# CHAPTER 13

## Zargon lattices

## 13.1. Invitation to the reader.

The reader may find consulting the reference *Sphere packings, lattices and groups* (SPLAG) by J.H. Conway and N.J.A. Sloane [CS13] useful for this chapter.

Groups are the most investigated topic for superstructures with one operation. The theory of zargonions leads us in a natural way to ask what impact zargonions have on the theory of groups, even the classification of simple groups, for which the claim has been made that a complete classification was achieved after monumental efforts finally in 2008.

The application of the zargonion idea to these groups is a new subject for which there are currently few experts, and not initially the writer. Nevertheless, there are logical deductions which can be made, and I shall be making them. Since the conclusions do not tie in with what is believed to be a correct proof of the classification, which encompasses work by many individuals some of whom have diligently spent a lifetime on the study of simple groups, a question arises as to the source of this discrepancy. But it is our belief that the origin of this has now been isolated, which we describe to begin with informally in the next section.

The expert in group theory is invited to correct any misconceptions so that revisions can be made, and so the mathematician in the street can be better informed.

## 13.2. Introduction and history.

We introduce an analysis of group theory via the theory of novanions, developed in [Ad15], in the mathematical chapter XVII of *Investigations into universal physics* [Ad18b], by Graham Ennis and me. Adonion groups are found as a generalisation of the way octonions lead to special Lie groups. Using standard theory, we describe the way in which octonions lead to such groups and then extend these ideas to zargonion groups. In our initial attempt to study this subject without a proper understanding of its background, we looked at group structures derived from the nonassociative octonions. We noticed that the octonions have a (mod 12) Jacobi identity mapping to algebras, where octonions are related to some exceptional Lie algebras, like $E_8$, with corresponding mapping to groups. We investigate whether the standard approach and our own more limited methods give the same results here.

The beginning of this study arises out of a speculation of Daniel Hajas that I call the Hajas conjecture, in which novanions are connected with the heterotic string in physics, 10-novanions, to be described, are related to the 10-dimensional fermionic string in physics, and my extension of that idea, that the 26-dimensional bosonic string is related to 26-novanions. We will see we need to go beyond this. The heterotic string has a fermionic dimension of 10 and a bosonic dimension of 26, so that in the conventional picture $26 - 10 = 16$ dimensions in the universe are compactified. We have shown in the above references that 26-novanions, containing octonion type algebras are also present.

We note that there are two distinct zargonion algebras of dimension 26, the 26-adonions and the 26-novanions.

Picking up the work of Borcherds linking the monster simple group to orbifolds in the 26 dimensions of the heterotic string, we investigate whether there are simple groups beyond the monster, in direct conflict with current understandings. Our motivation is as follows. On page 938 of 'Quantum field theory' by Eberhard Zeidler, with reference to the Thompson series, he writes

"Borcherds calculated this series using the monster Lie algebra. This Lie algebra is constructed as the space of physical states of a bosonic string moving in a $\mathbb{Z}_2$ orbifold M/$\mathbb{Z}_2$ of a 26-dimensional torus M".

By this means the classification of simple groups is derived. Preliminary investigation of the work of Borcherds led credence to the Hajas conjecture, but subsequent more detailed investigations using (mod 12) algebra give mismatches in the computed size, called the order, of simple groups derived from novanions by our own methods. According to conventional wisdom this classification is finite. We wish to investigate whether or not a spanner can be thrown in the works, and note that the number of distinct n-novanionic algebras is infinite, given that the override condition in sections 8.13 and 8.14 can always be allocated, which corresponds non-trivially when the n-novanion contains octonionic components. We will look at this proof of the finiteness condition and provide more in-depth investigations to compare standard reasoning with what is available from our own methods.

However, we note an interesting nonstandard conclusion we have arrived at with the Hajas identification, given essentially in [Ad18b] chapter XI, namely that all n-novanions can be allocated bosonic and fermionic parts. Thus both the 10- and 26-novanions may be allocated these components, and this also holds in general for n-novanions.

In chapter XX of [Ad18b] we employ the 26-novanions in Heim theory extended to gluons and quarks. There exist other possible universes with n > 26. So the 26-novanions do indeed contain a bosonic algebra.

A conjecture is: can we apply the unbounded zargonion algebra to derive a Thompson series so that the number of simple groups is not bounded?

We make this statement because override conditions for novanions of dimension greater than 26 lead to simple groups via Jacobi identity mappings to algebras, since overrides mix different quaternions together, creating one overarching structure. However, $E_8$ has order linked to that of the monster. The simple groups we are about to investigate have a size greater than that of the monster.

Although simple groups of similar order of magnitude to the monster have been obtained by zargon methods, they differ. Moreover, there are sequences of simple zargon groups without termination, violating the finiteness condition on the classification of sporadic groups. The groups corresponding to zargon algebras are not initially simple, but quotients with zargon subalgebras lead to factor groups which are simple.

Recent investigation reveals the reason for why the Borcherds and zargon constructions differ. The monster can be derived from vertex operator algebras with infinite quadratic form. Novanion algebras sometimes have zero quadratic form for a zero scalar component, t. These zero forms are not obtained as diagonalised forms which can be obtained by multiplication of the novanion by its conjugate. The Lie algebra construction forbids algebras of zero quadratic form. The scalar t = 0 component is realised in the final novanion algebra in conjunction with other additive elements to form a composite element, but we can map t = 0 → t = ±1, and so

novanions are allowable objects in such algebra constructions, although this has not hitherto been recognised. From this point of view the mapping from 10-novanions to fermion strings and 26-novanions to bosonic strings used in the Borcherds construction is spurious.

