

## CHAPTER IV

### The novanion simple group construction

#### 4.1. Invitation to the reader.

Groups are the most investigated topic for superexponential structures with one operation. The theory of novanions leads us in a natural way to ask what impact novanions have on the theory of groups, even the classification of simple groups, for which the claim has been made that a complete classification was achieved after monumental efforts finally in 2008.

The novanion idea for these groups is a new subject for which there are currently few experts, and not the writer. Nevertheless, there are logical deductions which can be made, and I shall be making them. Since the conclusions do not tie in with what is believed to be a correct proof of the classification, a question is whether ignorance and delusion lies with the author, or with society at large, which encompasses many individuals some of whom have diligently spent a lifetimes work on the study of simple groups. But it is our belief that the origin of the discrepancy has now been isolated, which is described in the text.

The expert in group theory is invited to correct any misconceptions so that revisions can be made, and so the mathematician in the street can be better informed.

#### 4.2. Introduction and history.

We introduce an analysis of group theory via the theory of novanions, developed in [Ad15], and also in the mathematical chapter XVII of *New Physics* [Ad18] by Graham Ennis and me with applications to universal physics. Novanions are a generalisation of octonions. Novanion groups are found as a generalisation of the way octonions lead to special Lie groups. The octonions have a (mod 12) Jacobi identity mapping to algebras, and octonions are related to some exceptional Lie algebras, like  $E_8$ , with corresponding mapping to groups. We will investigate whether these two methods give the same results.

The beginning of this study arises out of a speculation of Daniel Hajas that I call the Hajas conjecture, in which novanions are connected with the heterotic string in physics, 10-novanions, to be described, are related to the 10-dimensional fermionic string in physics, and my extension of that idea, that the 26-dimensional bosonic string is related to 26-novanions. We will see we need to go beyond this. The heterotic string has a fermionic dimension of 10 and a bosonic dimension of 26, so that in the conventional picture  $26 - 10 = 16$  dimensions in the universe are compactified. We have shown in the above references that 26-novanions, containing octonion type algebras are also present.

Picking up the work of Borchers linking the monster simple group to orbifolds in the 26 dimensions of the heterotic string, we investigate whether there are simple groups beyond the monster, in direct conflict with current understandings. Our motivation is as follows. On page 938 of ‘Quantum field theory’ by Eberhard Zeidler, with reference to the Thompson series, he writes

“Borchers calculated this series using the monster Lie algebra. This Lie algebra is constructed as the space of physical states of a bosonic string moving in a  $\mathbb{Z}_2$  orbifold  $M/\mathbb{Z}_2$  of a 26-dimensional torus  $M$ ”.

By this means the classification of simple groups is derived. Preliminary investigation of the work of Borchers led credence to the Hajas conjecture, but subsequent more detailed investigations give mismatches in the computed size, called the order, of simple groups derived from novanions. This classification is finite, according to conventional wisdom. We wish to investigate whether or not a spanner can be thrown in the works, and note that the number of distinct  $n$ -novanionic algebras is infinite, given that the override condition in sections 8.13 and 8.14 can always be allocated, which corresponds non-trivially when the  $n$ -novanion contains octonionic components. The reader is invited to investigate the proof of the finiteness condition. More in-depth investigation is needed to compare standard reasoning with what is given here.

However, we note an interesting nonstandard conclusion we have arrived at with the Hajas identification, given essentially in [Ad18] chapter XI, namely that all  $n$ -novanions can be allocated bosonic and fermionic parts. Thus both the 10- and 26-novanions may be allocated these components, and this also holds in general for  $n$ -novanions.

In chapter XX of [Ad18] we employ the 26-novanions in Heim theory extended to gluons and quarks. There exist other possible universes with  $n > 26$ . So the 26-novanions do indeed contain a bosonic algebra.

The conjecture is: can we apply the unbounded  $n$ -novanionic algebra to derive a Thompson series so that the number of simple groups is not bounded?

We make this statement because override conditions for novanions of dimension greater than 26 lead to simple groups via Jacobi identity mappings to algebras, since overrides mix different quaternions together, creating one overarching structure. However,  $E_8$  has order linked to that of the monster. The simple groups we are about to investigate have a size greater than that of the monster.

Although simple groups of similar order of magnitude to the monster have been obtained by novanion methods, they differ. Moreover, there are sequences of simple novanion groups without termination, violating the finiteness condition on the classification of sporadic groups. The groups corresponding to novanion algebras are not initially simple, but quotients with novanion subalgebras lead to factor groups which are simple.

Recent investigation reveals the reason for why the Borchers and novanion constructions differ. The monster can be derived from vertex operator algebras with infinite quadratic form. Novanion algebras sometimes have zero quadratic form for a zero scalar component,  $t$ . These zero forms are not obtained as diagonalised forms which can be obtained by multiplication of the novanion by its conjugate. The Lie algebra construction forbids algebras of zero quadratic form. The scalar  $t = 0$  component is realised in the final novanion algebra in conjunction with other additive elements to form a composite element, but we can map  $t = 0 \rightarrow t = \pm 1$ , and so novanions are allowable objects in such algebra constructions, although this has not hitherto been recognised. From this point of view the mapping from 10-novanions to fermion strings and 26-novanions to bosonic strings used in the Borchers construction is spurious.

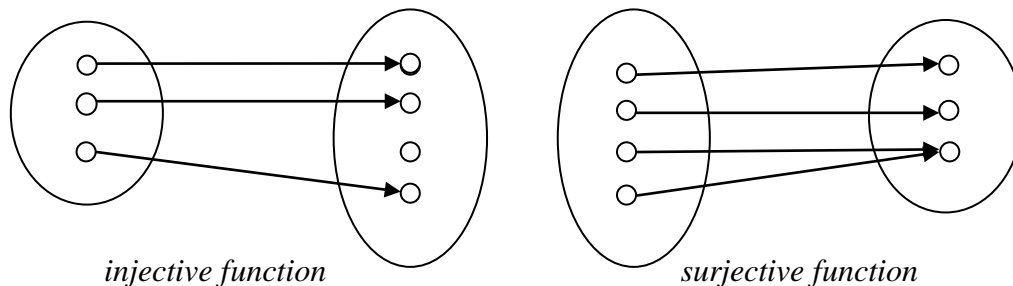
I am unclear at this stage whether other aspects of my reasoning are a novelty. These ideas are used to derive novanion groups, and can equally be used for division algebras like the octonions. The groups corresponding to novanion algebras are not initially simple, but quotients with novanion subalgebras lead to factor groups which are simple.

The development proceeds by first describing axiom systems which give mathematical structures for numbers and their generalisations. The intention is to provide the background for the discussion which follows. We discuss the classification of simple groups and their order. Various basic terms used in the classification such as the center of a group, factor groups, normal subgroups, Lagrange's subgroup theorem that the order of a subgroup divides the order of its group, and the Schur multiplier are introduced. We derive various finite groups, which are not subgroups of the quaternions, from the quaternion Lie algebra.

We next give a detailed discussion of novanions, which are division algebras when the scalar part is not zero. Octonions are division algebras. We introduce 10-, 26- and more general novanions. In the case of octonions, we show that they have a finite representation as a Lie algebra (mod 12), from which we may obtain a group. Products in the group which would otherwise be 0 (mod 12) are converted to  $\pm e^{2\pi i n/3}$ . This maintains the Lie algebra structure and means that the group derived from it is provided with inverses.

### 4.3. Concepts in group theory.

A function  $f: S \rightarrow T$  is called *injective* (or *one-to-one*, or an injection) if  $f(a) \neq f(b)$  for any two different elements  $a$  and  $b$  of the domain. It is called *surjective* (or *onto*) if  $f(S) = T$ . That is, it is surjective if for every element  $y$  in the codomain there is an  $x$  in the domain such that  $f(x) = y$ . The function  $f$  is called *bijective* if it is both injective and surjective.



A *dual* (or *opposite*) map reverses all arrows. If the original function is injective, then some of the elements of the opposite map may not have values in the codomain, so this is not a function on elements. Likewise, if the original function is surjective, the opposite map for an element in its domain may have not one but a set of elements corresponding to this element in its codomain, and again it is not a function on elements.

An *inverse function* (or *fiber*) of a bijective map  $x \leftrightarrow f(x)$  is the map  $f^{-1}: f(x) \leftrightarrow x$ .

Sets of numbers may be combined under operations like '+' or '×' to form other numbers.

A *magma* is the most general structure combining sets and an operation. A magma is a set  $M$  with a single binary operation  $M \times M \rightarrow M$ , combining elements in pairs of the magma, with each pair forming another element belonging to the magma. No other properties are specified in this definition.

A *group* is a magma with the following structure. The operation on the magma can be written either additively or multiplicatively without brackets, the two choices being equivalent within the group. For an *abelian group* ( $a + b = b + a$ ) the *identity*,  $e$ , for a group if written additively is the element 0, or 1 written multiplicatively, where  $a + 0 = a = 0 + a$ . Groups have *inverses* ( $-a$ ) of a written additively or  $a^{-1}$  written multiplicatively with the additive rule

$$a + (-a) = 0,$$

alternatively if written multiplicatively

$$a(a^{-1}) = 1 = (a^{-1})a.$$

A group is *nonabelian* or *noncommutative*, generally written multiplicatively, if some  $ab \neq ba$ .

In any group the integral *power* of an element  $a$  can be defined as the element

$$a^m = a.a. \dots a \text{ (m terms)}.$$

Negative powers can be defined by

$$a^m a^{-m} = 1.$$

A group  $G$  is called *cyclic* if it contains an element the powers of which exhaust  $G$ . Cyclic groups are abelian. It is often understood that cyclic groups are finite. When this is not so, we will explicitly state that they are infinite.

A *permutation* is a bijection of a finite set to itself. A permutation which interchanges cyclically  $m$  objects of a set  $\{1, 2, \dots, m\}$  forms an abelian group called a cycle of degree  $m$ . This permutation is obtained from a power by specifying that  $a^{m-1}$  is the  $m^{\text{th}}$  element and the cyclic permutation consists of multiplying by  $a$ . It can be represented by

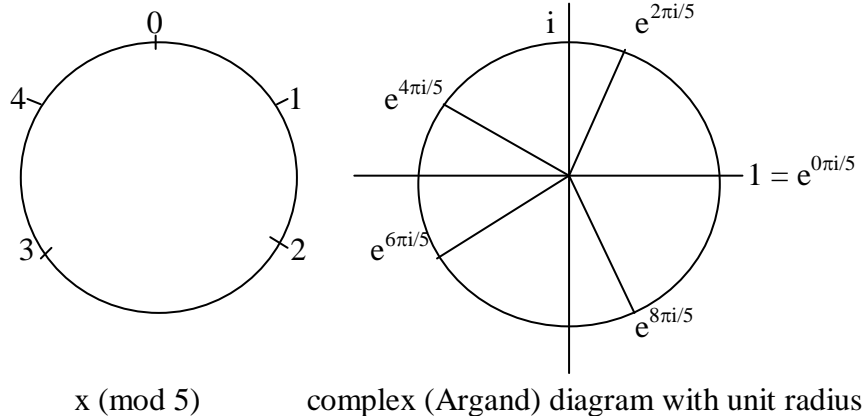
$$\begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ 2 & 3 & \dots & m & 1 \end{pmatrix},$$

or in contracted notation by  $(1\ 2\ \dots\ m)$ .

