

CHAPTER VII

Nonconformal analysis

7.1. Introduction.

The hyperintricate Cauchy-Riemann equations and the complex Cauchy integral formula are considered.

7.2. Fourier series and convolutions (~).

7.3. The Maurer-Cartan equation (~).

7.4. The intricate analytic Cauchy-Riemann equations.

Suppose that

$$P(g) = w(g) + x(g)i + y(g)\alpha + z(g)\phi \quad (1)$$

is a function of an intricate number, $g \in \mathfrak{A}$. The *intricate derivative* of P at a point g_0 is defined by

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathfrak{A}}} \frac{P(g_0 + h) - P(g_0)}{h}, \quad (2)$$

wherever this limit is path independent of h to 0. Alternatively, to obtain this derivative, we can regard h as a (hyper)infinitesimal number and discard (hyper)infinitesimal parts of the division performed in (2). Note that when an intricate numerator is divided by the intricate denominator h , the result depends on whether the reciprocal h^{-1} is multiplied first or last, so we assume two types of derivative. When h comes first we write $\lim_{h \rightarrow 0}$, otherwise the limit is from the right, denoted by $\lim_{0 \leftarrow h}$.

A standard argument indicates for this left limit to be unique, intricate polynomials satisfy the condition of being intricate left analytic, that is we have

$$P(w, x, y, z) = P(w + xi + y\alpha + z\phi, 0, 0, 0). \quad (3)$$

If the limit (2) exists, it can be computed by taking the limit $h \rightarrow 0$ along the real, imaginary, actual or phantom axes. In all cases it should give the same result. Approaching along the real axis, we find for the limit on the left

$$\lim_{\substack{h \rightarrow 0 \\ h \in U}} \frac{P(g_0 + h) - P(g_0)}{h} = \frac{\partial P(g_0)}{\partial w}, \quad (4)$$

whereas along the imaginary axis

$$\lim_{\substack{h \rightarrow 0 \\ h \in U}} \frac{P(g_0 + hi) - P(g_0)}{hi} = \frac{1}{i} \frac{\partial P(g_0)}{\partial x}, \quad (5)$$

along the actual axis

$$\lim_{\substack{h \rightarrow 0 \\ h \in U}} \frac{P(g_0 + h\alpha) - P(g_0)}{h\alpha} = \frac{1}{\alpha} \frac{\partial P(g_0)}{\partial y}, \quad (6)$$

and for the phantom axis

$$\lim_{\substack{h \rightarrow 0 \\ h \in U}} \frac{P(g_0 + h\phi) - P(g_0)}{h\phi} = \frac{1}{\phi} \frac{\partial P(g_0)}{\partial z}. \quad (7)$$

Then to be intricate left analytic the equality of the results (4) to (7) gives

$$\frac{\partial P}{\partial x} = i \frac{\partial P}{\partial w} \quad (8)$$

$$\frac{\partial P}{\partial y} = \alpha \frac{\partial P}{\partial w} \quad (9)$$

$$\frac{\partial P}{\partial z} = \phi \frac{\partial P}{\partial w}, \quad (10)$$

which are the 3 left intricate Cauchy–Riemann equations at the point g_0 . \square

If we change the basis by means of continuous first derivatives to $P = r + si + t\alpha + u\phi$, then

$$\frac{\partial P}{\partial w} = \frac{\partial r}{\partial w} + i \frac{\partial s}{\partial w} + \alpha \frac{\partial t}{\partial w} + \phi \frac{\partial u}{\partial w}$$

$$\frac{\partial P}{\partial x} = \frac{\partial r}{\partial x} + i \frac{\partial s}{\partial x} + \alpha \frac{\partial t}{\partial x} + \phi \frac{\partial u}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial r}{\partial y} + i \frac{\partial s}{\partial y} + \alpha \frac{\partial t}{\partial y} + \phi \frac{\partial u}{\partial y}$$

$$\frac{\partial P}{\partial z} = \frac{\partial r}{\partial z} + i \frac{\partial s}{\partial z} + \alpha \frac{\partial t}{\partial z} + \phi \frac{\partial u}{\partial z},$$

giving, using equations (8) to (10), on multiplying on the left

$$\frac{\partial P}{\partial w} = \frac{\partial r}{\partial w} + i \frac{\partial s}{\partial w} + \alpha \frac{\partial t}{\partial w} + \phi \frac{\partial u}{\partial w}$$

$$\frac{\partial P}{\partial w} = i \frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} - \phi \frac{\partial t}{\partial x} + \alpha \frac{\partial u}{\partial x}$$

$$\frac{\partial P}{\partial w} = \alpha \frac{\partial r}{\partial y} + \phi \frac{\partial s}{\partial y} + \frac{\partial t}{\partial y} + i \frac{\partial u}{\partial y}$$

$$\frac{\partial P}{\partial w} = \phi \frac{\partial r}{\partial z} - \alpha \frac{\partial s}{\partial z} - i \frac{\partial t}{\partial z} + \frac{\partial u}{\partial z},$$

so that the intricate Cauchy-Riemann equations become, on equating intricate parts

