CHAPTER V

Polynomial comparison

5.1. Introduction.

The objective in this chapter is to document further aspects of polynomial comparison theory the concrete results of which violate the Galois solvability model. We have proved in [Ad15], Volume II that this model fails in the case of dependent roots and matrix roots, and the group automorphism model does not generally leave other complex roots fixed when two roots are swapped, that this multiplicative theory does not describe solvability of polynomials, since a polynomial in multiplicative form is already solved, and when the automorphism model is extended to ring automorphisms, with + and ×, then for complex roots these automorphisms are involutions which cannot in general be represented by permutation groups, as is claimed to happen in Galois solvability theory. Ring automorphism theory is also defective, in that it does not incorporate linear transformations, which are of necessity used in the standard solutions of polynomial equations of degree up to the quartic. However, we have shown that under the additional condition that zeros of polynomial equations are obtained by killing central terms in a method of descent, that a theory of degrees of freedom, in other words an independency result, so this method is not contingent on group theory, shows that solutions of polynomial equations in radicals are not available when the degree of the equation is greater than four. This might explain the persistence with which the teaching of the Galois solvability model is held in mathematics departments at universities. Nevertheless, we are able to show the total falsity of the Galois model in respect of the absence of solutions by radicals by using comparison methods, which give solutions in radicals of polynomial equations of degree 6 and 8. We will begin in section 3 a discussion of the comparison method for a cubic variety, where there is an interesting geometric realisation available in violation of Galois solvability theory. The sextic is tackled in section 4, the octic in section 5. It is natural to ask whether there exist solutions of polynomial equations of general degree using this method.

5.2. Galois representation theory and Galois solvability.

On one level of description Galois *representation* theory connects the theory of groups with the binomial theorem. Not only are the coefficients of the expansion of the binomial theorem in integer powers related to numbers of permutations, and permutations form groups, but also Fermat’s little theorem is a consequence of the binomial theorem, and this theorem connects directly with many features of finite arithmetic, called congruence theory. An alternative terminology for Fermat’s little theorem uses the phrase Frobenius automorphism. Essentially, the theory of Galois representations is correct in its programme and outline. We will need Galois representation theory to prove the Weil conjectures and the modularity theorem.

The issue with Galois *solvability* theory, which goes in additive format beyond the binomial theorem to encompass polynomials of arbitrary degree and arbitrary complex coefficients, as we have mentioned in [Ad15] Vol II, is that the model it provides for the solvability, that is the zeros, of a polynomial equation is erroneous. This creates multiple issues in mathematics which are currently unresolved. These issues may be divided into the social communication of mathematics, what it teaches, what it admits as the truth, and how this knowledge is passed from one generation to another, including through the examination system, and then the body of knowledge and its consequences, because we are saying that proofs in other areas of mathematics which depend on this theory are false, and need re-examination.

That this situation has arisen is a tragedy for mathematical culture, but it is not an isolated feature of accepted but false mathematics. We need therefore to examine the social structures for the communication of mathematics. It is my contention that the journal system dominated by Reed Elsevier is corrupt, this is obvious and a scandal, and that the peer review process, the language for which it insists is necessary, and other forms of acceptance are also lacking in what is desirable. Mathematics and other sciences need to reach out in plain language to the general scholar so that its features can be inspected in full and knowledgeable detail, and so that the portals of this knowledge are not guarded by a clique restricting inspection of its contents and creating barriers to communication through the excessive use of jargon and results only explained in an interminable trail of references, or with no explanation at all.

As a separate issue, I wish to raise again here the work of Nathan Jacobson (1910 – 1999, who taught at Yale from 1947 and was president of the AMS 1971 – 1973). It follows from the work of Gentzen that all proofs may be put in the form of a tree, where the top of the tree contains the assumptions of the proof, and the root contains the conclusion. So far as I can gather, all proofs by Jacobson contain internal loops, that is, they cannot be reduced to tree form. It may happen that a proof branches off into other proofs which themselves contain circuits, or nodes which are ambiguous or absent. It is difficult to prove motivation here. I would make the suggestion that all proofs by Jacobson cannot be modified to axiomatic form. Thus it appears that whereas Jacobson's work contains many true theorems (and a few false ones), the proofs are invalid. For instance, by these means he proves that all functions are associative. But a nonassociative function can be defined by multiplication of an octionian by another octonian to form an octonion function, and octonions are nonassociative. This is significant because there is only one purported theorem on the correctness of the Galois model, and this is provided by Jacobson, which he calls the Jacobson-Bourbaki theorem, although it does not appear in Bourbaki.