I am unclear at this stage whether other aspects of my reasoning are a novelty. These ideas are used to derive zargon groups, and can equally be used for division algebras like the octonions. The groups corresponding to zargon algebras are not initially simple, but quotients with zargon subalgebras lead to factor groups which are simple.

The development has proceeded in chapter 4 by first describing axiom systems which give mathematical structures for numbers and their generalisations. The intention is to provide the background for the discussion which follows. We discuss the classification of simple groups and their order. Various basic terms used in the classification such as the center of a group, factor groups, normal subgroups, Lagrange's subgroup theorem that the order of a subgroup divides the order of its group, and the Schur multiplier were introduced in chapter 6. We derive various finite groups, which are not subgroups of the quaternions, from the quaternion Lie algebra.

We next give a detailed discussion of novanions, which are division algebras when the scalar part is not zero. Octonions are division algebras. We introduce 10-, 26- and more general zargonions. In the case of octonions, we show that they have a finite representation as a Lie algebra (mod 12), from which we may obtain a group. Products in the group which would otherwise be 0 (mod 12) can be converted to $\pm 1$. This maintains the Lie algebra structure and means that the group derived from it is provided with inverses. Other zargonions may have no such Lie algebra structure, nevertheless a mapping of 0 (mod m) to $\pm 1$ may still be made.

## 13.3. Two dimensional lattices and theta functions.

SPLAG 101-102.


One can associate to any (positive-definite) lattice $\Lambda$ a theta function given by

The theta function of a lattice is then a holomorphic function on the upper half-plane. Furthermore, the theta function of an even unimodular lattice of rank n is actually a modular form of weight n/2. The theta function of an integral lattice is often written as a power series

in        so that the coefficient of $q^n$ gives the number of lattice vectors of norm n.

Up to normalization, there is a unique modular form of weight 4: the Eisenstein series $G_4(\tau)$. The theta function for the $E_8$ lattice must then be proportional to $G_4(\tau)$. The normalization can

be fixed by noting that there is a unique vector of norm 0. This gives

where $\sigma_3(n)$ is the divisor function. It follows that the number of $E_8$ lattice vectors of norm 2n is 240 times the sum of the cubes of the divisors of n. The first few terms of this series are given by (sequence A004009 in the OEIS):

The E$_8$ theta function may be written in terms of the Jacobi theta functions as follows:

where

Theta series[edit]

One can associate to any (positive-definite) lattice $\Lambda$ a theta function given by

The theta function of a lattice is then a holomorphic function on the upper half-plane. Furthermore, the theta function of an even unimodular lattice of rank n is actually a modular form of weight $n/2$ for the full modular group $PSL(2,\mathbf{Z})$. The theta function of an integral

lattice is often written as a power series in         so that the coefficient of $q^n$ gives the number of lattice vectors of squared norm 2n. In the Leech lattice, there are 196560 vectors of squared norm 4, 16773120 vectors of squared norm 6, 398034000 vectors of squared norm 8 and so on. The theta series of the Leech lattice is

where         is the normalized Eisenstein series of weight 12,         is the modular discriminant,         is the divisor function for exponent 11, and         is the Ramanujan tau function. It follows that for $m\geq1$ the number of vectors of squared norm $2m$ is

## 13.4. Sphere packings and kissing numbers.

SPLAG Ch 1.

The E$_8$ lattice is remarkable in that it gives optimal solutions to the sphere packing problem and the kissing number problem in 8 dimensions.

The sphere packing problem asks what is the densest way to pack (solid) n-dimensional spheres of a fixed radius in $\mathbf{R}^n$ so that no two spheres overlap. Lattice packings are special types of sphere packings where the spheres are centered at the points of a lattice. Placing spheres of radius $1/\sqrt{2}$ at the points of the E$_8$ lattice gives a lattice packing in $\mathbf{R}^8$ with a density of

It has long been known that this is the maximum density that can be achieved by a lattice packing in 8 dimensions.[6] Furthermore, the E$_8$ lattice is the unique lattice (up to isometries and rescalings) with this density.[7] Mathematician Maryna Viazovska has recently shown that this density is, in fact, optimal even among irregular packings.[8][9]

The kissing number problem asks what is the maximum number of spheres of a fixed radius that can touch (or "kiss") a central sphere of the same radius. In the $E_8$ lattice packing mentioned above any given sphere touches 240 neighboring spheres. This is because there are 240 lattice vectors of minimum nonzero norm (the roots of the $E_8$ lattice). It was shown in 1979 that this is the maximum possible number in 8 dimensions.[10][11]

The sphere packing problem and the kissing number problem are remarkably difficult and optimal solutions are only known in 1, 2, 3, 8, and 24 dimensions (plus dimension 4 for the kissing number problem). The fact that solutions are known in dimensions 8 and 24 follows in part from the special properties of the $E_8$ lattice and its 24-dimensional cousin, the Leech lattice.

## 13.5. Hamming codes.

SPLAG Ch 3.

