to a pig farm in the cultural revolution in China, on adjustment to the collapse of my excessive ambition, I would probably have become an exemplary worker exceeding all quotas, without a thought to my personal promotion. But that would have been a waste. I have received an excellent education that I did not request, and did not find myself succeeding particularly well within it, but having adjusted to the culture of science, I became fascinated by its ideas and content. The wonderful mathematician Ramanujan did not succeed fully at university because he was too interested in his investigations of the subject to follow properly the course work, and he was not interested in social promotion through the system. In the pursuit of my obsessive interests in science, in my life this has befallen me too.

Aesop's fable of the tortoise and the hare ends with the tortoise winning the race. In research a certain level of intelligence is useful, but what is important is persistent curiosity. I am the tortoise. In my research I have exceeded in duration, but despite my wishes, not in collective effort, many Long Marches. The result of this work is no longer mine, it belongs to human culture, but social promotion through the system as a consequence of this is of no interest to me, except where it might promote general social welfare.

At the start of my life my attitude was to ignore the rules and do the right thing. But now I have come to the conclusion that it is very useful to know what the rules are. They can be used to define acceptable and unacceptable behaviour, and to navigate through the social system to attain objectives.

I started my scientific career with an innate conservatism towards scientific conventional wisdom and an aspiration to reach the summit of its understandings. In this respect I treated science as a religion. But at university I began to have doubts as to the correctness of some theories, although I did not have the means or resources to question them properly.

In order to succeed well in the academic system, it is necessary to give the correct answer to questions. Many mathematicians now teaching the subject freely admit that they crammed for examinations, and did not properly understand results, although they were concerned to memorise and replicate them. I believe a number of mathematicians writing on the subject even today do so from the point of view that a proof has to be memorised, and if it is viewed abstractly it is not important to understand properly what is going on, provided the conclusion is reached rigorously.

In pursuit of these interests, time and again in mathematics I am confronted with the fact that I do not understand a proof. When I inspect my own proofs and come to the same conclusion, the answer is easy. However much I am attached to an idea behind a proof, which sometimes comes from experience of the material with which it is concerned, if on reviewing what I am trying to say, the result is no nearer after all the effort, the theorem has to be jettisoned.

There are standard proofs in mathematics which I have not understood after considerable effort at coming to grips with what they are saying. These proofs are sometimes technically erudite and sometimes extremely long. They can be surprising or counterintuitive. I do not believe the majority of mathematicians spend time on inspecting them, but rely on the peer review process to check their validity. Any doubts are allayed if the result is well-established and a considerable confirmatory literature surrounds it.

I have learnt that if, after considerable investigations, such a result is no clearer, that initial doubts are confirmed, and that the theorem is counterintuitive, a not unreasonable strategy is to assume that this does not arise from my own innate stupidity, but that the result is wrong. This at least allows an entry into other approaches to investigate what really might be the situation, since there is a logical mode of deduction which states that to prove a result, first assume the opposite and then prove a contradiction. Very often when I try this approach, the result is not a contradiction but an escape route, and if after much research an escape route cannot be closed off, persistent investigation has led to a refutation of current findings.

To begin with, this astonished me, and an implication going beyond the proclaimed rigorous methods of mathematics to subjects of weaker intent is that much human reasoning of the current day is suspect. But now in new work I have taken this conclusion not as an end view in my investigations, but its starting point.

As a word of encouragement, to make progress in mathematics it is best to assume that the truth can be arrived at by a process of successive corrections of theory. If I look at the work of Aristotle, whose deep thinking and wisdom ranged over history, logic, physics, biology and philosophy in a way that seems impossible because of intense specialisation today, we see someone who is concerned to find the truth when others of his generation were not. There is an erudite and difficult passage in his work where he deals with the motion of the moon. It is apparent 2200 years later that, not to put too fine a point on it, it is a load of codswallop. I feel that some intimidated students might have been examined on it, and heaven help them if they got the wrong answer. It has been said that although Aristotle had many false ideas, that the history of much of scientific thought has consisted in correcting them. But it is not possible to make much progress when no prior thinking has been made of a topic. That progress has been made in mathematics has been contingent on the fact that those studying the subject had some wrong ideas but did not stay silent.