5.3. A geometric realisation of the cubic.

Our theories have deconstructed Galois solvability theory, where we have replaced a theory of group symmetries by a theory of dependencies, and obtained the unsolvability of the quintic by techniques of killing central terms which are independent of group theory. We now explore a case of comparison theory in which there is no killing of central terms, but a polynomial with appended roots is equated to a comparison equation which is a nested polynomial within another, and this polynomial is solvable. We are able to extend the type of comparison equation essentially to a nested variety, and this will allow us to express a cubic equation with an appended root in terms of an extended comparison equation. As pointed out by Doly García, this has implications for the representation of a cube root of a number in terms of square roots which are geometrically realisable. This is the complete negation of the classical result on the impossibility of such a construction and some other no-go results which are also derived from Galois theory.

We will show that the equation

$$x^3 + Kx + L = 0 \tag{1}$$

is solvable by only using square roots. The method does not involve 'killing central terms' but uses a type of comparison method where the comparison equation is written not in the form of a polynomial, but a variety with two variables. Let

$$(x^3 + Kx + L)(x + m) = 0 \tag{2}$$

so

$$x^4 + mx^3 + Kx^2 + (Km + L)x + Lm = 0. \tag{3}$$

We now consider a comparison equation, where detailed work shows (x + c) is not feasible as the second variable, so we substitute (x^2 + c) instead

$$(x^2 + ax + b)^2 + p(x^2 + ax + b)(x^2 + c) + q(x^2 + c)^2 = 0, \quad (4)$$

which can be expressed as the solvable quadratic equation

$$y^2 + pyz + qz^2 = 0$$

with

$$y = x^2 + ax + b$$
$$z = x^2 + c,$$

with solution that of

$$(x^2 + ax + b) = \left[\frac{-p \pm\sqrt{p^2 - 4q}}{2}\right](x^2 + c), \quad (5)$$

giving

$$\left[1 + \frac{p \mp\sqrt{p^2 - 4q}}{2}\right]x^2 + ax + b + \left[\frac{p \mp\sqrt{p^2 - 4q}}{2}\right]c = 0,$$

which using

$$G = 1 + \frac{p \mp\sqrt{p^2 - 4q}}{2} \quad (6)$$

$$H = b + \frac{p \mp\sqrt{p^2 - 4q}}{2} \quad (7)$$

has solution

$$x = \frac{-a \pm\sqrt{a^2 - 4GH}}{2G}. \quad (8)$$

Expanding out (4) gives

$$(1 + p + q)x^4 + (2a + pa)x^3 + (2b + a^2 + p(c + b) + 2qc)x^2$$
$$+ (2ab + pac)x + b^2 + pbc + qc^2 = 0, \quad (9)$$

comparing with equation (3)

$$m = a(2 + p)/(1 + p + q) \quad (10)$$
$$K(1 + p + q) = 2b + a^2 + p(c + b) + 2qc \quad (11)$$
$$Km + L = (2ab + pac)/(1 + p + q)$$
$$Lm = (b^2 + pbc + qc^2)/(1 + p + q),$$

and on eliminating m from (10)

$$Ka(2 + p) + L(1 + p + q) = 2ab + pac \quad (12)$$
$$La(2 + p) = b^2 + pbc + qc^2. \quad (13)$$

We will put for convenience c = 1, giving

$$K(1 + p + q) = 2b + a^2 + p(1 + b) + 2q \quad (14)$$
$$Ka(2 + p) + L(1 + p + q) = 2ab + pa \quad (15)$$
$$La(2 + p) = b^2 + pb + q, \quad (16)$$

and eliminate q from, say, (14) to give

$$q = [–K(1 + p) + 2b + a^2 + p(1 + b)]/(K – 2) \quad (17)$$
$$Ka(2 + p) + L(1 + p) + L[–K(1 + p) + 2b + a^2 + p(1 + b)]/(K – 2) =$$
$$a(2b + p) \quad (18)$$
$$La(2 + p) = b^2 + pb + [–K(1 + p) + 2b + a^2 + p(1 + b)]/(K – 2). \quad (19)$$

We will use (18) and (19) to give two expressions for p.