The $E_8$ lattice is very closely related to the (extended) Hamming code $H(8,4)$ and can, in fact, be constructed from it. The Hamming code $H(8,4)$ is a binary code of length 8 and rank 4; that is, it is a 4-dimensional subspace of the finite vector space $(\mathbf{F}_2)^8$. Writing elements of $(\mathbf{F}_2)^8$ as 8-bit integers in hexadecimal, the code $H(8,4)$ can by given explicitly as the set

{00, 0F, 33, 3C, 55, 5A, 66, 69, 96, 99, A5, AA, C3, CC, F0, FF}.

The code $H(8,4)$ is significant partly because it is a Type II self-dual code. It has a minimum Hamming weight 4, meaning that any two codewords differ by at least 4 bits. It is the largest length 8 binary code with this property.

One can construct a lattice $\Lambda$ from a binary code $C$ of length $n$ by taking the set of all vectors $x$ in $\mathbf{Z}^n$ such that $x$ is congruent (modulo 2) to a codeword of $C$.[12] It is often convenient to rescale $\Lambda$ by a factor of $1/\sqrt{2}$,

Applying this construction a Type II self-dual code gives an even, unimodular lattice. In particular, applying it to the Hamming code $H(8,4)$ gives an $E_8$ lattice. It is not entirely trivial, however, to find an explicit isomorphism between this lattice and the lattice $\Gamma_8$ defined above.

## 13.6. The Golay code.

References: p 649 SPLAG.

The binary Golay code, independently developed in 1949, is an application in coding theory. More specifically, it is an error-correcting code capable of correcting up to three errors in each 24-bit word, and detecting a fourth. It was used to communicate with the Voyager probes, as it is much more compact than the previously-used Hadamard code.

Quantizers, or analog-to-digital converters, can use lattices to minimise the average root-mean-square error. Most quantizers are based on the one-dimensional integer lattice, but

using multi-dimensional lattices reduces the RMS error. The Leech lattice is a good solution to this problem, as the Voronoi cells have a low second moment.

The vertex algebra of the two-dimensional conformal field theory describing bosonic string theory, compactified on the 24-dimensional quotient torus $\mathbf{R}^{24}/\Lambda_{24}$ and orbifolded by a two-element reflection group, provides an explicit construction of the Griess algebra that has the monster group as its automorphism group. This **monster vertex algebra** was also used to prove the monstrous moonshine conjectures.

## 13.7. The novanion $\Pi_{9,1}$ Lorentzian lattice.

Except as 13.8 does not seem to be represented in SPLAG.

## 13.8. The $E_8$ lattice.

SPLAG 10-, 120-, 340-, 406-, 427-, 461-.

For any basis of $\mathbb{R}^n$, the subgroup of all linear combinations with integer coefficients of the basis vectors forms a lattice, which can be viewed as a regular tiling of a space by a primitive cell called the fundamental parallelotope. A *unimodular* lattice is an integral lattice with determinant or hypervolume 1 or −1.

The $E_8$ lattice is a special lattice in $\mathbb{R}^8$. It can be characterised as the unique positive-definite, even unimodular lattice of rank 8, so that it can be generated by the columns of an 8×8 matrix where the determinant of the fundamental parallelotope of the lattice is ±1. The name derives from the fact that it is the root lattice of the $E_8$ root system, described in [Ad15], chapter IV.

The *norm* of the $E_8$ lattice (divided by 2) is a positive definite even unimodular quadratic form in 8 variables, and conversely such a quadratic form can be used to construct a positive-definite, even, unimodular lattice of rank 8. The existence of such a form was first shown by H. Smith in 1867 and the first explicit construction of this quadratic form was given by A. Korkin and G. Zoltarev in 1873. The $E_8$ lattice is also called the *Gosset lattice* after T. Gosset who was one of the first to study the geometry of the lattice around 1900.

The $E_8$ lattice is a discrete subgroup of $\mathbb{R}^8$ which spans all of $\mathbb{R}^8$. It can be given explicitly by the set of points $\Gamma_8 \subset \mathbb{R}^8$ such that
- all the coordinates are integers or half-integers but not a mixture of these
- the sum of the eight coordinates is an even integer.

The sum of two lattice points is another lattice point, so that $\Gamma_8$ is indeed a subgroup.

An alternative description of the $E_8$ lattice is the set of all points in $\Gamma'_8 \subset \mathbb{R}^8$ such that
- all the coordinates are integers and the sum of the coordinates is even, or
- all the coordinates are half-integers and the sum of the coordinates is odd.

The lattices $\Gamma_8$ and $\Gamma'_8$ are isomorphic – we can pass from one to the other by changing the signs of any odd number of coordinates. The lattice $\Gamma_8$ is called the *even coordinate system* for $E_8$ while the lattice $\Gamma_8'$ is called the *odd coordinate system*. Unless we specify otherwise we will work in the even coordinate system.

Even unimodular lattices occur only in dimensions divisible by 8. In dimension 16 there are two such lattices: $\Gamma_8 \oplus \Gamma_8$, and $\Gamma_{16}$ constructed analogously to $\Gamma_8$. In dimension 24 there are 24 such lattices, called Niemeier lattices, the most important of which is the Leech lattice.

A possible basis for $\Gamma_8$ is given by the columns of the upper triangular matrix

$$\Gamma_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$\Gamma_8$ is then the integral span of these vectors. All other possible bases are obtained from this one by right multiplication by elements of GL(8, $\mathbb{Z}$).