I wish to conclude this section with a remark on learning and unlearning. In general, I have found no problems in presenting my ideas to undergraduates in their twenties who are studying for degrees in a science subject, particularly mathematics and physics. These ideas are absorbed and accepted without trauma. The reason that revolution has its current meaning is that Copernicus's book *On the revolutions of the heavenly spheres* could not be cognitively accepted by those who had been trained in a different outlook and had accepted it in their earlier years. I have found that those who have been taught the subject of mathematics in their youth, have reached maturity and maybe have been teaching this too, and have accepted this teaching cannot accommodate to the double burden of learning this new material which is at variance to what they have accepted, and concurrently unlearning what has been embedded in their way of thinking for very many years. A response for some such younger people already trained is to find at every opportunity ways of undermining these ideas, always framed from the point of view of total acceptance of what they have been taught, and for such people who are older, of refusing to engage in any way with these ideas, and even the refusal to read anything about them. Thus there is a task I face which is a difficult one. I do not know enough about psychology to understand the proper techniques which people can use to unlearn a subject. I know that when learning a piece of music, if you introduce an error into the performance of it, it becomes necessary to unlearn the error, and this is difficult, because the error becomes embedded in the way the mind operates in producing automatic responses. Have I then been able to escape from this situation myself? Perhaps I have an immediate readiness, frequently applied, of being able to change my conclusions in the face of evidence, together with the determination to find the facts and long term to maintain my stance or suspend my judgement when I believe the evidence is unproven. The only other way I can think that the mind uses to unlearn responses is to cry, but this is a speculation. I think this situation is about natural behaviour and needs to be viewed with compassion and understanding. In the long term, the logic of the viewpoint I am presenting will win out. That it may take a long time is unfortunate, but given the human condition this is understandable.

The author's responsibility

This work is provided in the Mathematics section of the website www.jimhadams.org, where a brief synopsis of each chapter is provided. It is the sequel to *Superexponential algebra* [Ad15], displayed after *Number, space and logic* in the Mathematics section. Nevertheless, I have made *Number, space and logic* depend less on [Ad15] by taking some material from it.

A central objective of this work is to encourage the graduate to produce mathematical ideas, and also to give assistance by providing a minimally fussy exposition of the research into number theory of the last fifty years. We are seeking to foster insight so that the reader can pursue further investigations into the technical literature in a spirit of an understanding of its background.

Except for marketing, our inclination is that a proof is only the final stage in the presentation of a mathematical idea. Proofs are necessary, but not sufficient.

Since I am aware that the first language of the reader may not be English, I have looked at the text and removed high literary style. Technical terms are I hope well explained, and examples are given.

I am surrounded by people who definitely dislike and do not relate to abstraction, particularly mathematical abstraction. Some writers in our subject use abstract methods of reasoning to obscure simple ideas and clothe them in a system of thought which is remote. But abstraction can also reveal, and even attain ends which cannot be achieved by other means. Mathematics is best written in a style appropriate to its audience. For a graduate text I am confronted with the need and indulgent necessity, if given enough time, of writing in a rigorous mathematical style that will appeal to a mature mathematical audience. I have attempted, after much effort in attaining results, to rewrite this text in a clear classical style that reveals the subject as well as developing it.

Exercises are not provided to work through the text, the reason being that these are no more needed for the development of creative skills than the memorisation of words and standard literary works is necessary for an active participation in becoming an author. My hope is that the diligent reader will provide what is necessary for constructive work, and that it is no longer necessary to give direction.