$$\{Ka – a + L + L[–K + 1 + b]/(K – 2)\}p =$$
$$\{–2Ka – L – L[–K + 2b + a^2]/(K – 2) + 2ab\} \quad (20)$$
$$\{La – b – [–K + (1 + b)]/(K – 2)\}p =$$
$$\{–2La + b^2 + [–K + 2b + a^2]/(K – 2)\}, \quad (21)$$

and then set, for the number D

$$\{Ka - a + L + L[-K + 1 + b]/(K - 2)\} = D\{La - b - [-K + 1 + b]/(K - 2)\},$$

giving a linear relationship between a and b, for, say, D = 1

$$[K - 1 - L]a = [(-L - K + 1)b + (-L + K + 1)]/(K - 2) \qquad (22)$$

giving for a^2

$$[K - 1 - L]^2a^2 = [(-L - K + 1)b + (-L + K + 1)]^2/(K - 2)^2 \qquad (23)$$

whereas equations (20) and (21) combine to give

$$-2Ka - L - L[-K + 2b + a^2]/(K - 2) + 2ab =$$
$$-2La + b^2 + [-K + 2b + a^2]/(K - 2),$$
$$2[-K + L + b]a - L - [L - 1][-K + 2b + a^2]/(K - 2) = b^2, \qquad (24)$$

which means for instance that the term in b^2 is nontrivially

$$\{(-L - K + 1)^2/[(K - L - 1)^2(K - 2)^2] + 2(-L - K + 1)/[(K - L - 1)(K - 2)] - 1\}b^2,$$

so that substituting for a in (22) and a^2 in (23) into (24) gives a solvable quadratic for b, where the full equation is

$$\left[2(-L - K + 1) - \frac{(L-1)(-L-K+1)^2}{(K-2)^2(K-L-1)} - (K-2)(K-L-1)\right]b^2$$
$$+ 2[(-K + L)(-L - K + 1) + (-L + K + 1) - (L - 1)(K - L - 1)]b$$
$$- 2\left[\frac{(-L-K+1)^2}{(K-2)(K-L-1)}\right]b$$
$$+ [2(-K + L)(-L + K + 1) - L(K - 2)(K - L - 1) - (L - 1)(-K)(K - L - 1)]$$
$$+ \frac{(-L+K+1)^2}{(K-2)^2(K-L-1)} = 0. \qquad (25)$$

which allows further simplification. It then determines a in (22), thus p in (20), q in (17), m from (10) and we have set c = 1. By these means we are able to solve for x in (8), providing the solution of essentially the cubic (1) entirely in terms of square roots. ☐

An Argand diagram for complex numbers containing a real and imaginary axis represents these numbers geometrically. So a Pythagoras theorem representation of a right-angled triangle can be used to represent a square root. This arises because it is possible geometrically to bisect a line, and if $\sqrt{q}$ is a number we wish to represent geometrically, then

$$(q - 1)^2 + 4q = (q + 1)^2$$
$$(q - 1)^2 + (2\sqrt{q})^2 = (q + 1)^2,$$

so that if q can be constructed, so can $\sqrt{q}$.

If we choose K = 0 and L = –2 in (1) so

$$x^3 = 2, \qquad (26)$$

then we find from (25) for example that

$$b = \frac{-6 \pm\sqrt{3201}i}{59},$$

with similar evaluations for other variables, and we find that the cube root of 2 given by (26) is geometrically realisable. ☐

5.4. The comparison solution in radicals of the sextic.

Although we have shown that there is no solution of the sextic by killing central terms, our theory of dependencies does not extend to comparison methods where the degree does not descend. Thus we have no overarching theory in this case, and the author has been reduced to looking at specific cases. As discussed in [Ad15], the method of using (x + d) as the second variable in the variety does not work, but the choice $(x^2 + d)$ is effective in finding a solution by radicals for the sextic.