The shortest nonzero vectors in $\Gamma_8$ have norm 2. There are 240 such vectors.
- All half-integer: (can only be $\pm 1/2$)
  - All positive or all negative: 2
  - Four positive, four negative: $(8 \times 7 \times 6 \times 5)/(4 \times 3 \times 2 \times 1) = 70$
  - Two of one, six of the other: $2 \times (8 \times 7)/(2 \times 1) = 56$
- All integer: (can only be 0, $\pm 1$)
  - Two $\pm 1$, six zeroes: $4 \times (8 \times 7)/(2 \times 1) = 112$

These form a root system of type $E_8$. The lattice $\Gamma_8$ is equal to the $E_8$ root lattice, which means that it is given by the integral span of the 240 roots. Any choice of 8 simple roots gives a basis for $\Gamma_8$.

<div align="center">**Subsequent text will be revised substantially**</div>

The automorphism group (or symmetry group) of a lattice in $\mathbb{R}^n$ is defined as the subgroup of the orthogonal group O(n) that preserves the lattice. The symmetry group of the $E_8$ lattice is the Weyl/Coxeter group of type $E_8$. This is the group generated by reflections in the hyperplanes orthogonal to the 240 roots of the lattice. Its order is given by

The $E_8$ Weyl group contains a subgroup of order $128 \cdot 8!$ consisting of all permutations of the coordinates and all even sign changes. This subgroup is the Weyl group of type $D_8$. The full $E_8$ Weyl group is generated by this subgroup and the block diagonal matrix $H_4 \oplus H_4$ where $H_4$ is the Hadamard matrix

## 13.9. The adonion $\Pi_{13,1}$ Lorentzian lattice.

Except as 13.10 not identified in SPLAG.

### 13.10. The Coxeter-Todd lattice.

SPLAG 13-, 127-, 159-.

### 13.11. The novanion and adonion $\Pi_{25,1}$ Lorentzian lattices.

The Leech lattice can also be constructed as        where $w$ is the Weyl vector:

in the 26-dimensional even Lorentzian unimodular lattice $\text{II}_{25,1}$.

By Faulhaber's formula
$$1^2 + 2^2 + \cdots n^2 = \frac{2n^3 + 3n^2 + n}{6}.$$
The existence of such an integral vector of norm zero relies on the fact that $1^2 + 2^2 + ... + 24^2$ is a perfect square (in fact $70^2$); the number 24 is the only integer bigger than 1 with this property. This was conjectured by Édouard Lucas, but the proof came much later, based on elliptic functions.

The vector        in this construction is really the Weyl vector of the even sublattice $D_{24}$ of the odd unimodular lattice $I^{25}$. More generally, if L is any positive definite unimodular lattice of dimension 25 with at least 4 vectors of norm 1, then the Weyl vector of its norm 2 roots has integral length, and there is a similar construction of the Leech lattice using L and this Weyl vector.

### 13.12. The Leech lattice.

The *Leech lattice* is an even unimodular lattice $\Lambda_{24}$ in 24-dimensional Euclidean space, which is one of the best models for the kissing number problem. It was discovered by John Leech (1967). It may also have been discovered (but unpublished) by Ernst Witt in 1940.

# Characterization[edit]

The Leech lattice $\Lambda_{24}$ is the unique lattice in $\mathbf{E}^{24}$ with the following list of properties:

* It is unimodular; i.e., it can be generated by the columns of a certain 24×24 matrix with determinant 1.
* It is even; i.e., the square of the length of each vector in $\Lambda_{24}$ is an even integer.
* The length of every non-zero vector in $\Lambda_{24}$ is at least 2.

The last condition is equivalent to the condition that unit balls centered at the points of $\Lambda_{24}$ do not overlap. Each is tangent to 196,560 neighbors, and this is known to be the largest number of non-overlapping 24-dimensional unit balls that can simultaneously touch a single unit ball (compare with 6 in dimension 2, as the maximum number of pennies which can touch a central penny; see kissing number). This arrangement of 196560 unit balls centred about another unit ball is so efficient that there is no room to move any of the balls; this configuration, together with its mirror-image, is the *only* 24-dimensional arrangement where 196560 unit balls simultaneously touch another. This property is also true in 1, 2 and 8 dimensions, with 2, 6 and 240 unit balls, respectively, based on the integer lattice, hexagonal tiling and E8 lattice, respectively.

It has no root system and in fact is the first unimodular lattice with no *roots* (vectors of norm less than 4), and therefore has a centre density of 1. By multiplying this value by the volume

of a unit ball in 24 dimensions,    , one can derive its absolute density.

Conway (1983) showed that the Leech lattice is isometric to the set of simple roots (or the Dynkin diagram) of the reflection group of the 26-dimensional even Lorentzian unimodular lattice $II_{25,1}$. By comparison, the Dynkin diagrams of $II_{9,1}$ and $II_{17,1}$ are finite.

# Constructions[edit]

The Leech lattice can be constructed in a variety of ways. As with all lattices, it can be constructed by taking the integral span of the columns of its generator matrix, a 24×24 matrix with determinant 1.

Leech generator matrix [show]

[1]

## Using the binary Golay code[edit]

The Leech lattice can be explicitly constructed as the set of vectors of the form $2^{-3/2}(a_1, a_2, ..., a_{24})$ where the $a_i$ are integers such that

and for each fixed residue class modulo 4, the 24 bit word, whose 1s correspond to the coordinates $i$ such that $a_i$ belongs to this residue class, is a word in the binary Golay code. The Golay code, together with the related Witt design, features in a construction for the 196560 minimal vectors in the Leech lattice.

