CHAPTER VI

Modular Forms

6.1. Introduction.

Theta function[edit]

One can associate to any (positive-definite) lattice Λ a theta function given by

The theta function of a lattice is then a holomorphic function on the upper half-plane. Furthermore, the theta function of an even unimodular lattice of rank n is actually a modular form of weight $n/2$. The theta function of an integral lattice is often written as a power series

in so that the coefficient of q^n gives the number of lattice vectors of norm n.

Up to normalization, there is a unique modular form of weight 4: the Eisenstein series $G_4(\tau)$. The theta function for the E_8 lattice must then be proportional to $G_4(\tau)$. The normalization can be fixed by noting that there is a unique vector of norm 0. This gives

where $\sigma_3(n)$ is the divisor function. It follows that the number of E_8 lattice vectors of norm $2n$ is 240 times the sum of the cubes of the divisors of n. The first few terms of this series are given by (sequence A004009 in the OEIS):

The E_8 theta function may be written in terms of the Jacobi theta functions as follows:

where

Theta series[edit]

One can associate to any (positive-definite) lattice Λ a theta function given by

The theta function of a lattice is then a holomorphic function on the upper half-plane. Furthermore, the theta function of an even unimodular lattice of rank *n* is actually a modular form of weight *n*/2 for the full modular group *PSL*(2,**Z**). The theta function of an integral

lattice is often written as a power series in so that the coefficient of q^n gives the number of lattice vectors of squared norm 2*n*. In the Leech lattice, there are 196560 vectors of squared norm 4, 16773120 vectors of squared norm 6, 398034000 vectors of squared norm 8 and so on. The theta series of the Leech lattice is

where is the normalized Eisenstein series of weight 12, is the modular discriminant,

is the divisor function for exponent 11, and is the Ramanujan tau function. It follows that for *m*≥1 the number of vectors of squared norm 2*m* is