Another picture of a cyclic group is given by the ‘clock’ diagram  $x \pmod p$  and the Argand complex circle diagram, where there is a bijection for fixed  $r$

$$x \pmod p \leftrightarrow e^{r + (2\pi i x/p)},$$

which can be pictured in the example diagrams for  $p = 5$ :



A group derived from cyclic group generators which do not intersect, so the generators form a partition for the group, is also cyclic. For a finite cyclic group its number of elements, or *order*, which is the number of times it takes a generator to return to the identity permutation, is the least common multiple (l.c.m.) of the order of its cyclic components. An example of a cyclic permutation with the identity permutations present is

$$(1\ 2)(4\ 6\ 7)(3)(5)$$

which we can contract by removing the identity permutations to

$$(1\ 2)(4\ 6\ 7) = (4\ 6\ 7)(1\ 2).$$

The set of all permutations of  $m$  objects forms a group called the *symmetric group*, denoted by  $S_m$ . The name is derived from its origins in describing polynomial equations.

A noncommutative group can be generated by cycles which overlap somewhere. For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The symmetric group can be described by matrices. For example  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$  can be represented by the matrix with one 1 in each row and column, and zeros elsewhere

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

in which, say,  $2 \rightarrow 4$  is represented by a 1 in the second row and fourth column, with operations defined by matrix multiplication. As a further example, the cyclic group of order 4 is given by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

All elements of  $S_4$  can be obtained from the above by permuting rows or alternatively and equivalently by permuting columns.

A *subgroup*  $S$  of a group  $G$  is a group included in  $G$ . If  $S \neq G$ ,  $S$  is a *proper subgroup*. The number of elements in the subgroup is called the order of the subgroup. The complement of  $S$  in  $G$  cannot form a subgroup, since 1 does not belong to it.

The *center* of a group  $G$ , denoted  $Z(G)$ , is the set of elements that commute with every element of  $G$ .

The *commutator* of two group elements  $g$  and  $h$  is

$$[g, h] = g^{-1}h^{-1}gh.$$

The commutator  $[g, h]$  is equal to the identity element  $e$  if and only if  $hg = gh$ .

The *commutator subgroup*  $[G, G]$  (also called the *derived subgroup* of  $G$  and denoted  $G^{(1)}$ ) is the subgroup generated by all the commutators.

A *homomorphism*  $h$  of a group  $G$  to a group  $G'$  is a surjective map  $ab = g \rightarrow h(g)$ , such that  $h(ab) = h(a)h(b)$ .

An *automorphism* is a homomorphism  $G \rightarrow G$ .

**Theorem 4.5.1.** Under a homomorphism the identity  $e$  of  $G$  maps to the identity  $h(e)$  of  $G'$ , and maps inverses  $g^{-1}$  to  $h(g)^{-1} = h(g^{-1})$ .

*Proof.* The identity satisfies  $ee = e$ , so  $h(ee) = h(e) = h(e)h(e)$ . The inverse satisfies  $(g)(g^{-1}) = e$ , so  $h(g)h(g^{-1}) = h(e) = h(g)h(g)^{-1}$ , and multiplying on the left by  $h(g)^{-1}$  gives  $h(g^{-1}) = h(g)^{-1}$ .  $\square$

The set  $\{k\}$  is the *kernel* of a group homomorphism  $h: G \rightarrow G'$ , if it satisfies

$$h(a)h(k) = h(a) = h(k)h(a),$$

in other words it is the identity of  $G'$ .

A *right coset* or *right residue class* of a subgroup  $S$  of  $G$  is the set of elements  $Sa$ , with  $s \in S$  and  $a \in G$ . A *left coset* is the set  $aS$ , and when both coincide the set can be called a *coset*.

The *quotient group*  $G/S$  of  $G \text{ mod } S$  for  $G$  a group,  $S$  a subgroup, is the family of left cosets.

**Lemma 4.3.2.** *If  $S$  is finite, each right (or left) coset has as many elements as  $S$ . Two right (or left) cosets are either identical or have no common elements.*

*Proof.* The map  $a \rightarrow sa$  is a bijection, since each  $sa$  is the image of one and only one  $a$ , and if  $a \rightarrow sb$ , with  $b \neq a$  then  $1 = a(a^{-1})$  maps to  $sb(a^{-1}) = s(ba^{-1}) = s$ , so  $b = a$ . Further, if there were any intersection, then  $sb = sa$ , which we have shown is impossible unless  $b = a$ .  $\square$

If  $G$  is finite, it can be partitioned into a finite number of right or left cosets where each coset contains the same number of elements, and the conclusion is

**Theorem 4.3.3. (Lagrange's subgroup theorem).** *The order of a finite subgroup  $S \subseteq G$  divides the order of a finite group containing it.*  $\square$

A *conjugate* of an element  $x$  in a group  $G$  is an element  $a^{-1}xa$ .

**Theorem 4.3.4.** *For an element  $a$  of  $G$ , conjugation  $T_a: x \rightarrow a^{-1}xa$  is an automorphism of  $G$ .*

*Proof.*  $(a^{-1}xa)(a^{-1}ya) = a^{-1}(xy)a$ .  $\square$

An automorphism of the form  $a^{-1}xa$  is called an *inner automorphism*, otherwise it is called an *outer automorphism*.

It follows from what we have said that inner automorphisms form a subgroup of all the automorphisms of a group  $G$ .  $\square$

A subgroup  $S$  of  $G$  is *normal* in  $G$  if and only if it is invariant under all inner automorphisms of  $G$ . For example, consider the symmetric group  $S_3$  of all permutations of the set  $\{1, 2, 3\}$ . Then  $\{(1, (1\ 2\ 3), (1\ 3\ 2))\}$  is a normal subgroup of  $S_3$ , because we can verify the following statements.

$$(1\ 2)\{(1, (1\ 2\ 3), (1\ 3\ 2))\} = \{(1\ 2), (1\ 3), (2\ 3)\} = \{(1, (1\ 2\ 3), (1\ 3\ 2))\}(1\ 2)$$

$$(1\ 3)\{(1, (1\ 2\ 3), (1\ 3\ 2))\} = \{(1\ 3), (2\ 3), (1\ 2)\} = \{(1, (1\ 2\ 3), (1\ 3\ 2))\}(1\ 3)$$

$$(2\ 3)\{(1, (1\ 2\ 3), (1\ 3\ 2))\} = \{(2\ 3), (1\ 2), (1\ 3)\} = \{(1, (1\ 2\ 3), (1\ 3\ 2))\}(2\ 3).$$

**Theorem 4.3.5.** *A subgroup  $S$  is normal if and only if all of its right cosets are left cosets.*

*Proof.* Suppose  $S$  is normal. Then  $aSa^{-1} = (a^{-1})^{-1}S(a^{-1}) = S$ . Thus  $Sa = aS$ . Conversely, applying lemma 4.3.2, if two cosets are equal so that  $Sa = bS$ , then  $a = b$  and  $S$  is normal.  $\square$

It should be carefully noted that the equation  $Sa = aS$  does not claim that every element of  $S$  commutes with  $a$ , only that the cosets  $Sa$  and  $aS$  are the same.

A group  $G$  is called *simple* if its only normal subgroups are the identity and  $G$  itself.

The *general linear group*  $GL$  of degree  $n$  is the set of  $n \times n$  invertible matrices, meaning they have a multiplicative inverse, together with the operation of ordinary matrix multiplication.  $GL(n, \mathbb{C})$  are invertible matrices with complex number elements.

The *projective linear group*  $PGL$  is the induced action of the general linear group of a vector space  $V$  on the associated projective space  $P(V)$ . Explicitly, the projective linear group is the quotient group

$$PGL(V) = GL(V)/Z(V)$$

where  $GL(\mathbf{V})$  is the general linear group of  $\mathbf{V}$  and  $Z(\mathbf{V})$  is the subgroup of all nonzero scalar transformations of  $\mathbf{V}$ . These are quotiented out because they act trivially on the projective space and they form the kernel of the action. The notation "Z" is used because the scalar transformations form the center of the general linear group.

A group homomorphism from  $D$  to  $G$  is said to be a *Schur cover* of the finite group  $G$  if the kernel is contained both in the center and the commutator subgroup of  $D$ , and amongst all such homomorphisms, this  $D$  has maximal size.

The *Schur multiplier* of  $G$  is the kernel of any Schur cover. When the homomorphism is understood, the group  $D$  is often called the Schur cover.

Schur's motivation for studying the multiplier was to classify projective representations of a group. A projective representation is much like a group representation except that instead of a homomorphism into the general linear group  $GL(n, \mathbb{C})$ , one takes a homomorphism into the projective general linear group  $PGL(n, \mathbb{C})$ . In other words, a projective representation is a representation modulo the center.

#### 4.4. The standard classification of simple groups.

The classification of simple groups lists 26 sporadic simple groups. The number of elements of the sporadic simple groups, together with their Schur multipliers is listed below.

Simple group (p = pariah)	Order	Order of Schur multiplier
Monster $M$	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	1
Baby monster $B$	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	2
Thompson group $Th$	$2^{11} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	1
Lyons group $Ly$ (p)	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	1
Harada-Norton group $HN$	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	1
O'Nan group $O'N$ (p)	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	3
Suzuki sporadic group $Suz$	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	6
Rudvalis group $Ru$ (p)	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	2
Held group $He$	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	1
McLaughlin group $MCL$	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	3
Higman-Sims group $HS$	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2
Fischer group $Fi_{22}$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	6
Fischer group $Fi_{23}$	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	1
Fischer group $Fi_{24}$	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	3
Conway group $Co_1$	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	2
Conway group $Co_2$	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1
Conway group $Co_3$	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1
Janko group $J_1$ (p)	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	1
Janko group $J_2$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
Janko group $J_3$ (p)	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	3
Janko group $J_4$ (p)	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	1
Mathieu group $M_{11}$	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
Mathieu group $M_{12}$	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2
Mathieu group $M_{22}$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	12
Mathieu group $M_{23}$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
Mathieu group $M_{24}$	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1

The 20 sporadic groups which are subquotients of the monster are called the *happy family*. The remaining 6 are referred to as *pariahs*. 37 does not divide the order of the monster but divides  $L_7$  and  $J_4$ , which are therefore pariahs. Four other groups can be shown to be pariahs.

#### 4.5. The $E_8$ lattice.

For any basis of  $\mathbb{R}^n$ , the subgroup of all linear combinations with integer coefficients of the basis vectors forms a lattice, which can be viewed as a regular tiling of a space by a primitive cell called the fundamental parallelotope. A *unimodular* lattice is an integral lattice with determinant or hypervolume 1 or  $-1$ .

The  $E_8$  lattice is a special lattice in  $\mathbb{R}^8$ . It can be characterised as the unique positive-definite, even unimodular lattice of rank 8, so that it can be generated by the columns of an  $8 \times 8$  matrix where the determinant of the fundamental parallelotope of the lattice is  $\pm 1$ . The name derives from the fact that it is the root lattice of the  $E_8$  root system, described in [Ad15], chapter IV.

The *norm* of the  $E_8$  lattice (divided by 2) is a positive definite even unimodular quadratic form in 8 variables, and conversely such a quadratic form can be used to construct a positive-definite, even, unimodular lattice of rank 8. The existence of such a form was first shown by H. Smith in 1867 and the first explicit construction of this quadratic form was given by A. Korkin and G. Zoltarev in 1873. The  $E_8$  lattice is also called the *Gosset lattice* after T. Gosset who was one of the first to study the geometry of the lattice around 1900.

The  $E_8$  lattice is a discrete subgroup of  $\mathbb{R}^8$  which spans all of  $\mathbb{R}^8$ . It can be given explicitly by the set of points  $\Gamma_8 \subset \mathbb{R}^8$  such that

- all the coordinates are integers or half-integers but not a mixture of these
- the sum of the eight coordinates is an even integer.

The sum of two lattice points is another lattice point, so that  $\Gamma_8$  is indeed a subgroup.

An alternative description of the  $E_8$  lattice is the set of all points in  $\Gamma'_8 \subset \mathbb{R}^8$  such that

- all the coordinates are integers and the sum of the coordinates is even, or
- all the coordinates are half-integers and the sum of the coordinates is odd.