## Based on other lattices[edit]

Conway & Sloane (1982) described another 23 constructions for the Leech lattice, each based on a Niemeier lattice. It can also be constructed by using three copies of the E8 lattice, in the same way that the binary Golay code can be constructed using three copies of the extended Hamming code, $H_8$. This construction is known as the *Turyn* construction of the Leech lattice.

### As a laminated lattice[edit]

Starting with a single point, $\Lambda_0$, one can stack copies of the lattice $\Lambda_n$ to form an $(n + 1)$-dimensional lattice, $\Lambda_{n+1}$, without reducing the minimal distance between points. $\Lambda_1$ corresponds to the integer lattice, $\Lambda_2$ is to the hexagonal lattice, and $\Lambda_3$ is the face-centered cubic packing. Conway & Sloane (1982b) showed that the Leech lattice is the unique laminated lattice in 24 dimensions.

### As a complex lattice[edit]

The Leech lattice is also a 12-dimensional lattice over the Eisenstein integers. This is known as the *complex Leech lattice*, and is isomorphic to the 24-dimensional real Leech lattice. In the complex construction of the Leech lattice, the binary Golay code is replaced with the ternary Golay code, and the Mathieu group $M_{24}$ is replaced with the Mathieu group $M_{12}$. The $E_6$ lattice, $E_8$ lattice and Coxeter–Todd lattice also have constructions as complex lattices, over either the Eisenstein or Gaussian integers.

### Using the icosian ring[edit]

The Leech lattice can also be constructed using the ring of icosians. The icosian ring is abstractly isomorphic to the E8 lattice, three copies of which can be used to construct the Leech lattice using the Turyn construction.

### Witt's construction[edit]

In 1972 Witt gave the following construction, which he said he found in 1940 January 28.

Suppose that $H$ is an $n$ by $n$ Hadamard matrix, where $n=4ab$. Then the matrix          defines a bilinear form in $2n$ dimensions, whose kernel has $n$ dimensions. The quotient by this kernel is a nonsinguar bilinear form taking values in $(1/2)\mathbf{Z}$. It has 3 sublattices of index 2 that are integral bilinear forms. Witt obtained the Leech lattice as one of these three sublattices by taking $a=2$, $b=3$, and taking $H$ to be the 24 by 24 matrix (indexed by $\mathbf{Z}/23\mathbf{Z} \cup \infty$) with entries $X(m+n)$ where $X(\infty)=1$, $X(0)=-1$, $X(n)=$is the quadratic residue symbol mod 23 for nonzero $n$. This matrix $H$ is a Paley matrix with some insignificant sign changes.

### Using a Paley matrix[edit]

Chapman (2001) described a construction using a skew Hadamard matrix of Paley type. The

Niemeier lattice with root system          can be made into a module for the ring of integers of

the field          . Multiplying this Niemeier lattice by a non-principal ideal of the ring of integers gives the Leech lattice.

### Using octonions[edit]

If $L$ is the set of octonions with coordinates on the          lattice. Then the Leech lattice is the

set of triplets          such that:

where

# Symmetries[edit]

The Leech lattice is highly symmetrical. Its automorphism group is the Conway group $Co_0$, which is of order 8 315 553 613 086 720 000. The center of $Co_0$ has two elements, and the quotient of $Co_0$ by this center is the Conway group $Co_1$, a finite simple group. Many other sporadic groups, such as the remaining Conway groups and Mathieu groups, can be constructed as the stabilizers of various configurations of vectors in the Leech lattice.

Despite having such a high *rotational* symmetry group, the Leech lattice does not possess any hyperplanes of reflection symmetry. In other words, the Leech lattice is chiral.

The automorphism group was first described by John Conway. The 398034000 vectors of norm 8 fall into 8292375 'crosses' of 48 vectors. Each cross contains 24 mutually orthogonal vectors and their negatives, and thus describe the vertices of a 24-dimensional orthoplex. Each of these crosses can be taken to be the coordinate system of the lattice, and has the same symmetry of the Golay code, namely $2^{12} \times |M_{24}|$. Hence the full automorphism group of the Leech lattice has order $8292375 \times 4096 \times 244823040$, or 8 315 553 613 086 720 000.

# Geometry[edit]

Conway, Parker & Sloane (1982) showed that the covering radius of the Leech lattice is ; in other words, if we put a closed ball of this radius around each lattice point, then these just cover Euclidean space. The points at distance at least        from all lattice points are called the ***deep holes*** of the Leech lattice. There are 23 orbits of them under the automorphism group of the Leech lattice, and these orbits correspond to the 23 Niemeier lattices other than the Leech lattice: the set of vertices of deep hole is isometric to the affine Dynkin diagram of the corresponding Niemeier lattice.

The Leech lattice has a density of        . Cohn & Kumar (2009) showed that it gives the densest lattice packing of balls in 24-dimensional space. Henry Cohn, Abhinav Kumar, and Stephen D. Miller et al. (2016) improved this by showing that it is the densest sphere packing, even among non-lattice packings.

The 196560 minimal vectors are of three different varieties, known as *shapes*:

- 1104 vectors of shape $(4^2, 0^{22})$, for all permutations and sign choices;
- 97152 vectors of shape $(2^8, 0^{16})$, where the '2's correspond to octads in the Golay code, and there is an even number of minus signs;
- 98304 vectors of shape $(-3, 1^{23})$, where the changes of signs come from the Golay code, and the '3' can appear in any position.