I thank Graham Ennis, who has sustained and inspired me with his encouragement. I have found John Baez's website beautifully presented and a treasure-trove of links to interesting and accessible accounts of the research literature. Particularly useful to my mathematical development have been the books *A first course in modular forms* by Fred Diamond and Jerry Shurman [DS05], and *Sphere packings, lattices and groups* by John H. Conway and Neil Sloane [CN98]. Research needs ideas, and also the people to pull them back to Earth. As well as the usual fulsome acknowledgement to Doly García in her criticism of this work, who does not disagree with everything I say because I can quite easily adopt her suggestions, I wish to express especially my gratitude to the stimulating interest of Jim Hamilton whose calculational checks amounted to an interactive collaboration on the chapter on polynomial wheel methods and comparison techniques. The contents of this work are my own, as is the responsibility for any errors.

Jim H. Adams
Dublin
September 2017

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Definitions of mathematical terms

The following terms, symbols, ideas and definitions are used in the text. The arrangement is by ideas rather than alphabetic. This may be scanned as a more technical alternative to the details of the contents in the Foreword, or as a further introduction to the contents which follow in the eBook. The notation, for example (SA III), refers to *Superexponential algebra* chapter III, and (NSL I, IV) refers to this work, volume I, chapter IV.

1. *Sets* S, T (SA III).
$\varnothing$ (*the empty set*). The set with no members.
$\odot$ (*the void set*). The set satisfying a false condition.
$\in$ (*belongs to*). If x is a member of a set S then $x \in S$.
$\subset$ (*properly included in*). If a set S is included in a set T and S does not equal T.
$\subseteq$ (*included in*). Inclusion, when S = T is possible.
CS_T (*complement of* S *in* T). Those x not belonging to S but that belong to T, and $S \subset T$.
$\cup$ (*union of sets*). If x belongs to S or x belongs to T then x belongs to $S \cup T$.
$\cap$ (*intersection of sets*). If $x \in S$ and $x \in T$ then $x \in S \cap T$.

2. $\mathbb{N}$ is the set {1, 2, 3 ...} of *positive whole numbers*, also called *natural numbers*. If this set contains the element 0, we denote it in this eBook by $\mathbb{N}_{\cup 0}$. If we wish to emphasise that it does not contain zero, we use $\mathbb{N}_{\neq 0}$.
$\mathbb{Z}$ (from the German Zahl for number) is the set {..., -3, -2, -1, 0, 1, 2, ...} of negative, zero or positive *integers*.
$\mathbb{Q}$ is the set of *rational numbers* m/n, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, for example ½.
$\mathbb{A}$ is used in this eBook as the set of *algebraic numbers*, sums and differences of numbers of the form p^q, where $p, q \in \mathbb{Q}$, but p and q together are not both zero, for example $1 + 2\sqrt[3]{\frac{-1}{5}}$.
$\mathbb{M}_t$ is the set of *transnatural numbers* for index t, satisfying the rules for $\mathbb{N}$. $\mathbb{M}_1 = \mathbb{N}$, with proper injections $\mathbb{M}_t \to \mathbb{M}_{t+1}$, $\mathbb{M}_t \to \mathbb{M}'$ and surjections $\mathbb{M}' \to \mathbb{M}_{t+1}$ for distinct $\mathbb{M}'$.
$\mathbb{Z}_t$ *Transintegers* are positive, zero or negative transnatural numbers.
$\mathbb{Q}_t$ *Transrational numbers* are of the form m/n where $m \in \mathbb{Z}_t$ and $n \in \mathbb{M}_t$.
$\mathbb{A}_t$ *Transalgebraic numbers* are sums and differences of the form p^q, where $p, q \in \mathbb{Q}_t$, or alternatively formed in a similar way from superexponential operations.
$\mathbb{R}$ *Real numbers* have no imaginary component but are possibly not transalgebraic.
p-adic number. Arithmetic with a nonstandard idea of closeness between numbers.

3. *Peano axioms*. The rules for natural number arithmetic.
UCH. The false *uncountable continuum hypothesis* that $\mathbb{R}$ is bijective to $2^{\mathbb{N}}$.
ZFC. *Zermelo-Fraenkel* standard set theory with the axiom of choice.
mZFC. *Modified ZFC*, allowing a set satisfying a false condition (but not invalid choice).
Propositional calculus. The logic of truth tables for true and false.
Predicate calculus. Propositional calculus allowing the statement 'there exists'.
Untwisted logic. Logic for a set embedded in an oriented manifold.
Twisted logic. Logic for a set embedded in a twisted manifold.