We introduce the general sextic equation

$$x^6 + Gx^5 + Hx^4 + Kx^3 + Lx^2 + Mx + N = 0. \quad (1)$$

Our comparison equation is

$$(x^3 + ax^2 + bx + c)^2 + p(x^3 + ax^2 + bx + c)(x^2 + d) + q(x^2 + d)^2. \quad (2)$$

The solution of (23) this time is the solution of

$$x^3 + ax^2 + bx + c = \left[\frac{-p \pm \sqrt{p^2 - 4q}}{2}\right](x^2 + d), \quad (3)$$

the solvable cubic

$$x^3 + \left[a + \frac{p \mp \sqrt{p^2 - 4q}}{2}\right]x^2 + bx + c + \left[\frac{p \mp \sqrt{p^2 - 4q}}{2}\right]d = 0. \quad (4)$$

Expanded out, (2) is

$$x^6 + 2(ax^2 + bx + c)x^3 + (ax^2 + bx + c)^2$$
$$+ p(x^5 + ax^4 + (b + d)x^3 + (c + ad)x^2 + bdx + cd)$$
$$+ q(x^4 + 2dx^2 + d^2) = 0$$

or

$$x^6 + (2a + p)x^5 + (2b + a^2 + pa + q)x^4 + (2ab + 2c + p(b + d))x^3$$
$$+ (b^2 + 2ac + p(c + ad) + 2qd)x^2 + (2bc + pbd)x$$
$$+ (c^2 + pcd + qd^2) = 0. \quad (5)$$

Comparing (1) and (5)

$$G = 2a + p \quad (6)$$
$$H = 2b + a^2 + pa + q \quad (7)$$
$$K = 2c + 2ab + p(b + d) \quad (8)$$
$$L = b^2 + 2ac + p(c + ad) + 2qd \quad (9)$$
$$M = b(2c + pd) \quad (10)$$
$$N = c^2 + pcd + qd^2. \quad (11)$$

We now make the observation that M factorises, so that if $M = 0$, then if $b \neq 0$, we can divide by it and reduce in effect the degree of equation (10). But this is possible since a linear substitution in y of the equation

$$y^6 + ry^5 + sy^4 + ty^3 + uy^2 + vy + w = 0$$

can reduce the coefficient of y^5 to zero, and on substitution of $y = 1/x$, we obtain effectively equation (1) with $M = 0$, and bypass a solution of the quintic.

Using equation (6)

$$p = G – 2a, \quad (12)$$

which enables us to obtain q from equation (7), namely

$$H = a^2 + 2b + Ga – 2a^2 + q$$

or

$$q = H + a^2 – Ga – 2b. \quad (13)$$

Then equation (8) becomes

$$K = 2ab + 2c + (G – 2a)(d + b)$$
$$K = 2c + Gd + Gb – 2ad, \quad (14)$$

whereas (12) and (13) in (9) gives

$$L = 2ac + b^2 + (G – 2a)(ad + c) + 2d(H + a^2 – Ga – 2b)$$
$$L = b^2 – Gad + Gc + 2Hd – 4db. \quad (15)$$

From (10) using $M = 0$ and $b \neq 0$

$$c = \left(a - \frac{G}{2}\right)d, \quad (16)$$

$$c^2 = \frac{G^2d^2}{4} - Gad^2 + a^2d^2. \tag{17}$$

Equation (11) now becomes

$$N = c^2 + (G - 2a)cd + (H + a^2 - Ga - 2b)d^2$$

$$N = c^2 + Gcd - 2acd + Hd^2 + a^2d^2 - Gad^2 - 2bd^2. \tag{18}$$

Using (16) and (17) in (14), (15) and (18)

$$K = Gb, \tag{19}$$

$$L = b^2 - Gad + G\left(a - \frac{G}{2}\right)d + 2Hd - 4db$$

$$L = b^2 - \frac{G^2d}{2} + 2Hd - 4db, \tag{20}$$

$$N = \frac{G^2d^2}{4} - Gad^2 + a^2d^2 + (G - 2a)\left(a - \frac{G}{2}\right)d^2 + Hd^2 + a^2d^2 - Gad^2 - 2bd^2$$

$$N = -\frac{G^2d^2}{4} + Hd^2 - Gad^2 - 2bd^2. \tag{21}$$

Now substituting

$$b = \frac{K}{G} \tag{22}$$

from (19) in (20) gives

$$L = \left(\frac{K}{G}\right)^2 - \frac{G^2d}{2} + 2Hd - \frac{4Kd}{G}$$

$$d\left(-\frac{G^2}{2} + 2H - \frac{4K}{G}\right) = L - \left(\frac{K}{G}\right)^2$$

$$d = \frac{L - \left(\frac{K}{G}\right)^2}{\left(-\frac{G^2}{2} + 2H - \frac{4K}{G}\right)}, \tag{23}$$

and in (21)

$$N = -\frac{G^2d^2}{4} + Hd^2 - Gad^2 - \frac{2K}{G}d^2,$$

so that

$$aG = -\frac{N}{d^2} - \frac{G^2}{4} + H - \frac{2K}{G}, \tag{24}$$

which gives from (23)

$$a = -\frac{N\left(-\frac{G^2}{2} + 2H - \frac{4K}{G}\right)^2}{G\left(L - \left(\frac{K}{G}\right)^2\right)^2} - \frac{G}{4} + \frac{H}{G} - \frac{2K}{G^2}. \tag{25}$$

