CHAPTER IV

The novanion simple group construction

4.1. Invitation to the reader.

Groups are the most investigated topic for superexponential structures with one operation. The theory of novanions leads us in a natural way to ask what impact novanions have on the theory of groups, even the classification of simple groups, for which the claim has been made that a complete classification was achieved after monumental efforts finally in 2008.

The novanion idea for these groups is a new subject for which there are currently few experts, and not the writer. Nevertheless, there are logical deductions which can be made, and I shall be making them. Since the conclusions do not tie in with what is believed to be a correct proof of the classification, a question is whether ignorance and delusion lies with the author, or with society at large, which encompasses many individuals some of whom have diligently spent a lifetimes work on the study of simple groups. But it is our belief that the origin of the discrepancy has now been isolated, which is described in the text.

The expert in group theory is invited to correct any misconceptions so that revisions can be made, and so the mathematician in the street can be better informed.

4.2. Introduction and history.

We introduce an analysis of group theory via the theory of novanions, developed in [Ad15], and also in the mathematical chapter XVII of *New Physics* [Ad18] by Graham Ennis and me with applications to universal physics. Novanions are a generalisation of octonions. Novanion groups are found as a generalisation of the way octonions lead to special Lie groups. The octonions have a (mod 12) Jacobi identity mapping to algebras, and octonions are related to some exceptional Lie algebras, like E_8, with corresponding mapping to groups. We will investigate whether these two methods give the same results.

The beginning of this study arises out of a speculation of Daniel Hajas that I call the Hajas conjecture, in which novanions are connected with the heterotic string in physics, 10-novanions, to be described, are related to the 10-dimensional fermionic string in physics, and my extension of that idea, that the 26-dimensional bosonic string is related to 26-novanions. We will see we need to go beyond this. The heterotic string has a fermionic dimension of 10 and a bosonic dimension of 26, so that in the conventional picture 26 – 10 = 16 dimensions in the universe are compactified. We have shown in the above references that 26-novanions, containing octonion type algebras are also present.

Picking up the work of Borcherds linking the monster simple group to orbifolds in the 26 dimensions of the heterotic string, we investigate whether there are simple groups beyond the monster, in direct conflict with current understandings. Our motivation is as follows. On page 938 of ‘Quantum field theory’ by Eberhard Zeidler, with reference to the Thompson series, he writes

“Borcherds calculated this series using the monster Lie algebra. This Lie algebra is constructed as the space of physical states of a bosonic string moving in a $\mathbb{Z}_2$ orbifold $M/\mathbb{Z}_2$ of a 26-dimensional torus M”.

By this means the classification of simple groups is derived. Preliminary investigation of the work of Borcherds led credence to the Hajas conjecture, but subsequent more detailed investigations give mismatches in the computed size, called the order, of simple groups derived from novanions. This classification is finite, according to conventional wisdom. We wish to investigate whether or not a spanner can be thrown in the works, and note that the number of distinct n-novanionic algebras is infinite, given that the override condition in sections 8.13 and 8.14 can always be allocated, which corresponds non-trivially when the n-novanion contains octonionic components. The reader is invited to investigate the proof of the finiteness condition. More in-depth investigation is needed to compare standard reasoning with what is given here.

However, we note an interesting nonstandard conclusion we have arrived at with the Hajas identification, given essentially in [Ad18] chapter XI, namely that all n-novanions can be allocated bosonic and fermionic parts. Thus both the 10- and 26-novanions may be allocated these components, and this also holds in general for n-novanions.

In chapter XX of [Ad18] we employ the 26-novanions in Heim theory extended to gluons and quarks. There exist other possible universes with $n > 26$. So the 26-novanions do indeed contain a bosonic algebra.

The conjecture is: can we apply the unbounded n-novanionic algebra to derive a Thompson series so that the number of simple groups is not bounded?

We make this statement because override conditions for novanions of dimension greater than 26 lead to simple groups via Jacobi identity mappings to algebras, since overrides mix different quaternions together, creating one overarching structure. However, E_8 has order linked to that of the monster. The simple groups we are about to investigate have a size greater than that of the monster.

Although simple groups of similar order of magnitude to the monster have been obtained by novanion methods, they differ. Moreover, there are sequences of simple novanion groups without termination, violating the finiteness condition on the classification of sporadic groups. The groups corresponding to novanion algebras are not initially simple, but quotients with novanion subalgebras lead to factor groups which are simple.