4. *Congruence arithmetic* (*mod n*). Finite or ‘clock’ arithmetic where transnatural numbers come back to themselves, so its set is {0, 1, ... (n – 1)} and n = 0.
Prime number. A transnatural number which only when divided by 1 and itself gives a transnatural number. Example: 7.
Totient (φ(s)). For a transnatural number s as a product of primes p, q, ... r to powers j, k, ... m, if s = $(p^j)(q^k)\dots(r^m)$ then $\varphi(s) = s\left[\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\cdots\left(1-\frac{1}{r}\right)\right]$. Example: φ(20) = 8.

5. *Abelian*. Occurs for a set with a general operation + (not necessarily addition as usually understood) when a + b = b + a always.
Commutative. Abelian, but generally written for × rather than +.
Associative. Satisfying a + (b + c) = (a + b) + c, or a(bc) = (ab)c, etc.

6. *Eudoxus numbers*, 𝕌. Any number which multiplied by some integer can have a size within a range of finite positive natural numbers.
Complex numbers, ℂ. Numbers of the form a + bi, where a, b ∈ 𝕌 and i = √-1.
Gaussian integers. Complex numbers where a and b above are integers.
𝔽 is a *field* (SA III). It contains axioms (rules) for addition and multiplication. Examples could be the Eudoxus numbers 𝕌 and complex numbers ℂ.
𝕐 is a *zero algebra* (SA III). This is similar to a field except for the existence of multizeros.
Exponential algebra, (SA XII and XIII). Contains axioms for exponentiation.
𝔻w *exponential algebra*, (SA XIII). A nonstandard exponential algebra.
Superexponentiation (chapter III and SA XV). An operation, of which the first three are addition, multiplication and exponentiation, where the nth is found by repeating the (n – 1)th.
Superstructure. Contains axioms connecting superexponential nth operations for various n.

7. *Standard protocol*. (chapter I). The ordinal infinity $\mathbf{\Omega}_{\mathbb{N}} = \sum_{\text{all } \mathbb{N}} 1$. This is not a natural number, and is treated as being irreducible.
Ladder number. A superexponential expression in $\mathbf{\Omega}_{\mathbb{N}}$, with Eudoxus coefficients.
Strict transfer principle. The axioms for variables in a superexponential algebra also hold for the variable $\mathbf{\Omega}_{\mathbb{N}}$.
Capital Ξ function. Satisfies $1^{\Xi(a)}$ = a.

8. *Implies*. (exercise, SA XI). For statements A and B, A implies B is only false when A is true and B is false.
Sufficient. A is sufficient for B means A implies B.
Necessary. A is necessary for B means A is implied by B (the same as B implies A).

9. *Function* (SA III). A set of pairs, {x, f(x)}, all x of which have a value f(x).
Injection. A mapping from all the sets {a, b} to {f(a), f(b)}, where f(a) ≠ f(b) if a ≠ b.
Surjection. A mapping where every f(x) in the set {f(x)} has a value from an x.
Bijection. A mapping which is simultaneously injective and surjective.

10. *Magma*, M (SA III). A set with one binary operation, with no other properties specified.
Polymagma (SA XV). Maps a number of copies of a set to itself.

Group, G (SA III). This satisfies the multiplicative axioms for a field, except multiplication may be noncommutative: ab $\neq$ ba.
Subgroup. A set of elements in a group which satisfies within itself all the properties of the containing group.
Order of a group. The number of elements (or members) in a group.
Homomorphism of groups is a surjective map h: G $\to$ G$'$ of groups with h(ab) = h(a)h(b).
Isomorphism of groups. A bijective homomorphism.
Automorphism of a group is an isomorphism of a group to itself.
Inner automorphism of a group is an automorphism of the form x $\leftrightarrow$ $a^{-1}xa$.
Outer automorphism of a group. An automorphism which is not inner.
Normal subgroup is invariant under all inner automorphisms of the containing group G.
Simple group has no normal subgroups other than itself and 1.