The ternary Golay code, binary Golay code and Leech lattice give very efficient 24-dimensional spherical codes of 729, 4096 and 196560 points, respectively. Spherical codes are higher-dimensional analogues of Tammes problem, which arose as an attempt to explain the distribution of pores on pollen grains. These are distributed as to maximise the minimal angle between them. In two dimensions, the problem is trivial, but in three dimensions and higher it is not. An example of a spherical code in three dimensions is the set of the 12

vertices of the regular icosahedron.

# History[edit]

Many of the cross-sections of the Leech lattice, including the Coxeter–Todd lattice and Barnes–Wall lattice, in 12 and 16 dimensions, were found much earlier than the Leech lattice. O'Connor & Pall (1944) discovered a related odd unimodular lattice in 24 dimensions, now called the odd Leech lattice, one of whose two even neighbors is the Leech lattice. The Leech lattice was discovered in 1965 by John Leech (1967, 2.31, p. 262), by improving some earlier sphere packings he found (Leech 1964).

Conway (1968) calculated the order of the automorphism group of the Leech lattice, and, working with John G. Thompson, discovered three new sporadic groups as a by-product: the Conway groups, $Co_1$, $Co_2$, $Co_3$. They also showed that four other (then) recently announced sporadic groups, namely, Higman-Sims, Suzuki, McLaughlin, and the Janko group $J_2$ could be found inside the Conway groups using the geometry of the Leech lattice. (Ronan, p. 155)

> Bei dem Versuch, eine Form aus einer solchen Klasse wirklich anzugeben, fand ich mehr als 10 verschiedene Klassen in $\Gamma_{24}$
>
> *Witt (1941, p. 324)*

Witt (1941, p. 324), has a single rather cryptic sentence mentioning that he found more than 10 even unimodular lattices in 24 dimensions without giving further details. Witt (1998, p. 328–329) stated that he found 9 of these lattices earlier in 1938, and found two more, the Niemeier lattice with A24
1 root system and the Leech lattice (and also the odd Leech lattice), in 1940.

**13.13. Moonshine.**

**13.14. The adonion Π₃₁,₁ Lorentzian lattice.**

**13.15. Adonion moonshine.**

**13.16. Enveloping Zargon moonshine.**

# Geometry[edit]

See 5₂₁ honeycomb

The $E_8$ lattice points are the vertices of the 5₂₁ honeycomb, which is composed of regular 8-simplex and 8-orthoplex facets. This honeycomb was first studied by Gosset who called it a *9-ic semi-regular figure*[4] (Gosset regarded honeycombs in n dimensions as degenerate n+1 polytopes). In Coxeter's notation,[5] Gosset's honeycomb is denoted by 5₂₁ and has the Coxeter-Dynkin diagram:



This honeycomb is highly regular in the sense that its symmetry group (the affine       Weyl group) acts transitively on the $k$-faces for $k \leq 6$. All of the $k$-faces for $k \leq 7$ are simplices.

The vertex figure of Gosset's honeycomb is the semiregular $E_8$ polytope (4₂₁ in Coxeter's notation) given by the convex hull of the 240 roots of the $E_8$ lattice.

Each point of the $E_8$ lattice is surrounded by 2160 8-orthoplexes and 17280 8-simplices. The 2160 deep holes near the origin are exactly the halves of the norm 4 lattice points. The 17520 norm 8 lattice points fall into two classes (two orbits under the action of the $E_8$ automorphism group): 240 are twice the norm 2 lattice points while 17280 are 3 times the shallow holes surrounding the origin.

A hole in a lattice is a point in the ambient Euclidean space whose distance to the nearest lattice point is a local maximum. (In a lattice defined as a uniform honeycomb these points correspond to the centers of the facets volumes.) A deep hole is one whose distance to the lattice is a global maximum. There are two types of holes in the $E_8$ lattice:

- *Deep holes* such as the point (1,0,0,0,0,0,0,0) are at a distance of 1 from the nearest lattice points. There are 16 lattice points at this distance which form the vertices of an 8-orthoplex centered at the hole (the Delaunay cell of the hole).

- *Shallow holes* such as the point       are at a distance of       from the nearest lattice points. There are 9 lattice points at this distance forming the vertices of an 8-simplex centered at the hole.

# Applications[edit]

In 1982 Michael Freedman produced a bizarre example of a topological 4-manifold, called the $E_8$ manifold, whose intersection form is given by the $E_8$ lattice. This manifold is an example of a topological manifold which admits no smooth structure and is not even triangulable.

In string theory, the heterotic string is a peculiar hybrid of a 26-dimensional bosonic string and a 10-dimensional superstring. In order for the theory to work correctly, the 16 mismatched dimensions must be compactified on an even, unimodular lattice of rank 16. There are two such lattices: $\Gamma_8 \oplus \Gamma_8$ and $\Gamma_{16}$ (constructed in a fashion analogous to that of $\Gamma_8$). These lead to two version of the heterotic string known as the $E_8 \times E_8$ heterotic string and the SO(32) heterotic string.

## 13.17. The $E_8$ and novanion group constructions from octonions.

[To be modified.]