The lattices  $\Gamma_8$  and  $\Gamma'_8$  are isomorphic – we can pass from one to the other by changing the signs of any odd number of coordinates. The lattice  $\Gamma_8$  is called the *even coordinate system* for  $E_8$  while the lattice  $\Gamma'_8$  is called the *odd coordinate system*. Unless we specify otherwise we will work in the even coordinate system.

Even unimodular lattices occur only in dimensions divisible by 8. In dimension 16 there are two such lattices:  $\Gamma_8 \oplus \Gamma_8$ , and  $\Gamma_{16}$  constructed analogously to  $\Gamma_8$ . In dimension 24 there are 24 such lattices, called Niemeier lattices, the most important of which is the Leech lattice.

A possible basis for  $\Gamma_8$  is given by the columns of the upper triangular matrix



$$\Gamma_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$\Gamma_8$  is then the integral span of these vectors. All other possible bases are obtained from this one by right multiplication by elements of  $GL(8, \mathbb{Z})$ .

The shortest nonzero vectors in  $\Gamma_8$  have norm 2. There are 240 such vectors.

- All half-integer: (can only be  $\pm 1/2$ )
  - All positive or all negative: 2
  - Four positive, four negative:  $(8 \times 7 \times 6 \times 5) / (4 \times 3 \times 2 \times 1) = 70$
  - Two of one, six of the other:  $2 \times (8 \times 7) / (2 \times 1) = 56$
- All integer: (can only be 0,  $\pm 1$ )
  - Two  $\pm 1$ , six zeroes:  $4 \times (8 \times 7) / (2 \times 1) = 112$

These form a root system of type  $E_8$ . The lattice  $\Gamma_8$  is equal to the  $E_8$  root lattice, which means that it is given by the integral span of the 240 roots. Any choice of 8 simple roots gives a basis for  $\Gamma_8$ .

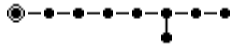
The automorphism group (or symmetry group) of a lattice in  $\mathbb{R}^n$  is defined as the subgroup of the orthogonal group  $O(n)$  that preserves the lattice. The symmetry group of the  $E_8$  lattice is the Weyl/Coxeter group of type  $E_8$ . This is the group generated by reflections in the hyperplanes orthogonal to the 240 roots of the lattice. Its order is given by

The  $E_8$  Weyl group contains a subgroup of order  $128 \cdot 8!$  consisting of all **permutations** of the coordinates and all even sign changes. This subgroup is the Weyl group of type  $D_8$ . The full  $E_8$  Weyl group is generated by this subgroup and the **block diagonal matrix**  $H_4 \oplus H_4$  where  $H_4$  is the **Hadamard matrix**

## Geometry[edit]

See **5<sub>21</sub> honeycomb**

The  $E_8$  lattice points are the vertices of the **5<sub>21</sub> honeycomb**, which is composed of regular **8-simplex** and **8-orthoplex facets**. This honeycomb was first studied by Gosset who called it a *9-ic semi-regular figure*<sup>[4]</sup> (Gosset regarded honeycombs in  $n$  dimensions as degenerate  $n+1$  polytopes). In **Coxeter's** notation,<sup>[5]</sup> Gosset's honeycomb is denoted by **5<sub>21</sub>** and has the **Coxeter-Dynkin diagram**:



This honeycomb is highly regular in the sense that its symmetry group (the affine Weyl group) acts transitively on the  $k$ -faces for  $k \leq 6$ . All of the  $k$ -faces for  $k \leq 7$  are simplices.

The vertex figure of Gosset's honeycomb is the semiregular  $E_8$  polytope ( $4_{21}$  in Coxeter's notation) given by the convex hull of the 240 roots of the  $E_8$  lattice.

Each point of the  $E_8$  lattice is surrounded by 2160 8-orthoplexes and 17280 8-simplices. The 2160 deep holes near the origin are exactly the halves of the norm 4 lattice points. The 17520 norm 8 lattice points fall into two classes (two orbits under the action of the  $E_8$  automorphism group): 240 are twice the norm 2 lattice points while 17280 are 3 times the shallow holes surrounding the origin.

A hole in a lattice is a point in the ambient Euclidean space whose distance to the nearest lattice point is a local maximum. (In a lattice defined as a uniform honeycomb these points correspond to the centers of the facets volumes.) A deep hole is one whose distance to the lattice is a global maximum. There are two types of holes in the  $E_8$  lattice:

- *Deep holes* such as the point  $(1,0,0,0,0,0,0,0)$  are at a distance of 1 from the nearest lattice points. There are 16 lattice points at this distance which form the vertices of an 8-orthoplex centered at the hole (the Delaunay cell of the hole).
- *Shallow holes* such as the point  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  are at a distance of  $\frac{1}{2}\sqrt{8}$  from the nearest lattice points. There are 9 lattice points at this distance forming the vertices of an 8-simplex centered at the hole.

## Sphere packings and kissing numbers[edit]

The  $E_8$  lattice is remarkable in that it gives optimal solutions to the sphere packing problem and the kissing number problem in 8 dimensions.

The sphere packing problem asks what is the densest way to pack (solid)  $n$ -dimensional spheres of a fixed radius in  $\mathbf{R}^n$  so that no two spheres overlap. Lattice packings are special types of sphere packings where the spheres are centered at the points of a lattice. Placing spheres of radius  $1/\sqrt{2}$  at the points of the  $E_8$  lattice gives a lattice packing in  $\mathbf{R}^8$  with a density of

It has long been known that this is the maximum density that can be achieved by a lattice packing in 8 dimensions.<sup>[6]</sup> Furthermore, the  $E_8$  lattice is the unique lattice (up to isometries and rescalings) with this density.<sup>[7]</sup> Mathematician Maryna Viazovska has recently shown that this density is, in fact, optimal even among irregular packings.<sup>[8][9]</sup>

The kissing number problem asks what is the maximum number of spheres of a fixed radius that can touch (or "kiss") a central sphere of the same radius. In the  $E_8$  lattice packing mentioned above any given sphere touches 240 neighboring spheres. This is because there are

240 lattice vectors of minimum nonzero norm (the roots of the  $E_8$  lattice). It was shown in 1979 that this is the maximum possible number in 8 dimensions.<sup>[10][11]</sup>

The sphere packing problem and the kissing number problem are remarkably difficult and optimal solutions are only known in 1, 2, 3, 8, and 24 dimensions (plus dimension 4 for the kissing number problem). The fact that solutions are known in dimensions 8 and 24 follows in part from the special properties of the  $E_8$  lattice and its 24-dimensional cousin, the **Leech lattice**.

## Other constructions[edit]

### Hamming code[edit]

The  $E_8$  lattice is very closely related to the (extended) **Hamming code**  $H(8,4)$  and can, in fact, be constructed from it. The Hamming code  $H(8,4)$  is a **binary code** of length 8 and rank 4; that is, it is a 4-dimensional subspace of the finite vector space  $(\mathbf{F}_2)^8$ . Writing elements of  $(\mathbf{F}_2)^8$  as 8-bit integers in **hexadecimal**, the code  $H(8,4)$  can be given explicitly as the set

$$\{00, 0F, 33, 3C, 55, 5A, 66, 69, 96, 99, A5, AA, C3, CC, F0, FF\}.$$

The code  $H(8,4)$  is significant partly because it is a **Type II self-dual code**. It has a minimum **Hamming weight** 4, meaning that any two codewords differ by at least 4 bits. It is the largest length 8 binary code with this property.

One can construct a lattice  $\Lambda$  from a binary code  $C$  of length  $n$  by taking the set of all vectors  $x$  in  $\mathbf{Z}^n$  such that  $x$  is congruent (modulo 2) to a codeword of  $C$ .<sup>[12]</sup> It is often convenient to rescale  $\Lambda$  by a factor of  $1/\sqrt{2}$ ,

Applying this construction a Type II self-dual code gives an even, unimodular lattice. In particular, applying it to the Hamming code  $H(8,4)$  gives an  $E_8$  lattice. It is not entirely trivial, however, to find an explicit isomorphism between this lattice and the lattice  $\Gamma_8$  defined above.

## Applications[edit]

In 1982 **Michael Freedman** produced a bizarre example of a topological **4-manifold**, called the  **$E_8$  manifold**, whose **intersection form** is given by the  $E_8$  lattice. This manifold is an example of a topological manifold which admits no **smooth structure** and is not even **triangulable**.

In **string theory**, the **heterotic string** is a peculiar hybrid of a 26-dimensional **bosonic string** and a 10-dimensional **superstring**. In order for the theory to work correctly, the 16 mismatched dimensions must be compactified on an even, unimodular lattice of rank 16. There are two such lattices:  $\Gamma_8 \oplus \Gamma_8$  and  $\Gamma_{16}$  (constructed in a fashion analogous to that of  $\Gamma_8$ ). These lead to two versions of the heterotic string known as the  $E_8 \times E_8$  heterotic string and the  $SO(32)$  heterotic string.



The **vertex algebra** of the **two-dimensional conformal field theory** describing **bosonic string theory**, compactified on the 24-dimensional **quotient torus**  $\mathbf{R}^{24}/\Lambda_{24}$  and **orbifolded** by a two-element reflection group, provides an explicit construction of the **Griess algebra** that has the **monster group** as its automorphism group. This **monster vertex algebra** was also used to prove the **monstrous moonshine** conjectures.

## Constructions[edit]

The Leech lattice can be constructed in a variety of ways. As with all lattices, it can be constructed by taking the **integral** span of the columns of its **generator matrix**, a 24×24 matrix with **determinant** 1.

Leech generator matrix [\[show\]](#)

[1]

### Using the binary Golay code[edit]

The Leech lattice can be explicitly constructed as the set of vectors of the form  $2^{-3/2}(a_1, a_2, \dots, a_{24})$  where the  $a_i$  are integers such that

and for each fixed residue class modulo 4, the 24 bit word, whose 1s correspond to the coordinates  $i$  such that  $a_i$  belongs to this residue class, is a word in the **binary Golay code**. The Golay code, together with the related Witt design, features in a construction for the 196560 minimal vectors in the Leech lattice.

### Using the Lorentzian lattice $\mathbf{II}_{25,1}$ [edit]

The Leech lattice can also be constructed as  $\frac{1}{2}(\mathbf{II}_{25,1} + w)$  where  $w$  is the Weyl vector:

in the 26-dimensional even Lorentzian **unimodular lattice**  $\mathbf{II}_{25,1}$ . The existence of such an integral vector of norm zero relies on the fact that  $1^2 + 2^2 + \dots + 24^2$  is a **perfect square** (in fact  $70^2$ ); the **number 24** is the only integer bigger than 1 with this property. This was conjectured by **Édouard Lucas**, but the proof came much later, based on **elliptic functions**.

The vector  $\frac{1}{2}w$  in this construction is really the **Weyl vector** of the even sublattice  $D_{24}$  of the odd unimodular lattice  $\mathbf{I}^{25}$ . More generally, if  $L$  is any positive definite unimodular lattice of dimension 25 with at least 4 vectors of norm 1, then the Weyl vector of its norm 2 roots has integral length, and there is a similar construction of the Leech lattice using  $L$  and this Weyl vector.

## Based on other lattices[edit]

Conway & Sloane (1982) described another 23 constructions for the Leech lattice, each based on a Niemeier lattice. It can also be constructed by using three copies of the E8 lattice, in the same way that the binary Golay code can be constructed using three copies of the extended Hamming code,  $H_8$ . This construction is known as the Turyn construction of the Leech lattice.