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial z} \quad (11)$$

$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x} = \frac{\partial u}{\partial y} = -\frac{\partial t}{\partial z} \quad (12)$$

$$\frac{\partial t}{\partial w} = \frac{\partial u}{\partial x} = \frac{\partial r}{\partial y} = -\frac{\partial s}{\partial z} \quad (13)$$

$$\frac{\partial u}{\partial w} = \frac{\partial t}{\partial x} = \frac{\partial s}{\partial y} = \frac{\partial r}{\partial z}, \quad (14)$$

the number of these latter equations being 16.

We represent the Jacobian matrix of equations (11) to (14) by

$$\begin{bmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [r \quad s \quad t \quad u] = \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ c & d & a & b \\ d & -c & -b & a \end{bmatrix}, \quad (15)$$

or as the 2-hyperintricate

$$a1_1 + b1_i + c\phi_\alpha + d\phi_\phi. \quad (16)$$

Conversely, if $P : \mathfrak{R} \rightarrow \mathfrak{R}$ is a differentiable function when regarded as a function on \mathbb{U}^4 , then P is a left intricate-valued real-differentiable function if and only if the 3 left intricate-differential Cauchy-Riemann equations hold. \square

From equation (11)

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x},$$

then if the second differential exists

$$\frac{\partial^2 r}{\partial w^2} = \frac{\partial^2 s}{\partial w \partial x},$$

and from equation (12)

$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x},$$

so

$$-\frac{\partial^2 r}{\partial x^2} = \frac{\partial^2 s}{\partial w \partial x},$$

giving

$$\frac{\partial^2 r}{\partial w^2} + \frac{\partial^2 r}{\partial x^2} = 0. \quad (17)$$

Further relations obtained in a similar way are

$$\frac{\partial^2 r}{\partial w^2} - \frac{\partial^2 r}{\partial y^2} = 0, \quad (18)$$

$$\frac{\partial^2 r}{\partial w^2} - \frac{\partial^2 r}{\partial z^2} = 0, \quad (19)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial x^2} = 0, \quad (20)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial y^2} = 0, \quad (21)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial z^2} = 0, \quad (22)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial x^2} = 0, \quad (23)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial y^2} = 0, \quad (24)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial z^2} = 0, \quad (25)$$

$$\frac{\partial^2 u}{\partial w^2} + \frac{\partial^2 u}{\partial x^2} = 0, \quad (26)$$

$$\frac{\partial^2 u}{\partial w^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad (27)$$

and

$$\frac{\partial^2 u}{\partial w^2} - \frac{\partial^2 u}{\partial z^2} = 0. \quad \square \quad (28)$$

A function with continuous second derivative satisfying (17) or (26) is said to be *harmonic*.

For the right limit to be unique, intricate polynomials P' satisfy the condition of being intricate right analytic, that is by taking the limit $0 \leftarrow h$ along the real, imaginary, actual or phantom axes, on approaching along the real axis we find for the limit on the right

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h) - P'(g_0)}{h} = \frac{\partial P'(g_0)}{\partial w}, \quad (29)$$

whereas along the imaginary axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + hi) - P'(g_0)}{hi} = \frac{\partial P'(g_0)}{\partial x} \frac{1}{i}, \quad (30)$$

along the actual axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h\alpha) - P'(g_0)}{h\alpha} = \frac{\partial P'(g_0)}{\partial y} \frac{1}{\alpha}, \quad (31)$$

and for the phantom axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h\phi) - P'(g_0)}{h\phi} = \frac{\partial P'(g_0)}{\partial z} \frac{1}{\phi}. \quad (32)$$

Then to be intricate right analytic the equality of the results (29) to (32) gives

$$\frac{\partial P'}{\partial x} = \frac{\partial P'}{\partial w} i \quad (33)$$

$$\frac{\partial P'}{\partial y} = \frac{\partial P'}{\partial w} \alpha \quad (34)$$

$$\frac{\partial P'}{\partial z} = \frac{\partial P'}{\partial w} \phi, \quad (35)$$

which are the 3 right intricate Cauchy–Riemann equations at the point g_0 . \square