The $E_8$ lattice is closely related to the nonassociative algebra of real octonions **O**. It is possible to define the concept of an integral octonion analogous to that of an integral quaternion. The integral octonions naturally form a lattice inside **O**. This lattice is just a rescaled $E_8$ lattice. (The minimum norm in the integral octonion lattice is 1 rather than 2). Embedded in the octonions in this manner the $E_8$ lattice takes on the structure of a nonassociative ring.

Fixing a basis $(1, i, j, k, \ell, \ell i, \ell j, \ell k)$ of unit octonions, one can define the integral octonions as a maximal order containing this basis. (One must, of course, extend the definitions of *order* and *ring* to include the nonassociative case). This amounts to finding the largest subring of **O** containing the units on which the expressions $x^*x$ (the norm of $x$) and $x + x^*$ (twice the real part of $x$) are integer-valued. There are actually seven such maximal orders, one corresponding to each of the seven imaginary units. However, all seven maximal orders are isomorphic. One such maximal order is generated by the octonions $i, j,$ and $\frac{1}{2}(i + j + k + \ell)$.

A detailed account of the integral octonions and their relation to the $E_8$ lattice can be found in Conway and Smith (2003).

### Example definition of integral octonions[edit]

Consider octonion multiplication defined by triads: 137, 267, 457, 125, 243, 416, 356. Then integral octonions form vectors:

1)       , i=0, 1, ..., 7

2)       , indexes abc run through the seven triads 124, 235, 346, 457, 561, 672, 713

3)  , indexes pqrs run through the seven tetrads 3567, 1467, 1257, 1236, 2347, 1345, 2456.

Imaginary octonions in this set, namely 14 from 1) and 7*16=112 from 3), form the roots of

the Lie algebra    . Along with the remaining 2+112 vectors we obtain 240 vectors that

form roots of Lie algebra    . See the Koca work on this subject.[13]

If we look at octonions, they have a multiplicative identity 1, and seven units $e_1$, $e_2$, ... , $e_7$. If we ignore their additive structure for the moment, we also have plus or minus these elements, and the plus and minus are distinguishable in the multiplicative group structure. The total number of elements of these types is now 16.

If we go over to a ring structure, additively it contains zero. We have also seen that the octonions with Lie brackets given by $[A, B] = AB - BA$, where for example this can be a matrix representation, although octonions are not matrices, since they are not associative, nevertheless they can be given the above Lie algebra structure (mod 12), satisfying the Jacobi identities.

The elements contain zero, for example $6e_7 \times 2e_7 = 0$ (mod 12). We therefore need to create a multiplicative group without zero. Note we can proceed not only (mod 12), but as we have mentioned, (mod 6), (mod 4), (mod 3) and (mod 2). If we take the case (mod 6), since for example if $3 \times 2 = 0$, then $3^{-1} \times 3 \times 2 = 0$, so $2 = 0$ and has no inverse. The only occurrence of multiplicative terms $\neq 0$ for an abelian group with result zero is $3 \times 2$ (mod 6). Then if we specify a modification of the group so that $3 \times 2 = 1$, since by prime factorisation this is the only pair $\neq 0$ that gives zero, we can define this new allocation in which $3^{-1} = 2$ and $2^{-1} = 3$, and this retains the structure of the Lie algebra, and the multiplicative group defined in this way does not contain zero, which we have excluded from it. A more general way of stating this is that we replace 0 (mod 6) in the Lie algebra by $\pm 1$ in its group.

To consider (mod 12), the products which are zero (mod 12) contain powers of factors 3 and 2, where 3 and $3^2$ occur (mod 12) and so do 2, $2^2$ and $2^3$. To analyse this, first look at (mod 9) where the only product of terms $\neq 0$ that has a product 0 (mod 9) is $3 \times 3$. So a Lie algebra (mod 9) can give rise to a multiplicative group where $3 \times 3 = 1$, in which 3 is its own inverse. Likewise for (mod 8) = (mod $2^3$), the only terms $\neq 0$ with abelian product zero are $4 \times 2$. The element 4 is now set with inverse 2, and on replacing zero in the (mod 8) Lie algebra by $\pm 1$ as an element in its group, we again obtain a multiplicative group. For (mod 12) we consider elements (mod $9 \times 8$) = (mod 72). Then there exists a corresponding group in which elements are classified multiplicatively by their (mod 9) or (mod 8) counterparts where 9 is irreducible in terms of 8. We could choose for example an element (mod 72) as $3^m \times 2^n \times$(prime < 72) where $3^m$ is in (mod 9), $2^n$ is in (mod 8) and the prime is considered (mod 72). Then from the Lie algebra (mod 12) derived from (mod 72), on setting zero to $\pm 1$ the mapping to the group is obtained.

If 9 is reducible in terms of 8, then consider again the case where 1 is identified with zero (mod 12), so that the multiplicative algebra at this stage contains 11 elements. Then $3 \times 4 = 1$, so $3 = 4^{-1} = 2^{-2}$. We have $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $5 = 2^4$, $10 = 2^5$, $9 = 2^6$, $7 = 2^7$, $3 = 2^8$, $6 = 2^9$ and $1 = 2^{10}$. To check consistency, for example $2^6 = 9 = 3 \times 3 = 2^8 \times 2^8 = 2^{16} = 2^{10} \times 2^6 = 2^6$. Then 2 is represented by the cyclic permutation (1 2 4 8 5 10 9 7 3 6), and the reduction of 3 in terms of

the generator 2 gives a multiplicative group on 10 elements. If we were to choose $3\times4 = -1$, then $3 = -2^{-2} = 2^8$, so $2^{10} = -1$, which is the mapping $2 \to 2i$ from the previous case.