## As a laminated lattice[edit]

Starting with a single point,  $\Lambda_0$ , one can stack copies of the lattice  $\Lambda_n$  to form an  $(n + 1)$ -dimensional lattice,  $\Lambda_{n+1}$ , without reducing the minimal distance between points.  $\Lambda_1$  corresponds to the integer lattice,  $\Lambda_2$  is to the hexagonal lattice, and  $\Lambda_3$  is the face-centered cubic packing. Conway & Sloane (1982b) showed that the Leech lattice is the unique laminated lattice in 24 dimensions.

## As a complex lattice[edit]

The Leech lattice is also a 12-dimensional lattice over the Eisenstein integers. This is known as the *complex Leech lattice*, and is isomorphic to the 24-dimensional real Leech lattice. In the complex construction of the Leech lattice, the binary Golay code is replaced with the ternary Golay code, and the Mathieu group  $M_{24}$  is replaced with the Mathieu group  $M_{12}$ . The  $E_6$  lattice,  $E_8$  lattice and Coxeter–Todd lattice also have constructions as complex lattices, over either the Eisenstein or Gaussian integers.

## Using the icosian ring[edit]

The Leech lattice can also be constructed using the ring of icosians. The icosian ring is abstractly isomorphic to the E8 lattice, three copies of which can be used to construct the Leech lattice using the Turyn construction.

## Witt's construction[edit]

In 1972 Witt gave the following construction, which he said he found in 1940 January 28.

Suppose that  $H$  is an  $n$  by  $n$  Hadamard matrix, where  $n=4ab$ . Then the matrix defines a bilinear form in  $2n$  dimensions, whose kernel has  $n$  dimensions. The quotient by this kernel is a nonsingular bilinear form taking values in  $(1/2)\mathbf{Z}$ . It has 3 sublattices of index 2 that are integral bilinear forms. Witt obtained the Leech lattice as one of these three sublattices by taking  $a=2$ ,  $b=3$ , and taking  $H$  to be the 24 by 24 matrix (indexed by  $\mathbf{Z}/23\mathbf{Z} \cup \infty$ ) with entries  $X(m+n)$  where  $X(\infty)=1$ ,  $X(0)=-1$ ,  $X(n)$  is the quadratic residue symbol mod 23 for nonzero  $n$ . This matrix  $H$  is a Paley matrix with some insignificant sign changes.

## Using a Paley matrix[edit]

Chapman (2001) described a construction using a skew Hadamard matrix of Paley type. The Niemeier lattice with root system can be made into a module for the ring of integers of

the field  $\mathbb{Z}[\sqrt{-23}]$ . Multiplying this Niemeier lattice by a non-principal ideal of the ring of integers gives the Leech lattice.

## Using octonions[edit]

If  $L$  is the set of octonions with coordinates on the  $E_8$  lattice. Then the Leech lattice is the set of triplets  $(x, y, z)$  such that:

where

## Symmetries[edit]

The Leech lattice is highly symmetrical. Its automorphism group is the Conway group  $Co_0$ , which is of order 8 315 553 613 086 720 000. The center of  $Co_0$  has two elements, and the quotient of  $Co_0$  by this center is the Conway group  $Co_1$ , a finite simple group. Many other sporadic groups, such as the remaining Conway groups and Mathieu groups, can be constructed as the stabilizers of various configurations of vectors in the Leech lattice.

Despite having such a high rotational symmetry group, the Leech lattice does not possess any hyperplanes of reflection symmetry. In other words, the Leech lattice is chiral.

The automorphism group was first described by John Conway. The 398034000 vectors of norm 8 fall into 8292375 'crosses' of 48 vectors. Each cross contains 24 mutually orthogonal vectors and their negatives, and thus describe the vertices of a 24-dimensional orthoplex. Each of these crosses can be taken to be the coordinate system of the lattice, and has the same symmetry of the Golay code, namely  $2^{12} \times |M_{24}|$ . Hence the full automorphism group of the Leech lattice has order  $8292375 \times 4096 \times 244823040$ , or 8 315 553 613 086 720 000.

## Geometry[edit]

Conway, Parker & Sloane (1982) showed that the covering radius of the Leech lattice is  $\sqrt{24}$ ; in other words, if we put a closed ball of this radius around each lattice point, then these just

cover Euclidean space. The points at distance at least  $\sqrt{24}$  from all lattice points are called the deep holes of the Leech lattice. There are 23 orbits of them under the automorphism group of the Leech lattice, and these orbits correspond to the 23 Niemeier lattices other than

the Leech lattice: the set of vertices of deep hole is isometric to the affine Dynkin diagram of the corresponding Niemeier lattice.

The Leech lattice has a density of  $\frac{1}{24}$ . [Cohn & Kumar \(2009\)](#) showed that it gives the densest lattice [packing of balls](#) in 24-dimensional space. Henry Cohn, Abhinav Kumar, and Stephen D. Miller et al. [\(2016\)](#) improved this by showing that it is the densest sphere packing, even among non-lattice packings.

The 196560 minimal vectors are of three different varieties, known as *shapes*:

- 1104 vectors of shape  $(4^2, 0^{22})$ , for all permutations and sign choices;
- 97152 vectors of shape  $(2^8, 0^{16})$ , where the '2's correspond to octads in the Golay code, and there is an even number of minus signs;
- 98304 vectors of shape  $(-3, 1^{23})$ , where the changes of signs come from the Golay code, and the '3' can appear in any position.

The [ternary Golay code](#), [binary Golay code](#) and Leech lattice give very efficient 24-dimensional [spherical codes](#) of 729, 4096 and 196560 points, respectively. Spherical codes are higher-dimensional analogues of [Tammes problem](#), which arose as an attempt to explain the distribution of pores on pollen grains. These are distributed as to maximise the minimal angle between them. In two dimensions, the problem is trivial, but in three dimensions and higher it is not. An example of a spherical code in three dimensions is the set of the 12

vertices of the regular icosahedron.

## History[[edit](#)]

Many of the cross-sections of the Leech lattice, including the [Coxeter–Todd lattice](#) and [Barnes–Wall lattice](#), in 12 and 16 dimensions, were found much earlier than the Leech lattice. [O'Connor & Pall \(1944\)](#) discovered a related odd unimodular lattice in 24 dimensions, now called the odd Leech lattice, one of whose two even neighbors is the Leech lattice. The Leech lattice was discovered in 1965 by [John Leech \(1967, 2.31, p. 262\)](#), by improving some earlier sphere packings he found ([Leech 1964](#)).

[Conway \(1968\)](#) calculated the order of the automorphism group of the Leech lattice, and, working with [John G. Thompson](#), discovered three new [sporadic groups](#) as a by-product: the [Conway groups](#),  $Co_1$ ,  $Co_2$ ,  $Co_3$ . They also showed that four other (then) recently announced sporadic groups, namely, [Higman-Sims](#), [Suzuki](#), [McLaughlin](#), and the [Janko group](#)  $J_2$  could be found inside the Conway groups using the geometry of the Leech lattice. (Ronan, p. 155)

Bei dem Versuch, eine Form aus einer solchen Klasse wirklich anzugeben, fand ich mehr als 10 verschiedene Klassen in  $\Gamma_{24}$

*Witt (1941, p. 324)*

[Witt \(1941, p. 324\)](#), has a single rather cryptic sentence mentioning that he found more than 10 even unimodular lattices in 24 dimensions without giving further details. [Witt \(1998, p. 328–329\)](#) stated that he found 9 of these lattices earlier in 1938, and found two more, the



Niemeier lattice with  $A_{24}$

1 root system and the Leech lattice (and also the odd Leech lattice), in 1940.

## 4.7. Novanion algebras.

We first give the axioms for a field, and successively weaken these axioms to give them for division algebras and novanion algebras.

The axioms for a *field*  $\mathbb{F}$ ,  $+$ ,  $\times$ , which we will denote simply by  $\mathbb{F}$ , satisfy for  $a, b, c \in \mathbb{F}$ , with  $a \times b$  being written as  $ab$

$$\text{additive closure:} \quad a + b \in \mathbb{F} \quad (1)$$

$$\text{associativity:} \quad a + (b + c) = (a + b) + c \quad (2)$$

$$\text{abelian addition:} \quad a + b = b + a \quad (3)$$

$$\text{existence of a zero: there exists a } 0 \in \mathbb{F} \text{ satisfying} \\ a + 0 = a \quad (4)$$

$$\text{existence of negative elements: there exists a } (-a) \in \mathbb{F} \text{ with} \\ a + (-a) = 0, \quad (5)$$

which we write introducing subtraction as

$$a - a = 0$$

$$\text{multiplicative closure:} \quad ab \in \mathbb{F} \quad (6)$$

$$\text{associativity:} \quad a(bc) = (ab)c \quad (7)$$

$$\text{commutativity:} \quad ab = ba \quad (8)$$

$$\text{existence of a 1: there exists a } 1 \in \mathbb{F} \text{ satisfying} \\ a1 = a \quad (9)$$

$$\text{existence of inverse elements: there exists an } a^{-1} \in \mathbb{F} \text{ for } a \neq 0 \text{ with} \\ a(a^{-1}) = 1, \quad (10)$$

which we write introducing division as

$$a/a = 1$$

$$\text{distributive law:} \quad a(b + c) = (ab) + (ac). \quad (11)$$

An associative division algebra satisfies all axioms for a field except the multiplicative commutativity rule (8). A nonassociative division algebra,  $D$ , also drops the multiplicative associative rule (7). It may introduce the following axiom:

$$\text{There do not exist } a, b \neq 0 \in D \text{ with} \\ ab = 0. \quad (12)$$

This rule is unnecessary for associative division algebras, since if  $1 \neq 0$  then (12) implies

$$ab \neq c, c \neq 0,$$

and therefore there exists no

$$a(b/c) = 1,$$

but  $b/c$  can be chosen to be the multiplicative inverse of  $a$ , a contradiction.  $\square$

A novanion algebra  $B$  drops axiom (12). It may substitute the following rule:

$$\text{If } a^2, b^2 > 0 \text{ for } a, b \in B \text{ and } a \text{ is in } c, b \text{ is in } d \in B, \text{ there do not exist any } c, d \text{ with} \\ cd = 0. \quad (13)$$

We can provide existence proofs for these structures. The complex numbers constitute a field. The quaternions form an associative division algebra. The octonions form a nonassociative division algebra. As we will prove in section 4.11, the 10-novanions, for example, form a novanion algebra.  $\square$

The consistency of the complex numbers is contingent on the consistency of analysis, given as a Gentzen-type proof in chapter III volume III. The consistency of the quaternions as an

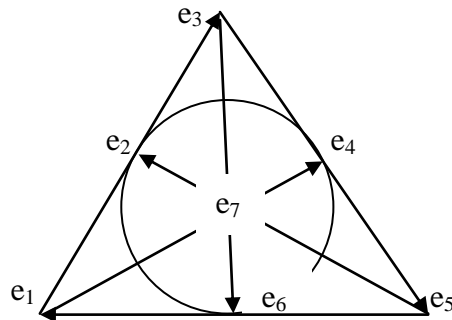
associative division algebra is given in reference to [Ad15], and we will prove in 4.11 the consistency of the 10-novonions as a novanion algebra. The consistency of the octonions now follows from their nature as novanion algebras of a specific type.  $\square$

#### 4.8. The classification of novanions.

J. F. Adams proved in 1960 that the only possible division algebras are at maximum those of the octonions, which are nonassociative, where quaternions and complex numbers are subalgebras of these. However, there exist nonassociative algebras, the n-novanions, with nonzero division, but when just the scalar part is zero two nonzero novanions may have a zero product. I obtain here the proof for  $n = 10$ , and extend it for  $n$  greater than 10. Novanions have a dimension of  $n = 1 + 3^f \prod_{i \in \mathbb{N}} (3^{g_i} - 2)_i$ , for which  $f \in \mathbb{N}_{\cup 0}$ ,  $g_i \in \mathbb{N}$ . Thus the n-novanions comprise a new type of quasi division algebra. We then introduce novanion rings. These do not have division, and thus can represent discrete structures.