11. *Ring*, A. Satisfies the additive and multiplicative axioms of a field, except there is no general division and multiplication may be noncommutative. Example: matrices.
Unital ring. A ring with a multiplicative identity, 1. We assume rings are unital.
Automorphism of a ring. (SA X). A bijective map of a ring A, H: A $\leftrightarrow$ A, where H(ab) = H(a)H(b) and H(a + b) = H(a) + H(b).

12. *Vector*, **v** (in bold). A matrix as one row (a row vector), or as one column (a column vector). Example: the row vector (x, y, z).
Vector space. Contains vectors with magnitude and direction, which can be added together and multiplied by scalars in a field.
Base point. The origin for a vector space.
Module. A module over a ring is a generalisation of a vector space over a field, being an additive abelian group like a vector space, where the scalars are the elements of a ring.
Scalar product of two vectors. The matrix product of multiplying each element of a row vector in turn with the corresponding elements of a column vector. Example: $x^2 + y^2 + z^2$.
Eigenvector. A vector **x** satisfying for matrix B, B**x** = λ**x**.
Eigenvalue. A value λ for the eigenvector **x** above. Example: λ is a complex root value.

13. *Matrix* (*plural matrices*). An array of numbers B = b_{jk}, where the element b_{ij} exists in the i^{th} row and j^{th} column. (SA I and II).
Symmetric matrix. U = $u_{jk} = u_{kj}$.
Antisymmetric matrix. V = $v_{jk} = -v_{kj}$.
Matrix transpose. If W = w_{jk} , then the transpose $W^T = w_{kj}$.
Unit diagonal matrix. Denoted by I = b_{jk}, where b_{jk} = 1 when j = k, otherwise b_{jk} = 0.
Trace of a matrix. The sum of all (main) diagonal entries b_{jk}, where j = k.
Determinant (*or hypervolume*) *of a matrix*, (*det* B). (SA I and II).
Singular matrix, D. Satisfies det D = 0.
Units, K*. The invertible elements of a ring, for example giving det $\neq$ 0.
Block. An n-dimensional array of numbers.
Block scalar product of two blocks. A scalar value obtained from blocks using scalar products of vectors and determinants of matrices.

14. *Intricate number*. A representation of 2×2 matrices, that is, with two rows and two columns, given by $a1 + bi + c\alpha + d\phi$. (SA I).
Intricate basis element. One of the vectors 1, i, α or ϕ above.
Real basis element. The number 1 in its intricate representation.
Imaginary basis element. The number i in its intricate representation.
Actual basis element. The number α in its intricate representation.
Phantom basis element. The number ϕ in its intricate representation.
Intricate conjugate. The number $a1 - bi - c\alpha - d\phi$.
J. $J = bi + c\alpha + d\phi$ in which $J^2 = 0$ or ± 1.
JAF. A changed basis for i, α and ϕ.

15. *Hyperintricate number*. A representation of $2^n \times 2^n$ matrices. (SA II).
Layer. For example, a hyperintricate number with a component in 3 layers is $A_{B,C}$ where A, B and C are intricate numbers, possibly intricate basis elements.
n-hyperintricate number. A hyperintricate number representable by sums of components in n layers. Sometimes denoted by $Я_n$.
n-hyperintricate conjugate, $Я^*_n$. Satisfies $Я^*_n Я_n = \det Я_n$.
J-abelian hyperintricate number. A number giving the example $A_B + ... + D_E$, where $A = p1 + qJ$, $B = p'1 + q'J'$, ..., $D = t1 + uJ$, $E = t'1 + u'J'$. Two such hyperintricate numbers with identical J and J′ commute.