If we multiply on the right we obtain

$$\begin{aligned} \frac{\partial P}{\partial w} &= \frac{\partial r}{\partial w} + i \frac{\partial s}{\partial w} + \alpha \frac{\partial t}{\partial w} + \phi \frac{\partial u}{\partial w} \\ \frac{\partial P}{\partial w} &= i \frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} + \phi \frac{\partial t}{\partial x} - \alpha \frac{\partial u}{\partial x} \\ \frac{\partial P}{\partial w} &= \alpha \frac{\partial r}{\partial y} - \phi \frac{\partial s}{\partial y} + \frac{\partial t}{\partial y} - i \frac{\partial u}{\partial y} \\ \frac{\partial P}{\partial w} &= \phi \frac{\partial r}{\partial z} + \alpha \frac{\partial s}{\partial z} + i \frac{\partial t}{\partial z} + \frac{\partial u}{\partial z}, \end{aligned}$$

so that the intricate Cauchy-Riemann equations now become, on equating intricate parts

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial z} \quad (36)$$

$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x} = -\frac{\partial u}{\partial y} = \frac{\partial t}{\partial z} \quad (37)$$

$$\frac{\partial t}{\partial w} = \frac{\partial u}{\partial x} = \frac{\partial r}{\partial y} = \frac{\partial s}{\partial z} \quad (38)$$

$$\frac{\partial u}{\partial w} = -\frac{\partial t}{\partial x} = -\frac{\partial s}{\partial y} = \frac{\partial r}{\partial z}, \quad (39)$$

We represent the Jacobian matrix of equations (36) to (39) as

$$\begin{bmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [r \quad s \quad t \quad u] = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ c & -d & a & -b \\ -d & c & b & a \end{bmatrix}, \quad (40)$$

where the 2-hyperintricate

$$a1_1 + b\alpha_i + c\phi_1 + di_i, \quad (41)$$

has been obtained from equation (40). \square

More generally, an intricate polynomial P'' may be allocated any combination of the following intricate mixed analytic expressions

$$\frac{\partial P''}{\partial x} = i \frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w} i, \quad (42)$$

$$\frac{\partial P''}{\partial y} = \alpha \frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w} \alpha, \quad (43)$$

$$\frac{\partial P''}{\partial z} = \phi \frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w} \phi. \quad (44)$$

Thus the structure of these transformations differs depending on whether multiplication occurs on the left or on the right. \square

7.5. J-diffeomorphisms and non-fixed J.

P may be expressed for intricate numbers in J format as

$$P = r + Jv, \quad (1)$$

It may or may not be the case that J is constant. Chapter IX uses the fact that for intricate numbers constant J implies J-abelian.

When a diffeomorphism is applied to $J = si + t\alpha + u\phi \rightarrow J + \delta J$ so that $J^2 = (J + \delta J)^2$, where $J^2 = 0$ or ± 1 , then

$$s = t(\partial t / \partial s) + u(\partial u / \partial s). \quad (2)$$

7.6. The intricate Cauchy-Riemann equations are nonconformal.

If we look at the Cauchy–Riemann equations 7.5.(11) to (14) written in 7.5.(15) as a 2-hyperintricate Jacobian matrix, and likewise for the Jacobian matrix 7.5.(40), a matrix of this form contains within it components of non-complex type (is not hyperimaginary), that is the matrix is not composed of layers containing only 1 and i . Geometrically, multiplying a complex number by a complex number is always the composition of a rotation with a scaling, and in particular preserves angles, but a non-complex matrix does not. Expressed more simply, for a complex number

$$e^p e^{i\theta} \cdot e^{i\theta'} = e^p e^{i(\theta + \theta')} = e^{p + i(\theta + \theta')}, \quad (1)$$

and this is the core idea of what we are saying. For a general intricate number (see *Superexponential algebra*, chapter XV, section 3), the intricate $e^{p + (bi + c\alpha + d\phi)K}$ cannot usually be expressed as $e^w e^{xi} e^{y\alpha} e^{z\phi}$ with $p = w$, $bK = x$, $cK = y$ and $dK = z$.

However, if two (hyper-)intricate numbers are J -abelian with the same $J^2 = -1$, then

$$e^p e^{J\theta} \cdot e^{J\theta'} = e^p e^{J(\theta + \theta')} = e^{p + J(\theta + \theta')}, \quad (2)$$

and the transformation is now conformal.

Consequently, a function satisfying the intricate Cauchy–Riemann equations, with a nonzero derivative does not in general preserve the angle between curves in the plane, so we say the intricate Cauchy–Riemann equations contain already the property for a function not to be conformal. \square

7.7. The novanionic analytic Cauchy-Riemann equations.

We wish to have a symbol designed for putting commas between ordered sets of coefficients of hyperintricate basis elements. For $0 < k \leq m \in \mathbb{N}$ define

$$(\cdot)_{k=1}^m w_k$$

to be (w_1) when $m = 1$, (w_1, w_2) when $m = 2$ and to have the properties for $m' \leq m$

$$(\cdot)_{k'=1}^{m'} w_{k'} = (\cdot)_{k=1}^{m'-k'+1} w_{k'}, \quad (1)$$

$$(\cdot)_{k=1}^m w_k = [(\cdot)_{k=1}^{m'} w_k][(\cdot)_{m'+1}^m w_k]. \quad (2)$$