#### 4.9. The nonassociative octonion division algebra.

The algebra of the octonions, also called the Cayley numbers, was discovered by J. Graves in 1843, and is given by the Fano plane for their basis elements  $1, e_1, e_2, e_3, e_4, e_5, e_6$  and  $e_7$  in the figure below.



Typical cyclic identities are

$$\begin{aligned} e_7 e_5 &= e_2 \\ e_5 e_6 &= e_1 \end{aligned} \tag{1}$$

and

$$e_4 e_2 = e_6.$$

Note what we have said here. The inner triple  $e_2 e_4 e_6$  acts like a quaternion, but the outer triple  $e_1 e_3 e_5$  does not. Nevertheless, we will need to allocate later a list ordered as right triple  $(e_2 e_4 e_6) +$  central triple  $(e_1 e_3 e_5) +$  one  $(e_7)$ . Octonions form a division algebra, in particular

$$e_c^2 = -1, \tag{2}$$

$$e_a e_b = -e_b e_a \quad (a \neq b),$$

and the inverse of

$$a1 + \sum_{n=1}^7 b_n e_n$$

is

$$(a1 - \sum_{n=1}^7 b_n e_n) / (a^2 + \sum_{n=1}^7 b_n^2). \tag{3}$$

The octonions,  $\mathbb{O}$ , are also generated by the Cayley-Dickson construction [Ba01]. This builds up algebras from the complex numbers, to the quaternions, to the octonions, to the sixteen dimensional sedenions, etc.

Define a T-algebra to be an algebra equipped with conjugation, a linear map T satisfying

$$a^{TT} = a, \tag{4}$$

$$(ab)^T = b^T a^T. \tag{5}$$

Starting from any T-algebra, the Cayley-Dickson construction gives a new algebra

$$(a, b)(c, d) = (ac - db^T, ad^T + cb), \tag{6}$$

with conjugation defined by

$$(a, b)^T = (a^T, -b). \tag{7}$$

Let us look at how this works in practice. The complex numbers have multiplication

$$(a, b)(c, d) = (ac - bd, ad + bc).$$

The complex conjugate

$$(a, b)^T = (a^T, -b) = (a, -b),$$

since the scalar part a is a real number with  $a^T = a$ .

The complex conjugate satisfies

$$(a, b)^T(a, b) = (aa + bb, ab - ba) = (a^2 + b^2, 0).$$

We define  $a^2 + b^2$  as the *norm* of the complex number. It is a non-negative real number.

For the noncommutative quaternions equations (6) and (7) hold. The order of the terms in (6) is important. We now have

$$\begin{aligned} (a, b)^T(a, b) &= (a^T, -b)(a, b) = (a^T a + b^T b, ba^T - ba^T) \\ &= (a^T a + b^T b, 0), \end{aligned}$$

where  $a^T a + b^T b$  is the norm.

The T-algebra generates the basis element multiplication table  $\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$  of the octonions

$\times$	$1$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$1$	1	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_1$	$e_1$	-1	$e_3$	$-e_2$	$e_5$	$-e_4$	$-e_7$	$e_6$
$e_2$	$e_2$	$-e_3$	-1	$e_1$	$e_6$	$e_7$	$-e_4$	$-e_5$
$e_3$	$e_3$	$e_2$	$-e_1$	-1	$e_7$	$-e_6$	$e_5$	$-e_4$
$e_4$	$e_4$	$-e_5$	$-e_6$	$-e_7$	-1	$e_1$	$e_2$	$e_3$
$e_5$	$e_5$	$e_4$	$-e_7$	$e_6$	$-e_1$	-1	$-e_3$	$e_2$
$e_6$	$e_6$	$e_7$	$e_4$	$-e_5$	$-e_2$	$e_3$	-1	$-e_1$
$e_7$	$e_7$	$-e_6$	$e_5$	$e_4$	$-e_3$	$-e_2$	$e_1$	-1

We can generate for each

$$\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}: e_i \times e_j \rightarrow e_k$$

a Cayley-Dickson construction of

$$e_i \times (-e_j) \rightarrow -e_k,$$

so that each of the 7 non-scalar basis elements in a row of the table can be multiplied by -1 to provide  $2^7 = 128$  possible Cayley-Dickson constructions.

The Fano plane has 7 non-scalar basis elements. The number of non-scalar quaternionic triplets is 7, each of which, since exquaternions are excluded, operates under a forward or a reversed orientation – again  $2^7$  possibilities. The following Fano triplets map bijectively to the standard Cayley-Dickson construction for the octonions

$$(e_1, e_2, e_3), (e_3, e_4, e_5), (e_1, e_4, e_6), (e_4, e_6, e_2), (e_1, e_7, e_6), (e_4, e_7, e_3), (e_5, e_7, e_2).$$

Thus distinct possibilities for the Fano plane match distinct instances of the Cayley-Dickson construction.  $\square$

## 4.10. Eigenvalues.

Our intention now is to introduce a generalisation of quaternions and octonions called n-novonions, which will be involved in the mathematics we wish to describe. In order to prove the consistency of the n-novonion algebra, we will need to introduce some further ideas on matrices. Like the octonions, the n-novonions cannot be described directly by matrices, since matrices are multiplicatively associative, meaning  $(AB)C = A(BC)$ , but novonions are not. Nevertheless we can characterise some of the properties of n-novonions by a set of linear equations. These can be described by a matrix equation. An important technique in analysing the matrix equations is to map the equation to a scalar equation involving complex numbers. These scalars are called eigenvalues, and satisfy a matrix equation like (1) below.

Within the field of complex numbers  $\mathbb{C}$ , the complex conjugate of  $c = a + ib$  is  $c^* = a - ib$ . For the corresponding matrix  $C$  with entries  $c_{jk} = a_{jk} + ib_{jk}$ , the conjugate  $C^* = a_{jk} - ib_{jk}$ . The transpose of a matrix  $C$  is denoted by  $C^T$  and has entries  $c_{kj}$ . The transpose is a contravariant (order reversing) operation:

$$(CD)^T = D^T C^T.$$

A matrix is defined as antisymmetric if  $C^T = -C$ .

The following proof is derived from chapter 11 of [Uh01].

**Theorem 4.10.1.** *All eigenvalues of a real antisymmetric matrix  $E = -E^T$  are pure imaginary.*

*Proof.* Consider the case of the eigenvalue  $\lambda$  and possibly complex column vector called an eigenvector  $\mathbf{x} \neq \mathbf{0}$ . According to the Fundamental Theorem of Algebra  $\lambda \in \mathbb{C}$ . Hence

$$E\mathbf{x} = \lambda\mathbf{x}. \quad (1)$$

If we take the complex conjugate of both sides of the eigenvalue-eigenvector equation (1), we obtain

$$(E\mathbf{x})^* = (\lambda\mathbf{x})^* = \lambda^*\mathbf{x}^*.$$

Transposing yields

$$(E\mathbf{x})^{*T} = \mathbf{x}^{*T} E^{*T} = \lambda^*\mathbf{x}^{*T}.$$

Define the norm as

$$\|E\mathbf{x}\|^2 = (\lambda^*)\lambda(\mathbf{x}^{*T})\mathbf{x}, \quad (2)$$

where for the set of real numbers  $\mathbb{R}$

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}} \in \mathbb{R}.$$

Since  $E^T = -E$  and  $E^{*T} = E^T$  for a real antisymmetric matrix  $E$ , we can write (2) as

$$\begin{aligned} \|E\mathbf{x}\|^2 &= \mathbf{x}^{*T} E^{*T} E^T \mathbf{x}, \\ &= \mathbf{x}^T E^T E \mathbf{x} \\ &= -\mathbf{x}^T E^2 \mathbf{x} \\ &= -\mathbf{x}^T \lambda^2 \mathbf{x}, \end{aligned}$$

because  $E^2\mathbf{x} = E(E\mathbf{x}) = E(\lambda\mathbf{x}) = \lambda(E\mathbf{x}) = \lambda^2\mathbf{x}$ , so

$$\|E\mathbf{x}\|^2 = -\lambda^2\mathbf{x}^T\mathbf{x}. \quad (3)$$

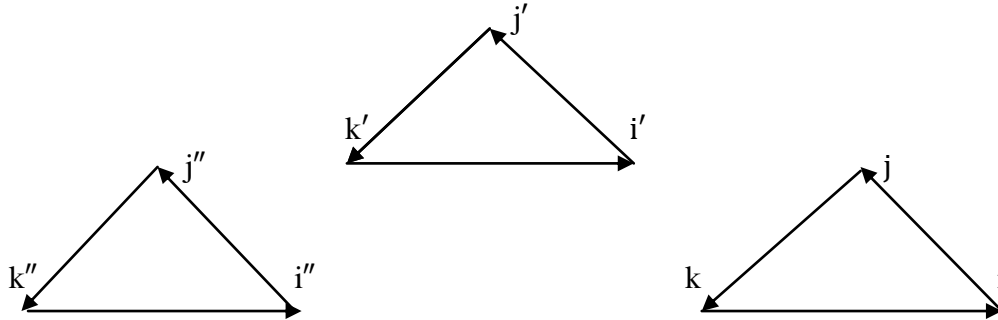
Now  $\mathbf{x}^T\mathbf{x} \neq 0$ , and thus  $\lambda^*\lambda = -\lambda^2$ , by comparing (2) and (3). Thus a real antisymmetric matrix  $E = -E^T$  can only have imaginary eigenvalues  $\lambda$ .  $\square$

### 4.11. The 10-novanions.

We have seen a novanion algebra  $B$  drops rule 4.7.(12). It may substitute the following rule:

$$\text{If } a^2, b^2 > 0 \text{ for } a, b \in B \text{ and } a \text{ is in } c, b \text{ is in } d \in B, \text{ there do not exist any } c, d \text{ with } cd = 0. \quad (1)$$

We now introduce the 10-novanions, represented by the set of triangle diagrams



where in general each triangle is a quaternion without 1.

The primed variables ( $'$ ), ( $'$ ) and ( $''$ ) act as holders of information concerning an algebra for them. When the variables all contain a common instance, for example ( $k$ ), ( $k'$ ) and ( $k''$ ), then the algebra is that of the quaternions, in which we have a cyclic algebra

$$kk' = k'' = -k'k. \quad (2)$$

When the variables contain different instances, such as  $k$  and  $i'$ , then the product contains the primed variable that does not belong to the first two elements, but the primed part commutes.

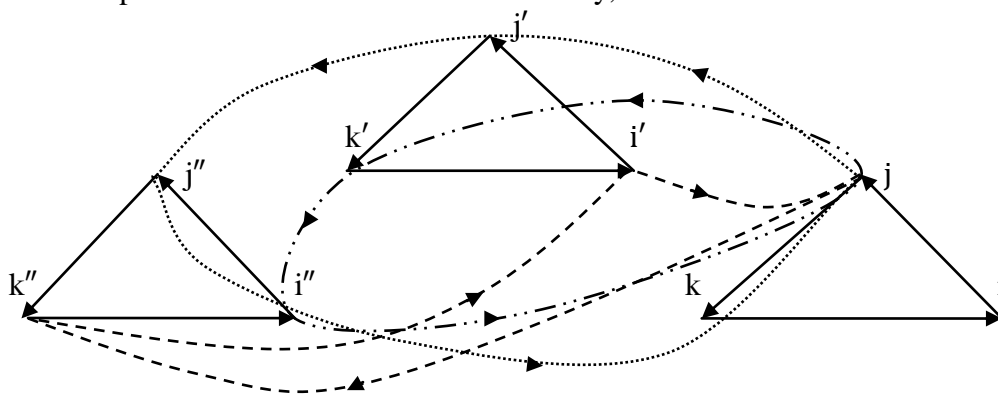
On top of this is the fact that the variables, say  $k$  and  $i$ , satisfy a quaternion algebra, so say

$$ki = j = -ik \quad (3)$$

and consequently

$$ki' = j'' = -i'k. \quad (4)$$

In order to picture the 10-novanions more closely, we will show the connections from node  $j$



Our claim is that the inverse of

$$a1 + \sum_{n=1}^3 \sum_{\text{primed } m=1}^3 b_n^m e_n$$

is

$$(a1 - (\sum_{n=1}^3 \sum_{\text{primed } m=1}^3 b_n^m e_n)) / (a^2 + (\sum_{n=1}^3 \sum_{\text{primed } m=1}^3 (b_n^m)^2)), \quad (5)$$

and this constitutes a type of division algebra with no divisors of zero provided  $a1 \neq 0$  – the 10 dimensional 10-novanions.