16. *Vulcannion algebra*, or *division algebra*. (SA III and V). A division ring where multiplication might be nonassociative. Multiplying two elements of such an algebra cannot give zero unless one of them is zero.
Zargonion algebra. A vulcannion algebra except possibly for zero scalar components.
Quaternions. (SA III and V). A type of associative division algebra.
Octonions, $\mathbb{O}$. (SA V). A nonassociative division algebra.
n-vulcannions. General nonassociative division algebras of dimension $n = 6k + 2$.
Vulcan number, v. The dimension of a vulcannion minus one – the number of its space components.
T-junction. A diagram used to classify vulcannions.
n-novanions. (SA V). An n dimensional nonassociative division algebra, but not when both the real parts in a multiplication are zero.
Zargonion. A combination of algebras obtained from vulcannions and novanions.

17. *Norm*. Applied to complex numbers a + bi, the norm is $\sqrt{(a^2 + b^2)}$. For intricate numbers $a1 + bi + c\alpha + d\phi$ the norm squared is $a^2 + b^2 - c^2 - d^2$. Applied to a $n \times n$ matrix B, the norm is the positive nth root of det B. Applied to n-zargonions $a1 + bi + c\alpha + d\phi + b'i' + c'\alpha' + d'\phi' + ...$, the norm is $\sqrt{(a^2 + b^2 + c^2 + d^2 + b'^2 + c'^2 + d'^2 + ...)}$.

18. *Additive format of a polynomial equation*. The form $ax^n + bx^{n-1} + ... + d = 0$. (SA VII and VIII).
Monic polynomial. Example in the case of a polynomial equation: when a above = 1.
Fundamental theorem of algebra. The complex polynomial in additive format given by $ax^n + bx^{n-1} + ... + d$ always has some values which are zero.

Multiplicative format of a polynomial equation. The form $(x - p)(x - q) \ldots (x - t) = 0$.
Zero of a polynomial. A value of a polynomial $f(x) = ax^n + bx^{n-1} + \ldots + d$ so that $f(x) = 0$.
Root of a polynomial equation. The roots of a polynomial equation $f(x) = 0$ are the values of x satisfying this.
Equaliser of two polynomials. The intersection of their values.
Degree of a polynomial. The value of n for f(x).
Duplicate root. A root of the equation $(x + a)^2 = 0$.
Antiduplicate root. A root of the equation $(x + a)(x - a) = 0$.
Independent roots. Occur when no known dependency relation is used in the solution of a polynomial equation.
Dependent roots. Occur when a known dependency relation is used in the solution of a polynomial equation.
Multivariate polynomial. A polynomial in a number of variables.
Variety. A polynomial equation in a number of variables. Example: $3x^2y + xyz + 4x^2z^2 = 0$.

19. *Equivalence relation ≡ in a set S*. Satisfies $a \equiv a$ (*reflexive*), if $a \equiv b$ then $b \equiv a$ (*symmetric*) and if $a \equiv b$ and $b \equiv c$ then $a \equiv c$ (*transitive*), for $a, b, c \in S$.
Equivalence class. A partition of a set where an equivalence relation between elements defines membership of the partition.
Partial order ≤ of a set S. Satisfies $a \leq a$, if $a \leq b$ and $b \leq a$ then $a = b$ (*antisymmetric*) and if $a \leq b$ and $b \leq c$ then $a \leq c$, for $a, b, c \in S$.
Poset. A partially ordered set.
Total order ≤ of a set S is a partial order existing for all $a, b, c \in S$.
Well-ordering ≤ of a set S. A total order where every nonempty subset has a least element.

20. *Left* (*or right*) *coset* of a subgroup S of G is the set of elements aS (or respectively Sa), with $s \in S$ and $a \in G$.
Quotient group G/S *of* G mod S. The family of left cosets of the group G with subgroup S, sG, $s \in S$.