We now have a from (25), d from (23), b from (22), c from the a and d in (16), q from a and b in (13) and p from a in (12), giving all coefficients in the solvable cubic (4) as the solution of an equation equivalent to the general sextic (1) as (2), where we have performed a reciprocal Tschirnhaus transformation on (1) to set M = 0. ☐

5.5. The comparison method for the octic.

Consider

$$x^8 + Ex^7 + Fx^6 + Gx^5 + Hx^4 + Kx^3 + Lx^2 + Mx + N = 0. \tag{1}$$

to be compared with

$$(x^4 + ax^3 + bx^2 + cx + d)^2 + p(x^4 + ax^3 + bx^2 + cx + d)(tx^4 + ux^2 + v)$$
$$+ q(tx^4 + ux^2 + v)^2. \tag{2}$$

For the coefficient of x^8 we want

$$1 + pt + qt^2 = 1,$$

or

$$t = -\left(\frac{p}{q}\right). \tag{3}$$

The reason we want t is that t = 0 leaves u and v dependent on p and q, and thus u is not an independent variable, but with t present, we bind the variable u in 2qtu = –2pu , qu^2 and 2quv in the expression involving q in (2), three independent terms in u for fixed v, since we have (qu)u and 2(qu)v.

Expanded out, equation (2) becomes

$$x^8 + (2a + pat)x^7 + (2b + a^2 + p(u + bt) + 2qtu)x^4 + (2ab + 2c + p(au + ct))x^5$$
$$+ (2d + 2ac + b^2 + p(bu + v + td) + q(2tv + u^2))x^4$$
$$+ (2ad +2bc + p(av + cu))x^3 + (c^2 + 2bd + p(bv + du) + 2quv)x^2$$
$$+ c(2d + pv)x + (d^2 + pdv + qv^2),$$

which using (3) is

$$x^8 + a\left(2 - \frac{p^2}{q}\right)x^7 + \left(a^2 + 2b + p\left(-u - \frac{pb}{q}\right)\right)x^6$$
$$+ \left(2ab + 2c + p\left(au - \frac{pc}{q}\right)\right)x^5$$
$$+ \left(2d + 2ac + b^2 + p\left(bu - v - \frac{pd}{q}\right) + qu^2\right)x^4$$
$$+ \left(2ad + 2bc + p(av + cu)\right)x^3$$
$$+ (c^2 + 2bd + p(bv + du) + 2quv)x^2$$
$$+ (2d + pv)x + (d^2 + pdv + qv^2) = 0. \qquad (4)$$

We now compare the coefficients of (1) and (4) to get

$$E = a\left(2 - \frac{p^2}{q}\right) \qquad (5)$$
$$F = a^2 + 2b + p\left(-u - \frac{pb}{q}\right) \qquad (6)$$
$$G = 2ab + 2c + p\left(au - \frac{pc}{q}\right) \qquad (7)$$
$$H = 2d + 2ac + b^2 + p\left(bu - v - \frac{pd}{q}\right) + qu^2 \qquad (8)$$
$$K = 2ad + 2bc + p(av + cu) \qquad (9)$$
$$L = c^2 + 2bd + p(bv + du) + 2quv \qquad (10)$$

(as we did for the sextic, the next equation uses M = 0, and assumes c ≠ 0)

$$0 = 2d + pv \qquad (11)$$
$$N = d^2 + pdv + qv^2. \qquad (12)$$

We will start with the simplest equation, (11)

$$d = -\frac{pv}{2} \qquad (13)$$

and introduce it in (8) – (10) and (12).

$$H = -2pv + 2ac + b^2 + p\left(bu - \frac{p^2v}{2q}\right) + qu^2 \qquad (14)$$
$$K = 2bc + pcu \qquad (15)$$
$$L = c^2 - \frac{p^2uv}{2} + 2quv \qquad (16)$$
$$N = -\frac{p^2v^2}{4} + qv^2. \qquad (17)$$