Recent investigation reveals the reason for why the Borcherds and novanion constructions differ. The monster can be derived from vertex operator algebras with infinite quadratic form. Novanion algebras sometimes have zero quadratic form for a zero scalar component, t. These zero forms are not obtained as diagonalised forms which can be obtained by multiplication of the novanion by its conjugate. The Lie algebra construction forbids algebras of zero quadratic form. The scalar $t = 0$ component is realised in the final novanion algebra in conjunction with other additive elements to form a composite element, but we can map $t = 0 \rightarrow t = \pm 1$, and so novanions are allowable objects in such algebra constructions, although this has not hitherto been recognised. From this point of view the mapping from 10-novanions to fermion strings and 26-novanions to bosonic strings used in the Borcherds construction is spurious.

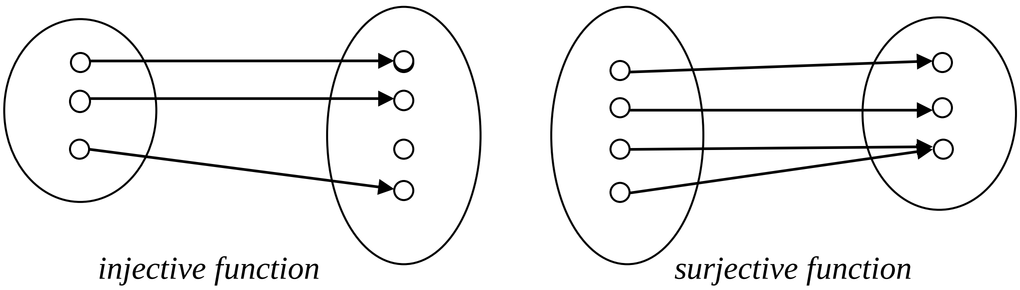
I am unclear at this stage whether other aspects of my reasoning are a novelty. These ideas are used to derive novanion groups, and can equally be used for division algebras like the octonions. The groups corresponding to novanion algebras are not initially simple, but quotients with novanion subalgebras lead to factor groups which are simple.

The development proceeds by first describing axiom systems which give mathematical structures for numbers and their generalisations. The intention is to provide the background for the discussion which follows. We discuss the classification of simple groups and their order. Various basic terms used in the classification such as the center of a group, factor groups, normal subgroups, Lagrange's subgroup theorem that the order of a subgroup divides the order of its group, and the Schur multiplier are introduced. We derive various finite groups, which are not subgroups of the quaternions, from the quaternion Lie algebra.

We next give a detailed discussion of novanions, which are division algebras when the scalar part is not zero. Octonions are division algebras. We introduce 10-, 26- and more general novanions. In the case of octonions, we show that they have a finite representation as a Lie algebra (mod 12), from which we may obtain a group. Products in the group which would otherwise be 0 (mod 12) can be converted to ± 1. This maintains the Lie algebra structure and means that the group derived from it is provided with inverses. Other novanions may have no such Lie algebra structure, nevertheless a mapping of 0 (mod m) to ± 1 may still be made.

4.3. Concepts in group theory.

A function f: $S \rightarrow T$ is called *injective* (or *one-to-one*, or an injection) if $f(a) \neq f(b)$ for any two different elements a and b of the domain. It is called *surjective* (or *onto*) if f(S) = T. That is, it is surjective if for every element y in the codomain there is an x in the domain such that f(x) = y. The function f is called *bijective* if it is both injective and surjective.



injective function *surjective function*

A *dual* (or *opposite*) map reverses all arrows. If the original function is injective, then some of the elements of the opposite map may not have values in the codomain, so this is not a function on elements. Likewise, if the original function is surjective, the opposite map for an element in its domain may have not one but a set of elements corresponding to this element in its codomain, and again it is not a function on elements.

An *inverse function* (or *fiber*) of a bijective map $x \leftrightarrow f(x)$ is the map $f^{-1}: f(x) \leftrightarrow x$.

Sets of numbers may be combined under operations like '+' or '×' to form other numbers.

A *magma* is the most general structure combining sets and an operation. A magma is a set M with a single binary operation $M \times M \rightarrow M$, combining elements in pairs of the magma, with each pair forming another element belonging to the magma. No other properties are specified.