21. *Ideal*, C. (SA III and XI). A subset of a ring, A, with the rule that $\{c, d\} \in C$ and $a \in A$ implies $(c - d) \in C$ and both ac and ca $\in C$.
Principal ideal, (a). The ideal generated by one element, a, of the ring A. For every $r \in A$, (a) is ra. Example: for $a \neq 0$ belonging to the integers $\mathbb{Z}$, $(3a) \subset (a) \subset \mathbb{Z}$.
Prime ideal, P. If a and b are two elements of A such that their product ab is an element of P, then a or b is in P, and P is not equal to the whole ring A. Example: integers containing all the multiples of a given prime number, together with zero. Example: the zero ideal (0).
Maximal ideal, M. In any ring A, this is an ideal M contained in just two ideals of A, M itself and the entire ring A. Every maximal ideal is prime. Nonexistence: the zero ideal (0) is not a maximal ideal of $\mathbb{Z}$ because $(0) \subset (2) \subset \mathbb{Z}$, nor is the ideal (6), since $(6) \subset (2) \subset \mathbb{Z}$.
Nilradical, N(A). The intersection of all prime ideals of a ring.

22. *Open set*. Example: the interval $a < x < b$ with the end points a and b removed.
Closed set. Example: the interval $a \leq x \leq b$ with the end points a and b present.
Topology. A theory of space using open and closed sets.

23. *Exact sequence.* (SA III).
Homology. A theory of holes. The dimension of the nth homology is the number of holes in a space for dimension n.

24. *Euler-Poincaré characteristic*. In 3-space the number of vertex points – edges + faces of a space divided into n-dimensional polygons.
Möbius strip. A reconnected rectangle with a twist.
Handle. Obtained on a surface by cutting out two holes and gluing in a cylinder.
Crosscap. Obtained on a surface by cutting out a hole and gluing in a Möbius strip.

25. *Graph*. A set of vertices and arrows (or edges) with a mapping from origin to terminus, and provided with a reverse mapping changing orientation.
Path. A finite sequence of edges with the terminus of each edge connecting to an origin of the next edge.
Circuit. A path with its end vertices connected together (start origin = end terminus vertex).
A *graph is connected* if all vertices are contained in a path.
Tree. A connected nonempty graph without circuits.
Node. An origin or terminus in a tree.
Parent node. An origin in a tree.
Child node. A terminus in a tree.
Root of a tree. A child with no parent.
Leaf of a tree. A parent with no child.

26. *Homotopy*. A theory of paths through a topological space.
Winding number. The number of times a loop winds round a point.

27. *n-branched space*. A space where the removal of a point disconnects the space into n pieces.
Explosion. An n-branched space with an infinite or transinfinite number of points.
Supernorm. An evaluation of the magnitude of an explosion.
Branch number. An evaluation of the number of branches in an explosion.

28. *Deformation retract*. The set of all occurrences of a vector transported along another vector.
Branched vector. A directed tree, with arrows proceeding from its root and splitting to its branches.
Branched deformation retract. A branched vector transported and split into copies along another branched vector.
Amalgam. A branched retract with some of its nodes connected.

29. *Morphism*. An associative mapping with identity in category theory.
Category. A description of mathematics in terms of morphisms.
Object. The element where a morphism comes from or goes to.
Arrow. A morphism considered as a directed mapping between objects.
Hom-set. The description of categories as a collection of arrows.

Functor (*covariant*). Describes states and transformations as if they were on the same footing. If g and f are morphisms, and T is a functor, it is covariant if T(gf) = T(g)T(f).
Contravariant functor. A functor reversing the order of composition: T(gf) = T(f)T(g).
Morphism of functors (*natural transformation*). An example is the determinant, as a transformation from commutative rings to groups.
Topos. A categorical description of a set.
Adjoint functor. A specific type of interrelationship between functors, of general use in mathematics.
Kan extension. The combinations of mappings between two sets are described in terms of hom-sets by an exponential, and this can be differentiated or integrated. Kan extensions implement this idea.

30. *Explanation*. A theory or theorem using matrices or other combinatorial means.
Charade. The image of an explanation in homological algebra.
Deconstruction. The mathematical refutation of a generally accepted result.