Let an n -hyperintricate number be denoted by \mathcal{Y}_n , with coefficients of basis elements w_k . We define the hyperintricate analytic condition for a function P corresponding to 18.5.(3) to be

$$P[(\cdot)_{k=1}^{4^n} w_k] = P[\mathcal{Y}_n, (\cdot)_{k=1}^{4^n-1} 0]. \quad (3)$$

The derivative of hyperintricate numbers satisfies the same equation as 18.5.(2). Then if γ_j is a basis element with coefficient w_{γ_j} , the equations corresponding to 18.5.(8)-(10) are

$$\frac{\partial P}{\partial w} = \gamma_j^{-1} \frac{\partial P}{\partial w_{\gamma_j}}. \quad (4)$$

Set

$$P = \sum_{k=1}^{4^n} \gamma_k \Gamma_k, \quad (5)$$

so that if we change basis then

$$\frac{\partial P}{\partial w_{\gamma_j}} = \sum_{k=1}^{4^n} \gamma_k \frac{\partial \Gamma_k}{\partial w_{\gamma_j}}, \quad (6)$$

and multiplying this on the left by γ_j^{-1} , using (4)

$$\frac{\partial P}{\partial w} = \gamma_j^{-1} \sum_{k=1}^{4^n} \gamma_k \frac{\partial \Gamma_k}{\partial w_{\gamma_j}}. \quad (7)$$

For multiplying on the left, let $\langle \gamma_j^{-1}, \gamma_k \rangle_L$ equal the sign of $\gamma_j^{-1} \gamma_k$ in that order when $\gamma_j^{-1} = \pm \gamma_k$, and otherwise be 0, so that the hyperintricate left derivative Cauchy-Riemann equations become, on equating hyperintricate parts

$$\langle \gamma_{j'}^{-1}, \gamma_k \rangle_L \left[\langle \gamma_j^{-1}, \gamma_k \rangle_L \frac{\partial r_k}{\partial w_{\gamma_j}} \right] = \langle \gamma_{j'}^{-1}, \gamma_k \rangle_L \left[\langle \gamma_{j'}^{-1}, \gamma_{k'} \rangle_L \frac{\partial r_{k'}}{\partial w_{\gamma_{j'}}} \right]. \quad (8)$$

For multiplying on the right, a similar situation obtains, on defining $\langle \gamma_j^{-1}, \gamma_k \rangle_R$ equal to the sign of $\gamma_k \gamma_j^{-1}$ (in reversed order from previously) when $\gamma_j^{-1} = \pm \gamma_k$, otherwise 0. The right derivative hyperintricate Cauchy-Riemann equations are obtained by substituting \langle, \rangle_L in (8) with \langle, \rangle_R . \square

As we have seen, the quaternions are an instance of hyperintricate numbers, to which these considerations apply. The proof that quaternion analytic Cauchy-Riemann equations are conformal and the extension of this idea to the octonions and n-novonions is now discussed.

7.8. The nonconformal split representation (~).

For an intricate number

$$H = a1 + bi + c\alpha + d\phi$$

this can be represented as

$$H = (a1 + bi) + \alpha(c1 + di).$$

For hyperimaginary numbers an example might be

$$H' = (a1_1 + bi_1) + (c1_1 + di_1),$$

where we have collected together with terms in the first brackets with an even number of i 's in their hyperintricate representation, and the second brackets containing an odd number of i 's. We might now expand this example to contain a general hyperintricate number, for example

$$H'' = [a1_1 + b\alpha_1](c1_1 + di_1) + [e1_1 + f\alpha_1](g1_1 + hi_1) + [j\alpha_1 + k1_1](m1_1 + ni_1) + [p\alpha_1 + q1_1](r1_1 + si_1). \quad (1)$$

We will use the representations of type (1) to describe conformal components of Cauchy-Riemann type and corresponding nonconformal components.

7.9. The complex Cauchy integral formula (~).

Holomorphic functions are the central objects of study in complex analysis. A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighbourhood of every point in its domain. The existence of a complex derivative in a neighbourhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal to its own Taylor series.

The term analytic function is often used interchangeably with holomorphic function, although the word "analytic" is also used in a broader sense to describe any function (real, complex, or of more general type) that can be written as a convergent power series in a neighbourhood of each point in its domain. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.

Holomorphic functions are also sometimes referred to as conformal maps. A holomorphic function whose domain is the whole complex plane is called an entire function. The phrase

"holomorphic at a point z_0 " means not just differentiable at z_0 , but differentiable everywhere within some neighbourhood of z_0 in the complex plane.

7.10. The nonvanishing Cauchy integral formula (~).

7.11. The nonconformal split Cauchy integral formula (~).

7.12. Picard's theorems (~).