A *group* is a magma with the following structure. The operation on the magma can be written either additively or multiplicatively without brackets, the two choices being equivalent within the group. For an *abelian group* (a + b = b + a) the *identity*, e, for a group if written additively is the element 0, or 1 written multiplicatively, where a + 0 = a = 0 + a. Groups have *inverses* (-a) of a written additively or a^{-1} written multiplicatively with the additive rule

$$a + (-a) = 0,$$

alternatively if written multiplicatively

$$a\,(a^{-1}) = 1 = (a^{-1})a.$$

A group is *nonabelian* or *noncommutative*, generally written multiplicatively, if some $ab \neq ba$.

In any group the integral *power* of an element a can be defined as the element

$$a^m = a.a.\ ...\ a \text{ (m terms)}.$$

Negative powers can be defined by

$$a^m a^{-m} = 1.$$

A group G is called *cyclic* if it contains an element the powers of which exhaust G. Cyclic groups are abelian. It is often understood that cyclic groups are finite. When this is not so, we will explicitly state that they are infinite.

A *permutation* is a bijection of a finite set to itself. A permutation which interchanges cyclically m objects of a set {1, 2, ..., m} forms an abelian group called a cycle of degree m. This permutation is obtained from a power by specifying that a^{m-1} is the m^{th} element and the cyclic permutation consists of multiplying by a. It can be represented by

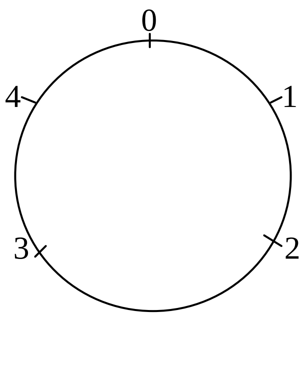
$$\begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix},$$

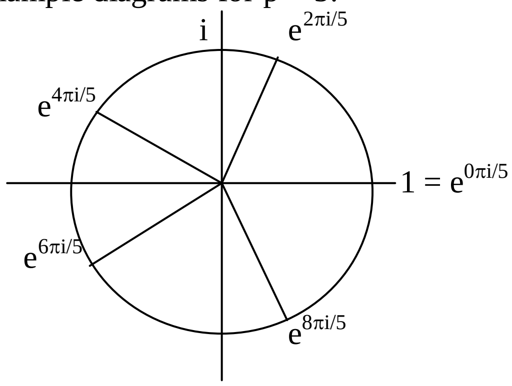
or in contracted notation by (1 2 ... m).

Another picture of a cyclic group is given by the 'clock' diagram x (mod p) and the Argand complex circle diagram, where there is a bijection for fixed r

$$x \text{ (mod p)} \leftrightarrow e^{r + (2\pi i x/p)},$$

which can be pictured in the example diagrams for p = 5:





x (mod 5) complex (Argand) diagram with unit radius

A group derived from cyclic group generators which do not intersect, so the generators form a partition for the group, is also cyclic. For a finite cyclic group its number of elements, or *order*, which is the number of times it takes a generator to return to the identity permutation, is the least common multiple (l.c.m.) of the order of its cyclic components. An example of a cyclic permutation with the identity permutations present is

(1 2)(4 6 7)(3)(5)

which we can contract by removing the identity permutations to

(1 2)(4 6 7) = (4 6 7)(1 2).

The set of all permutations of m objects forms a group called the *symmetric group*, denoted by S_m. The name is derived from its origins in describing polynomial equations.

A noncommutative group can be generated by cycles which overlap somewhere. For example

$$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}3&4\\4&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix} \neq \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix} = \begin{pmatrix}3&4\\4&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}.$$

The symmetric group can be described by matrices. For example $\begin{pmatrix}1&2&3&4&5\\2&4&1&3&5\end{pmatrix}$ can be represented by the matrix with one 1 in each row and column, and zeros elsewhere

$$\begin{bmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{bmatrix},$$

in which, say, $2 \rightarrow 4$ is represented by a 1 in the second row and fourth column, with operations defined by matrix multiplication. As a further example, the cyclic group of order 4 is given by the matrices

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix},\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix} \text{ and } \begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}.$$

All elements of S_4 can be obtained from the above by permuting rows or alternatively and equivalently by permuting columns.

A *subgroup* S of a group G is a group included in G. If S ≠ G, S is a *proper subgroup*. The number of elements in the subgroup is called the order of the subgroup. The complement of S in G cannot form a subgroup, since 1 does not belong to it.

The *center* of a group G, denoted Z(G), is the set of elements that commute with every element of G.