We will then use (5) to eliminate a, the information via (6) to eliminate b, and likewise via (7) to eliminate c.

$$a = \frac{Eq}{(2q - p^2)} \qquad (18)$$

$$F = \frac{E^2q^2}{(2q-p^2)^2} + 2b - pu - \frac{p^2b}{q} \tag{19}$$

$$G = \frac{2Eqb}{(2q-p^2)} + 2c + pau - \frac{p^2c}{q} \tag{20}$$

$$H = -2pv + \frac{2Eqc}{(2q-p^2)} + b^2 + pbu - \frac{p^3v}{2q} + qu^2. \tag{21}$$

From (19)

$$\left(2 - \frac{p^2}{2}\right)b = F - \frac{E^2q^2}{(2q-p^2)^2} + pu$$

$$b = \frac{q}{(2q-p^2)}\left[F - \frac{E^2q^2}{(2q-p^2)^2} + pu\right], \tag{22}$$

from (20)

$$G = \frac{2Eq^2}{(2q-p^2)^2} + 2c + \frac{Eqpu}{(2q-p^2)} - \frac{p^2c}{q}, \tag{23}$$

from (21)

$$H = -2pv + \frac{2Eqc}{(2q-p^2)} + \frac{q^2}{(2q-p^2)^2}\left[F - \frac{E^2q^2}{(2q-p^2)^2} + pu\right]^2 + \frac{qpu}{(2q-p^2)}\left[F - \frac{E^2q^2}{(2q-p^2)^2} + pu\right] - \frac{p^3v}{2q} + qu^2 \tag{24}$$

and (15) gives

$$K = c\left[\frac{2q}{(2q-p^2)}\left[F - \frac{E^2q^2}{(2q-p^2)^2} + pu\right] + pu\right]. \tag{25}$$

We will now use (23) to eliminate c.

$$\left(\frac{2q}{q} - \frac{p^2}{q}\right)c = G - \frac{2Eq^2}{(2q-p^2)^2} - \frac{Eqpu}{(2q-p^2)}$$

$$c = \frac{q}{(2q-p^2)}\left[G - \frac{2Eq^2}{(2q-p^2)^2} - \frac{Eqpu}{(2q-p^2)}\right]. \tag{26}$$

Using (26), (24) becomes

$$H = -2pv + \frac{2Eq^2}{(2q-p^2)^2}\left[G - \frac{2Eq^2}{(2q-p^2)^2} - \frac{Eqpu}{(2q-p^2)}\right] + \frac{q^2}{(2q-p^2)^2}\left[F - \frac{E^2q^2}{(2q-p^2)^2} + pu\right]^2 + \frac{qpu}{(2q-p^2)}\left[F - \frac{E^2q^2}{(2q-p^2)^2} + pu\right] - \frac{p^3v}{2q} + qu^2 \tag{27}$$

and using (26) in (25) gives

$$K = \frac{q}{(2q-p^2)}\left[G - \frac{2Eq^2}{(2q-p^2)^2} - \frac{Eqpu}{(2q-p^2)}\right]\left[\frac{2q}{(2q-p^2)}\left[F - \frac{E^2q^2}{(2q-p^2)^2} + pu\right] + pu\right], \tag{28}$$

whereas (26) in (16) gives

$$L = \frac{q^2}{(2q-p^2)^2}\left[G - \frac{2Eq^2}{(2q-p^2)^2} - \frac{Eqpu}{(2q-p^2)}\right]^2 + \left(2q - \frac{p^2}{2}\right)uv. \tag{29}$$

We now have to solve simultaneously (27), (28), (29) and (17). Our main concern is the degree of the equation which we have finally to solve, since we can only go up to the sextic, which we solved in section 4.

The first part of our strategy might be to look at these equations in order to reduce the degree of the variables we are using by changing these variables to composite ones. However we will bypass these allocations for the moment and state plainly that although each individual equation can be reduced by these means to the form of a sextic or less, collectively under substitutions of one equation in others to reduce the number of equations, this strategy fails, and the degree is too high. We have therefore to reappraise our circumstances and come up with an alternative.

A possible stratagem, on looking at the detailed form of these equations, is that equation (28) for K, and this is easier to see this from equation (15) from which it is derived, factorises on the right hand side, being a product with c. Thus if it were possible to set K = 0 and maintain the generality of our equation at the same time, then this could be (and is) effective in yielding a solution. We will go directly into a discussion of this issue.