We see that the n-novanions are nonassociative, since they have more than 4 basis elements; more explicitly

$$(j''k')j' = ij' = k'' \neq j''(k'j') = -j''i' = k.$$

We wish to enquire under what conditions there exist two 10-novanion numbers multiplied together giving zero:

$$(a1 + bi + cj + dk + b'i' + c'j' + d'k' + b''i'' + c''j'' + d''k'') \times (p1 + qi + rj + tk + q'i' + r'j' + t'k' + q''i'' + r''j'' + t''k'') = 0. \quad (6)$$

Their product is

real part:

$$ap - bq - cr - dt - b'q' - c'r' - d't' - b''q'' - c''r'' - d''t'' = 0, \quad (7)$$

i part:

$$bp + aq - dr + ct - b''q' - d''r' + c''t' + b'q'' - d'r'' + c't'' = 0, \quad (8)$$

j part:

$$cp + dq + ar - bt + d''q' - c''r' - b''t' + d'q'' + c'r'' - b't'' = 0, \quad (9)$$

k part:

$$dp - cq + br + at - c''q' + b''r' - d''t' - c'q'' + b'r'' + d't'' = 0, \quad (10)$$

i' part:

$$b'p + b''q - d''r + c''t + aq' - d'r' + c't' - bq'' - dr'' + ct'' = 0, \quad (11)$$

j' part:

$$c'p + d''q + c''r - b''t + d'q' + ar' - b't' + dq'' - cr'' - bt'' = 0, \quad (12)$$

k' part:

$$d'p - c''q + b''r + d''t - c'q' + b'r' + at' - cq'' + br'' - dt'' = 0, \quad (13)$$

i'' part:

$$b''p - b'q - d'r + c't + bq' - dr' + ct' + aq'' - d''r'' + c''t'' = 0, \quad (14)$$

j'' part:

$$c''p + d'q - c'r - b't + dq' + cr' - bt' + d''q'' + ar'' - b''t'' = 0, \quad (15)$$

k'' part:

$$d''p - c'q + b'r - d't - cq' + br' + dt' - c''q'' + b''r'' + at'' = 0. \quad (16)$$

**Alternative definition 4.11.1.** D is a (possibly nonassociative) division algebra whenever for any element a in D and any nonzero element b in D there exists just one element x in D with  $a = bx$  and only one element y in D with  $a = yb$ .

If  $a = 0$ , the 10-novanions contain possibilities for two nonzero 10-novanions giving a product which is zero. We give an example due to Doly García, showing that the 10-novanions satisfying  $a = 0$  do not form a division algebra of standard type

$$(i + i' + i'')(j + j' - 2j'') = 0. \quad (17)$$

Thus the 10-novanions are not a division algebra given by the condition for equation 8.7.(12), nor do they satisfy definition 8.11.1 since for an arbitrary real number g

$$(i + i' + i'')[j + j' - 2j'']g = 0. \quad (18)$$

From now on we will assume  $a \neq 0$ . By a symmetrical argument applied also to the following reasoning, we need to assume with this that  $p \neq 0$ .

Equations (7) to (16) form a matrix  $E + aI$ , where E is an antisymmetric matrix and I is the unit diagonal, multiplied on the right by the eigenvector  $(p, q, r, t, q', r', t', q'', r'', t'')$ . We have already given a proof that the eigenvalues of a real antisymmetric matrix are entirely imaginary, so these correspond to  $-a$ , which is real, whereas we are now excluding the only possibility for this,  $a = 0$ . So 10-novanions form a novanion algebra satisfying the conditions of equation (1).  $\square$

## 4.12. n-novonions.

The octonions given by a Fano plane under suitable orientations may be given three copies in primed variables ( ), (') and (''), and by an analogous procedure this constitutes a  $1 + (3 \times 7) = 22$  dimensional division algebra.

Extending these ideas further to multiple occurrences of the three or seven primed variables, say ('), (')' and (')'', we obtain in general an  $n = 1 + 3^{f7^g}$  dimensional novanion algebra, the n-novonions, in which if a common variable, k, is employed, then the lowest value within brackets of say (k), (k)'' and (k)'''' is evaluated.  $\square$

## 4.13. Discreteness and novanion rings.

We relate our discussion on quaternions to Eli Cartan's *The theory of spinors* [Ca66] where he claims a bijection between spinors and the relativistic (Dirac) equation of the electron which maps to quaternions. We will find these spinors do not map to quaternions and we cannot adjoin spinors to quaternions in a matrix formalism, because the dimension of this combination is greater than 4, so by a result [Ad15], the algebra cannot be a matrix one, and is nonassociative.

Cartan introduces spinors via the observable coefficients of vectors with zero norm

$$x_1^2 + x_2^2 + x_3^2 = 0 \quad (1)$$

satisfied by two numbers  $h_0, h_1$  given by

$$\begin{aligned} x_1 &= h_0^2 - h_1^2 \\ x_2 &= i(h_0^2 + h_1^2) \\ x_3 &= -2h_0h_1, \end{aligned}$$

with possible solutions

$$h_0 = \pm \sqrt{\frac{x_1 - ix_2}{2}}, \quad (2)$$

$$h_1 = \pm \sqrt{\frac{-x_1 - ix_2}{2}}. \quad (3)$$

Now choose a vector dependent only on the coordinates  $x_1$  and  $x_2$ , say  $x_1 - ix_2$ . If this is rotated about the axis  $x_3$  to a vector  $e^{i\theta}(x_1 - ix_2)$ , then  $h_0$  is transformed to  $e^{i\theta/2}h_0$ , and likewise  $h_1$  is transformed to  $e^{i\theta/2}h_1$ . Then when  $\theta = 2\pi$ , a complete rotation,  $x_3$  is transformed to  $-x_3$ . Thus the spinors  $h_0$  and  $h_1$  are *locally fermionic*.

He considers in the notation of [Ad15] the matrix basis

$$\phi_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad -i_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so that

$$\phi_1^2 = (-i_1)^2 = \alpha_1^2 = 1_1, \quad \phi_1(-i_1) = -(-i_1\phi_1), \quad -i_1\alpha_1 = -\alpha_1(-i_1), \quad \alpha_1\phi_1 = -\phi_1\alpha_1,$$

which he states are related to the quaternions on multiplication by  $-1_i$

$$I_1 = -1_i\phi_1, \quad I_2 = -1_i(-i_1), \quad I_3 = -1_i\alpha_1. \quad (4)$$

This is closely related to the basis already given by us in [Ad15],  $1_1, \alpha_i, i_1$  and  $\phi_i$ , which matches (4) except for minus signs whose square is 1 (it is only possible to change the basis left- or right-handedness by this means). From it Cartan is able to deduce the quaternion algebra

$$I_1^2 = I_2^2 = I_3^2 = -1, \quad I_1I_2 = -I_2I_1, \quad I_2I_3 = -I_3I_2, \quad I_3I_1 = -I_1I_3, \quad (5)$$

which he relates to the Dirac equation.  $\square$



However we must note that the transformations derived in (2) and (3) involve complex numbers, which are commutative, whereas equation (5) involves quaternions which are noncommutative. Thus there is no equation as given by Cartan of this type.  $\square$

In this system we can choose to embed this local quaternion structure in a global manifold which is oriented, that is say, an ix vector moving through  $2\pi$  radians in a jy and kz circle returns to itself with the ix vector pointing in the same direction. When this happens, we say the observable derived from the quaternion is *globally bosonic*. We can also implement nonoriented global manifolds in which an ix vector moving through  $2\pi$  radians in a jy and kz circle returns to itself with the ix vector pointing in the opposite direction. We then say the quaternion is *globally fermionic*.

Thus we are reduced to considering globally bosonic or globally fermionic quaternion structures. We will see that the globally bosonic and globally fermionic idea can be extended to the octonions and n-novonions. Our escape route from contradiction is that the extended quaternion object given by the n-novonions is not representable by a matrix, and a conclusion is that it is nonassociative.  $\square$

Since the n-novonions contain the quaternions or the octonions, and the octonions also contain quaternions, it follows that the arguments of [Ad15] chapter III on Wedderburn's little theorem apply also to such subalgebras, and therefore

**Theorem 4.13.1.** *Any finite novanion division ring is commutative.*

Thus novanion rings cannot in general be finite, although they may be discrete. Discrete rings without division can be bounded from below by finite elements ignoring the sign, for example by zero, and otherwise be infinite. If a ring is bounded from above and below by the mapping of, say, plus infinity to the integer novanion part 1 and minus infinity to -1, then an infinite example can be given over real components.

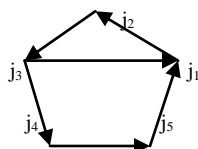
If novanion rings are given a lower and an upper bound in each of their n coordinates,  $x_0, x_1, \dots, x_{n-1}$  as  $\pm a_0, \pm a_1, \dots, \pm a_{n-1}$ , then the pair  $(+a_i, +a_j)$  and the pair  $(-a_i, -a_j)$  can be glued in two ways as  $a_i$  to  $-a_i$  and  $a_j$  to  $-a_j$ , or as  $a_i$  to  $-a_j$  and  $a_j$  to  $-a_i$ . The first defines a global bosonic structure and the second a global fermionic structure.

We will label bosonic structures by b and fermionic structures by f. If we look at just the novanionic imaginary components of the novanion, then for say a quaternion, there are configurations  $\{b, b, b\}$ ,  $\{f, b, b\}$ ,  $\{f, f, b\}$  and  $\{f, f, f\}$ .  $\square$

#### 4.14. The search for other novanion algebras.

Are there other novanion algebras of a type not already covered? This question has been stimulated by a first-year Sussex University student's identification of novanions with strings in physics, [Ad18], where the allocation  $n = 10$  is the same number as the dimensionality of the heterotic string, and for which we wish to investigate the bosonic allocation  $1 + 5^2 = 26$ .

In the pentagonal diagram shown next, an initial attempt depicts only one out of five subtriangles.



The pentagon can be enumerated cyclically, so that

$$j_1j_2 = j_3, j_2j_3 = j_4, j_3j_4 = j_5, j_4j_5 = j_1, j_5j_1 = j_2, \quad (1)$$

and jumping a vertex we evaluate the closest triangle

$$j_3j_1 = j_2, j_4j_2 = j_3, j_5j_3 = j_4, j_1j_4 = j_5, j_2j_5 = j_1, \quad (2)$$

where on inverting the orientation, we get a minus sign.