Consider the original elements $e_7$, $e_6$, ... , 1, –1, –$e_1$, ... , –$e_7$, 16 elements in all, and zero. If we allow new elements $0e_r = 0$, $1e_r = e_r$, $2e_r$, ..., $11e_r$, with $12e_r = 0e_r$, so that this exists in a (mod 12) arithmetic, then the Lie algebra derived from the octonions in this way has first a set of 11 elements to choose from (that is, not 0). Then if we create new elements by adding a second set, because the addition is $ae_r + be_s$ with $e_r \neq e_s$, but $ae_r + a(-e_r) = 0$, the paired $e_r$ and $(-e_r)$ elements (mod 12) have 12 values (including $0e_r + 0(-e_r) = 0$) that we must exclude, but $-[0e_r + 0(-e_r)] = 0e_r + 0(-e_r) = 0$ is the same case. Thus we have $12 - 6$ elements that are not zero. Further, we have values
(1)     $1e_7 + 0(-e_7) = 1e_7$
(2)     $2e_7 + 1(-e_7) = 1e^7$
...     ...
(11)   $11e_7 + 10(-e_7) = 1e_7$
(12)   $12e_7$ [which is $0(e_7)$] $+ 11(-e_7) = 1e_7$,
which we must also collect together as one item, but case (1) (mod 12) is the same as case (12), etc, with case (6) the same as case (7). Thus there are $12/2 = 6$ distinct cases for $1e_7$ and the total number of exclusions for differences $2e_7$, $3e_7$, ..., $11e_7$ is the same. Thus for the pair $e_7$, (–$e_7$) we obtain $12^2 - 12\times6 = 12\times6$ possibilities. There are 8 values $e_7$, $e_6$, ... $e_1$, 1 that we can account for in a similar way. Then the total number of distinct possibilities is $(12\times6)^8$, but we must exclude the cases where we have a sum of all the $e_r$ equal to zero. For $e_7$ this is 12 cases, as we have seen, and likewise for $e_6$, ... $e_1$, 1. Thus we exclude $(12)^8$ items, and the total number of elements of the octonion algebra is
$$(12\times6)^8 - 12^8 = 12^8(6^8 - 1).$$

This is the order of the Lie algebra we have derived from the octonions. We obtain the order of the Lie group from it. □


## 13.18. Novanion simple group constructions. [To be modified.]

Our objective now is to convert novanion algebras, with novanion brackets (mod m) like a Lie bracket, to novanion algebras, and then map these novanion algebras to groups. These groups, being multiplicatively invertible, do not contain zero, so we map $0 \to \pm1$, which retains the novanion algebra structure. The resulting novanion group thus does not have the same structure as the multiplicative part of the ring mapped directly to the novanion algebra.

For the 10-novanions consider the elements $e_9$, $e_8$, ... , 1, –1, –$e_1$, ... , –$e_9$, 20 elements in all, and zero. Using elements $0e_r = 0$, $1e_r = e_r$, $2e_r$, ..., $11e_r$ again, then in the case when we are not using $\pm1$, the extended Lie group derived from the 10-novanions has in a calculation similar to one given for the octonions, $(12\times6)^9$ elements. If we assume the scalar part is not zero, the available scalar values are $\pm1$, $\pm2$, ..., $\pm11$, amounting to 22 components, but we have $1 = -11$ (mod 12), etc., so effectively there are 11 components. We know that when the scalar part is not zero, we cannot get a zero result on multiplying 10-novanions together, provided we ignore the (mod 12) restriction. Thus the order of the 10-novanion group derived from this algebra is $(12\times6)^9\times11$ elements. If we obtain 0 (mod 12) in any multiplication, then as before we will map this result to $\pm1$ without changing the Lie-type bracket algebra.

For the 26-novanions or indeed any n-novanion algebra, the case is similar with $(12\times6)^n\times11$ elements. We know that overrides derived from the octonion structure operate for all 10-novanion components in a 26-novanion, thus external to the 10-novanion or an octonion or

their subalgebras, there are no other proper subentities in this algebra, which implies that the groups defined by a quotient of a 10-novanion group or an octonion group with the 26-novanion group are simple.

Note that we have demonstrated in section 12 that both bosonic and fermionic structures can be implemented in novanions. Thus any argument which says that a bosonic 26-novanion cannot contain a 10-novanion is incorrect. Since the bosonic 10-novanion is a maximal proper subgroup of the 26-novanions, the 26-novanions do not generate a simple group directly, only via quotients. $\square$

Note that $26 - 10 = 16$, but we have no indication that this is two copies of $E_8$. Indeed, it cannot be, since the structure is not closed, encroaching in overrides onto the 10-novanion. So we show once again that the novanion groups do not lead to the compactification assumed in string theory of the identification of the 16 dimensions with $E_8 \times E_8$. $\square$

These considerations may be extended without limit, for example to the 80-novanions, or more generally the $(3^n - 1)$-novanions, $n > 0$, which are those hypercube novanions with one corner containing not a triplet, but a singlet. In particular, for the 80-novanions there exists a 28-novanion slice group which has no containing groups other than the 80-novanion group itself, so there is an analogous structure to that for the 26-novanion group. The remaining dimensions are $79 - 27 = 52$, and this generates a simple group via the quotient with the 28-novanion group. $\square$