This latter fact implies we have an inbuilt norm and inverse; the inverse of

$$a1 + \sum_{n=1}^5 b_n j_n$$

is

$$a1 - \sum_{n=1}^5 b_n j_n / (a^2 + \sum_{n=1}^5 b_n^2), \quad (3)$$

which is nonassociative, as is demonstrated by

$$(j_3j_1)j_4 = -j_3 \neq j_3(j_1j_4) = -j_4.$$

The question arises as to whether this constitutes a novanion algebra, which would now be extended from previous considerations to include the dimensions

$$n = 1 + 3^f 5^g 7^h.$$

The possibility of the existence of the division algebra violating equation

$$(a1 + bj_1 + cj_2 + dj_3 + ej_4 + fj_5) \times (p1 + qj_1 + rj_2 + tj_3 + uj_4 + vj_5) = 0 \quad (4)$$

will now be investigated. Under the constraints (1) and (2) we obtain the set of equations

real part:

$$ap - bq - cr - dt - eu - fv = 0, \quad (5)$$

j<sub>1</sub> part:

$$bp + aq - fr + 0 - fu + (c + e)v = 0, \quad (6)$$

j<sub>2</sub> part:

$$cp + (f + d)q + ar - bt + 0 - bv = 0, \quad (7)$$

j<sub>3</sub> part:

$$dp - cq + (b + e)r + at - cu + 0 = 0, \quad (8)$$

j<sub>4</sub> part:

$$ep + 0 - dr + (c + f)t + au - dv = 0, \quad (9)$$

j<sub>5</sub> part:

$$fp - eq + 0 - et + (d + b)u + av = 0, \quad (10)$$

from which it follows that the E type matrix is not antisymmetric, but it may be represented as the sum of two matrices F and G, where F has all pure imaginary eigenvalues:

$$F = \begin{bmatrix} 0 & -b & -c & -d & -e & -f \\ b & 0 & -f & c & -f & e \\ c & f & 0 & -b & d & -b \\ d & -c & b & 0 & -c & e \\ e & f & -d & c & 0 & -d \\ f & -e & b & -e & d & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & c \\ 0 & d & 0 & 0 & -d & 0 \\ 0 & 0 & e & 0 & 0 & -e \\ 0 & -f & 0 & f & 0 & 0 \\ 0 & 0 & -b & 0 & b & 0 \end{bmatrix},$$

and we do not have pure imaginary eigenvalues for  $F + G - \lambda I$ .

Indeed a general matrix H may be represented as a sum of a symmetric part  $H_{\text{sym}}$  and an antisymmetric part  $H_{\text{anti}}$ . Then  $H_{\text{sym}}$  and  $H_{\text{anti}}$  are linearly independent over real coefficients, meaning there exist no real numbers c and d satisfying

$$cH_{\text{sym}} + dH_{\text{anti}} = 0.$$

By a demonstration analogous to that in section 4.10, and proved directly in [Uh01], the eigenvalues of a symmetric matrix are real. Further, a complex number with real coefficients  $h_1$  and  $h_2$  is linearly independent between  $h_1$  and  $h_2i$  over real coefficients. It now follows that the eigenvalue equation

$$H = hI \tag{11}$$

which has a unique set of  $n$  solutions is satisfied by

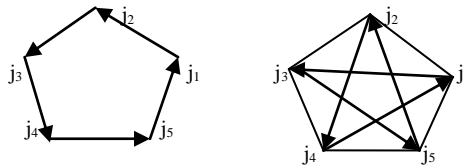
$$H_{\text{sym}} = h_1I$$

with  $n$  solutions and

$$H_{\text{anti}} = h_2I,$$

also with  $n$  solutions. These possible solutions are the only ones, since the solution set of (11) is unique. So if  $H_{\text{sym}}$  is not the zero matrix, a real  $h_1$  exists. This implies there is no novanion algebra available.

We find a similar type of situation for the pentagonal diagrams



since the diagram for  $j_1, j_2, j_3$  is unoriented and therefore does not constitute a quaternion, or in fact a division algebra, so we have failed on modifying equations (5) to (10) to come up with other solutions, where for pentagonal diagrams we can show these pentagons consist in the general case of combined oriented and unoriented quaternion diagrams.  $\square$

However, the identification relates to the number  $1 + 25$ , and we now wish to probe the allocation  $25 = 7 + 9 + 9$ , where 7 is the number of non-real basis elements of the octonions, and 9 is the number for the 10-novanions. There is an analogy here. The octonion non-real basis elements of 7 may be represented as  $1 + 3 + 3$ , where 3 is the number of such basis elements for the quaternions, and 1 for the complex numbers. We are forced for a number of reasons to decompose such an allocation into triplets, basically to retain the cyclic algebra for the quaternions.

The allocation will be as follows, where we subscript 3 and 1 to distinguish them

$$3_a, 3_b, 3_c \tag{i}$$

$$3_d, 3_e, 3_f \tag{ii}$$

$$3_g, 3_h, 1_u \tag{iii}$$

where allocations (i) and (ii) are internally similar to 10-novanions, and allocation (iii) is internally an octonion. We will explain why we use the word ‘similar’ later.

There are a number of possible configurations.

We want an algebra linking between (i), (ii) and (iii). Vertical allocations are present. We will choose next from straight lines going from left to right, for example the diagonal going upwards from  $3_g, 3_e$  to  $3_c$ . This is similar to a 10-novanion algebra. The descending line from  $3_a, 3_e$  to  $1_u$  is an octonion algebra. We then incorporate the algebra taking for example  $3_d, 3_b$  to  $1_u$ , an octonion algebra, or  $3_g, 3_b$  to  $3_f$ , this is similar to a 10-novanion algebra.

We have used the words ‘similar to a 10-novanion algebra’, and we now explain why. If we look at allocation (iii), this is part of the  $3_g 3_h 1_u$  octonion, where  $3_h$  and  $1_u$  are linked.

Although  $3_g$  is indeed a quaternion, we have already mentioned that  $3_h$  is not. Therefore the vertical allocation given by  $3_a 3_d 3_g$  is a 10-novanium, since it is made of genuine quaternions, but the vertical allocation  $3_b 3_e 3_h$  is not.  $3_b$ ,  $3_e$  and  $3_h$  occur in octonion representations. If we were to state that the central triple  $3_b$ ,  $3_e$  and  $3_h$  algebras were quaternions, we would have an inconsistency. Therefore for these allocations as part of a ‘similar to 10-novanium’ structure, we decide that the octonion structure overrides the 10-novanium one. Since there is only one special  $1_u$  part for the octonions, this part of the allocation is unique. The similar 10-novanium structure is now not a closed algebra within the 10-novaniums; part of it belongs to the octonions. The corresponding situation just for 10-novaniums with no octonionic overlap but with novaniumic overlap is described by the octonionic allocation already discussed.

Since there is no other mixing of allocations, the result is as consistent as the 10-novaniums and the octonions. This can be checked with equations like 4.11.(6) to (15), for which it is clear eigenvalues are pure imaginary. Finally a calculation like 4.11.(16) shows that this is a novanium algebra.  $\square$

The existence of 10-, 26- and 80-novaniums (the latter obtained by an array cube of items like (i) to (iii) – all configurations lie in planes of the cube, and an m-cube gives rise to a  $(3^m \pm 1)$ -novanium) implies that results derived for division algebras have a different extension for n-novaniums.  $\square$

#### 4.15. Sedenions and 64-novaniums.

The 16-dimensional sedenions are formed by the Cayley-Dickson construction [Ba01]. Since they are not alternative, they do not form a division algebra. That is, we do not have

$$x(xy) = (xx)y$$

and

$$(yx)x = y(xx)$$

for all  $x$  and  $y$  in the algebra, the proof using basis elements. Every associative algebra is alternative, but so too are some strictly non-associative algebras such as the octonions.

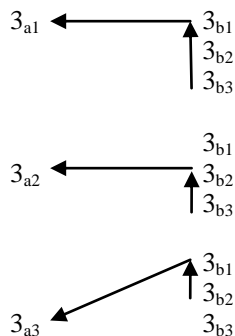
Whether there exist other 16-dimensional algebras not generated by the Cayley-Dickson construction was answered in the negative by J. F. Adams in 1960.

Irrespective of this result, the Cayley-Dickson construction generating a 64-dimensional algebra shows that this is not a division algebra, since in particular this contains the sedenions as a subalgebra. However, a 64-novanium has  $63 = 3^2 \times 7$  non-scalar basis elements, and we will see that novanium algebras of this type are consistent. A 64-novanium is given by the cube with slices

$$\begin{array}{lll} 3_a, 3_b, 1_p & 3'_a, 3'_b, 1'_p & 3''_a, 3''_b, 1''_p \\ 3_c, 3_d, 1_q & 3'_c, 3'_d, 1'_q & 3''_c, 3''_d, 1''_q \\ 3_e, 3_f, 1_r & 3'_e, 3'_f, 1'_r & 3''_e, 3''_f, 1''_r. \end{array}$$

To evaluate a typical slice algebra, if we take the leftmost array above, we know that  $3_b$  is not a quaternion, so we will build an override structure for the composition of two elements in  $3_b$ . Within this slice  $3_b$  belongs to three octonionic arrangements, those given by  $3_a, 3_b, 1_p$ , or  $3_c, 3_b, 1_r$ , or  $3_e, 3_b, 1_q$ , so we need to select an override on the nonquaternion  $3_b$ , so that when two elements are multiplied within it, just one allocation to an octonionic structure is selected.

We will need to look at this typical example in detail, so denote the three elements of  $3_b$  by  $3_{b1}$ ,  $3_{b2}$  and  $3_{b3}$ . We will display the 3 elements of  $3_b$  combining in pairs to form arrows with the following typical structures. We will choose at first, arbitrarily, a link to the  $3_a$ ,  $3_b$ ,  $1_p$  octonionic structure. Of course, two arrows shown below combine to give an oriented quaternion triple, for which reversal of arrows leads to a minus value.



The central triples  $3_d$  and  $3_f$  have similar structures, mapping to separate values in  $3_c$  and  $3_e$  respectively. We have stated the  $1_p$ ,  $1_q$  and  $1_r$  elements combined with  $3_b$  give on composition with one element of  $3_b$  the octonion structures  $(3_a, 3_b, 1_p)$ ,  $(3_c, 3_b, 1_r)$  and  $(3_e, 3_b, 1_q)$ . Because  $3_{b1}$  links to  $3_{a1}$ , we have to ensure that the link  $1_p$  to  $3_{b1}$  does not also link to  $3_{a1}$ , but this can be arranged.

Alternative structures can be considered. For example, if  $3_{b3}$ ,  $3_{b1}$  links to  $3_{a1}$  as before, we could also have  $3_{b3}$ ,  $3_{b2}$  linking to  $3_{c1}$  and  $3_{b2}$ ,  $3_{b1}$  linking to  $3_{e1}$ .  $\square$

#### 4.16. Further investigations.

The exceptional Lie algebras  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$  and  $E_8$  are related to the existence of division algebras limited in number to those embedded within the octonions [Wi09], [CSM95].

We have seen in chapter IV of [Ad15] that for matrices A, B and C the Lie bracket

$$[AB] = AB - BA$$

satisfies the Jacobi identity

$$[[AB]C] + [[BC]A] + [[CA]B] = 0. \quad (1)$$

For octonions, we do not have a matrix algebra, but we might wish to form Lie brackets from them satisfying (1). From section 8.7, for the octonions the only nonquaternion triple which is scanned from one set to another is  $e_1e_3e_5$  and

$$\begin{aligned} [[e_1 e_3] e_5] + [[e_3 e_5] e_1] + [[e_5 e_1] e_3] &= 2[e_2 e_5] + 2[e_4 e_1] + 2[e_6 e_3] \\ &= -12e_7, \end{aligned} \quad (2)$$

so this does not satisfy (1). However, there exist other nonquaternion triples, for example  $e_1e_7e_5$ , and this satisfies

$$\begin{aligned} [[e_1 e_7] e_5] + [[e_7 e_5] e_1] + [[e_5 e_1] e_7] &= 2[e_4 e_5] - 2[e_2 e_1] + 2[e_6 e_7] \\ &= -12e_3. \end{aligned} \quad (3)$$

We need to find out whether a factor of 12 is present in all special calculations. We have

$$\begin{aligned} [[e_3 e_7] e_1] + [[e_7 e_1] e_3] + [[e_1 e_3] e_7] &= 2[e_6 e_1] - 2[e_4 e_3] - 2[e_2 e_7] \\ &= 12e_5 \end{aligned} \quad (4)$$

and

$$\begin{aligned} [[e_5 e_7] e_3] + [[e_7 e_3] e_5] + [[e_3 e_5] e_7] &= 2[e_2 e_3] - 2[e_6 e_5] - 2[e_4 e_7] \\ &= 12e_1. \end{aligned} \quad (5)$$

In order to create a viable Lie bracket we note that we have constructed the octonions from the quaternions by the Cayley-Dickson construction in section 9. Thus, if we apply an inverse Cayley-Dickson construction to retrieve a pair of quaternions from the octonions, since the quaternions are representable by matrices, on each item of this pair we can create a Lie bracket satisfying (1). The information we might wish to keep in these Lie brackets could also be formed from the sum, difference, matrix product, and the product of a matrix by an inverse matrix of the pair, or any combination of these.

A better solution is to take Lie brackets including equations (2) to (5) (mod 12). We can also use a subgroup, that is, (mod 2), (mod 3), (mod 4) or (mod 6).  $\square$

The 10-novonions we have considered contain the quaternions as a subalgebra. Even when an override structure is imposed, its three elements constitute a quaternion. For the octonion structures we have considered, these also contain a quaternion, and this still applies if a nonstandard override is applied. Thus the n-novonions all contain quaternion subalgebras.  $\square$

Finally in this section we mention that, just as quaternions may be given globally bosonic or globally fermionic structures, as in section 4.13, so too can octonions and n-novonions. The analogous situation in the novonion algebra is that combined bosonic and fermionic structures may reside in the same n-novonion.  $\square$

#### 4.17. The García classification problem.

Doly García has given a factorisation of 10-novonions so that

$$(i + i' + i'')(j + j' - 2j'') = 0.$$

For n-novonions a classification problem is to find all factorisations of novonions such that for real  $a_k, b_k$ , the nonscalar basis elements  $i_k$  give

$$(a_1 i_1 + \dots a_n i_n)(b_1 i_1 + \dots b_n i_n) = 0.$$

In novonion physics these may be the generators of the universe at time  $t = 0$ .  $\square$

#### 4.18. The $E_8$ and novonion group constructions from octonions.

[To be modified.]

The  $E_8$  lattice is closely related to the **nonassociative algebra** of real **octonions**  $\mathbf{O}$ . It is possible to define the concept of an **integral octonion** analogous to that of an **integral quaternion**. The integral octonions naturally form a lattice inside  $\mathbf{O}$ . This lattice is just a rescaled  $E_8$  lattice. (The minimum norm in the integral octonion lattice is 1 rather than 2). Embedded in the octonions in this manner the  $E_8$  lattice takes on the structure of a **nonassociative ring**.

Fixing a basis  $(1, i, j, k, \ell, \ell i, \ell j, \ell k)$  of unit octonions, one can define the integral octonions as a **maximal order** containing this basis. (One must, of course, extend the definitions of *order* and *ring* to include the nonassociative case). This amounts to finding the largest **subring** of  $\mathbf{O}$  containing the units on which the expressions  $x^*x$  (the norm of  $x$ ) and  $x + x^*$  (twice the real part of  $x$ ) are integer-valued. There are actually seven such maximal orders, one corresponding to each of the seven imaginary units. However, all seven maximal orders are isomorphic. One such maximal order is generated by the octonions  $i, j$ , and  $\frac{1}{2}(i + j + k + \ell)$ .

A detailed account of the integral octonions and their relation to the  $E_8$  lattice can be found in Conway and Smith (2003).

### Example definition of integral octonions[edit]

Consider octonion multiplication defined by triads: 137, 267, 457, 125, 243, 416, 356. Then integral octonions form vectors:

- 1)  $e_i$ ,  $i=0, 1, \dots, 7$
- 2)  $e_{abc}$ , indexes abc run through the seven triads 124, 235, 346, 457, 561, 672, 713
- 3)  $e_{pqrs}$ , indexes pqrs run through the seven tetrads 3567, 1467, 1257, 1236, 2347, 1345, 2456.

Imaginary octonions in this set, namely 14 from 1) and  $7 \cdot 16 = 112$  from 3), form the roots of the Lie algebra  $e_6$ . Along with the remaining  $2 + 112$  vectors we obtain 240 vectors that form roots of Lie algebra  $e_7$ . See the Koca work on this subject.<sup>[13]</sup>

If we look at octonions, they have a multiplicative identity 1, and seven units  $e_1, e_2, \dots, e_7$ . If we ignore their additive structure for the moment, we also have plus or minus these elements, and the plus and minus are distinguishable in the multiplicative group structure. The total number of elements of these types is now 16.

If we go over to a ring structure, additively it contains zero. We have also seen that the octonions with Lie brackets given by  $[A, B] = AB - BA$ , where for example this can be a matrix representation, although octonions are not matrices, since they are not associative, nevertheless they can be given the above Lie algebra structure (mod 12), satisfying the Jacobi identities.

The elements contain zero, for example  $6e_7 \times 2e_7 = 0 \pmod{12}$ . The products which are zero (mod 12) contain powers of factors 3 and 2, where 3 and  $3^2$  occur (mod 12) and so do 2,  $2^2$  and  $2^3$ . To create a multiplicative group without zero, map an entire zero to  $\pm e^{2\pi i n/3}$ , where 3 maps to  $\pm 1$  and 2 to  $e^{2\pi i n/3}$ , so that the Lie brackets for  $\pm e^{2\pi i n/3}$  and zero are the same, maintaining the Lie algebra structure.

Because we are dealing with elements  $\pm e^{2\pi i n/3} e_r$ , we can relabel the eight octonion basis elements as 48, so that now the Lie bracket structure acts (mod 12) on these new elements.

Consider the original elements  $e_7, e_6, \dots, 1, -1, -e_1, \dots, -e_7$ , 16 elements in all, and zero. If we allow new elements  $0e_r = 0, 1e_r = e_r, 2e_r, \dots, 11e_r$ , with  $12e_r = 0e_r$ , so that this exists in a (mod 12) arithmetic, then the Lie algebra derived from the octonions in this way has first a set of 11 elements to choose from (that is, not 0). Then if we create new elements by adding a second set, because the addition is  $ae_r + be_s$  with  $e_r \neq e_s$ , but  $ae_r + a(-e_r) = 0$ , the paired  $e_r$  and  $(-e_r)$  elements (mod 12) have 12 values (including  $0e_r + 0(-e_r) = 0$ ) that we must exclude, but

$-[0e_r + 0(-e_r)] = 0e_r + 0(-e_r) = 0$  is the same case. Thus we have  $12 - 6$  elements that are not zero. Further, we have values

$$(1) \quad 1e_7 + 0(-e_7) = 1e_7$$

$$(2) \quad 2e_7 + 1(-e_7) = 1e_7$$

... ..

$$(11) \quad 11e_7 + 10(-e_7) = 1e_7$$

$$(12) \quad 12e_7 \text{ [which is } 0(e_7)\text{]} + 11(-e_7) = 1e_7,$$

which we must also collect together as one item, but case (1) (mod 12) is the same as case (12), etc, with case (6) the same as case (7). Thus there are  $12/2 = 6$  distinct cases for  $1e_7$  and the total number of exclusions for differences  $2e_7, 3e_7, \dots, 11e_7$  is the same. Thus for the pair  $e_7, (-e_7)$  we obtain  $12^2 - 12 \times 6 = 12 \times 6$  possibilities. There are 8 values  $e_7, e_6, \dots, e_1, 1$  that we can account for in a similar way. Then the total number of distinct possibilities is  $(12 \times 6)^8$ , but we must exclude the cases where we have a sum of all the  $e_r$  equal to zero. For  $e_7$  this is 12 cases, as we have seen, and likewise for  $e_6, \dots, e_1, 1$ . Thus we exclude  $(12)^8$  items, and the total number of elements of the octonion algebra is

$$(12 \times 6)^8 - 12^8 = 12^8(6^8 - 1).$$

This is the order of the Lie group via the Lie algebra we have derived from the octonions.  $\square$

#### 4.19. Novanion simple group constructions. [To be modified.]

Our objective now is to convert novanion algebras, with novanion brackets (mod 12) like a Lie bracket, to novanion Lie algebras, and then map these novanion Lie algebras to groups. These groups, being multiplicatively invertible, do not contain zero, so we map  $0 \rightarrow \pm 1$ , which retains the novanion Lie algebra structure. The resulting novanion group thus does not have the same structure as the multiplicative part of the ring mapped directly to the novanion Lie algebra.

For the 10-novanions consider the elements  $e_9, e_8, \dots, 1, -1, -e_1, \dots, -e_9, 20$  elements in all, and zero. Using elements  $0e_r = 0, 1e_r = e_r, 2e_r, \dots, 11e_r$  again, then in the case when we are not using  $\pm 1$ , the extended Lie group derived from the 10-novanions has in a calculation similar to one given for the octonions,  $(12 \times 6)^9$  elements. If we assume the scalar part is not zero, the available scalar values are  $\pm 1, \pm 2, \dots, \pm 11$ , amounting to 22 components, but we have  $1 = -11$  (mod 12), etc., so effectively there are 11 components. We know that when the scalar part is not zero, we cannot get a zero result on multiplying 10-novanions together, provided we ignore the (mod 12) restriction. Thus the order of the 10-novanion group derived from this algebra is  $(12 \times 6)^9 \times 11$  elements. If we obtain 0 (mod 12) in any multiplication, then as before we will map this result to  $\pm 1$  without changing the Lie-type bracket algebra.

For the 26-novanions or indeed any n-novanion algebra, the case is similar with  $(12 \times 6)^n \times 11$  elements. For the groups generated by these means note we can proceed not only (mod 12), but as we have mentioned, (mod 6), (mod 4), (mod 3) and (mod 2).

We know that overrides derived from the octonion structure operate for all 10-novanion components in a 26-novanion, thus external to the 10-novanion or an octonion or their subalgebras, there are no other proper subentities in this algebra, which implies that the groups defined by a quotient of a 10-novanion group or an octonion group with the 26-novanion group are simple.

Note that we have demonstrated in section 12 that both bosonic and fermionic structures can be implemented in novanions. Thus any argument which says that a bosonic 26-novanion



cannot contain a 10-novanium is incorrect. Since the bosonic 10-novanium is a maximal proper subgroup of the 26-novaniums, the 26-novaniums do not generate a simple group directly, only via quotients.  $\square$

Note that  $26 - 10 = 16$ , but we have no indication that this is two copies of  $E_8$ . Indeed, it cannot be, since the structure is not closed, encroaching in overrides onto the 10-novanium. So we show once again that the novanium groups do not lead to the compactification assumed in string theory of the identification of the 16 dimensions with  $E_8 \times E_8$ .  $\square$

These considerations may be extended without limit, for example to the 80-novaniums, or more generally the  $(3^n - 1)$ -novaniums,  $n > 0$ , which are those hypercube novaniums with one corner containing not a triplet, but a singlet. In particular, for the 80-novaniums there exists a 28-novanium slice group which has no containing groups other than the 80-novanium group itself, so there is an analogous structure to that for the 26-novanium group. The remaining dimensions are  $79 - 27 = 52$ , and this generates a simple group via the quotient with the 28-novanium group.  $\square$