The *commutator* of two group elements g and h is

$$[g, h] = g^{-1}h^{-1}gh.$$

The commutator [g, h] is equal to the identity element e if and only if hg = gh.

The *commutator subgroup* [G, G] (also called the *derived subgroup* of G and denoted $G^{(1)}$) is the subgroup generated by all the commutators.

A *homomorphism* h of a group G to a group G′ is a surjective map ab = g → h(g), such that

$$h(ab) = h(a)h(b).$$

An *automorphism* is a homomorphism G → G.

Theorem 4.5.1. *Under a homomorphism the identity* e *of* G *maps to the identity* h(e) *of* G′, *and maps inverses* g^{-1} *to* $h(g)^{-1} = h(g^{-1})$.

Proof. The identity satisfies ee = e, so h(ee) = h(e) = h(e)h(e). The inverse satisfies $(g)(g^{-1}) = e$, so $h(g)h(g^{-1}) = h(e) = h(g)h(g)^{-1}$, and multiplying on the left by $h(g)^{-1}$ gives $h(g^{-1}) = h(g)^{-1}$. ☐

The set {k} is the *kernel* of a group homomorphism h: G → G′, if it satisfies

$$h(a)h(k) = h(a) = h(k)h(a),$$

in other words it is the identity of G′.

A *right coset or right residue class* of a subgroup S of G is the set of elements Sa, with s ∈ S and a ∈ G. A *left coset* is the set aS, and when both coincide the set can be called a *coset*.

The *quotient group* G/S *of* G mod S for G a group, S a subgroup, is the family of left cosets.

Lemma 4.3.2. *If* S *is finite, each right* (*or left*) *coset has as many elements as* S. *Two right* (*or left*) *cosets are either identical or have no common elements*.

Proof. The map $a \to sa$ is a bijection, since each sa is the image of one and only one a, and if $a \to sb$, with $b \neq a$ then $1 = a(a^{-1})$ maps to $sb(a^{-1}) = s(ba^{-1}) = s$, so b = a. Further, if there were any intersection, then sb = sa, which we have shown is impossible unless b = a. ☐

If G is finite, it can be partitioned into a finite number of right or left cosets where each coset contains the same number of elements, and the conclusion is

Theorem 4.3.3. (**Lagrange's subgroup theorem**). *The order of a finite subgroup* $S \subseteq G$ *divides the order of a finite group containing it*. ☐

A *conjugate* of an element x in a group G is an element $a^{-1}xa$.

Theorem 4.3.4. *For an element* a *of* G, *conjugation* T_a: $x \to a^{-1}xa$ *is an automorphism of* G.

Proof. $(a^{-1}xa)(a^{-1}ya) = a^{-1}(xy)a$. ☐

An automorphism of the form a^{-1} xa is called an *inner automorphism*, otherwise it is called an *outer automorphism*.

It follows from what we have said that inner automorphisms form a subgroup of all the automorphisms of a group G. ☐

A subgroup S of G is *normal* in G if and only if it is invariant under all inner automorphisms of G. For example, consider the symmetric group S_3 of all permutations of the set {1, 2, 3}. Then {(1, (1 2 3), (1 3 2)} is a normal subgroup of S_3, because we can verify the following statements.

(1 2){(1, (1 2 3), (1 3 2)} = {(1 2), (1 3), (2 3)} = {(1, (1 2 3), (1 3 2)}(1 2)
(1 3){(1, (1 2 3), (1 3 2)} = {(1 3), (2 3), (1 2)} = {(1, (1 2 3), (1 3 2)}(1 3)
(2 3){(1, (1 2 3), (1 3 2)} = {(2 3), (1 2), (1 3)} = {(1, (1 2 3), (1 3 2)}(2 3).

Theorem 4.3.5. *A subgroup* S *is normal if and only if all of its right cosets are left cosets*.

Proof. Suppose S is normal. Then $aSa^{-1} = (a^{-1})^{-1}S(a^{-1}) = S$. Thus Sa = aS. Conversely, applying lemma 4.3.2, if two cosets are equal so that Sa = bS, then a = b and S is normal. ☐

It should be carefully noted that the equation Sa = aS does not claim that every element of S commutes with a, only that the cosets Sa and aS are the same.

A group G is called *simple* if its only normal subgroups are the identity and G itself.

The *general linear group* GL of degree n is the set of n×n invertible matrices, meaning they have a multiplicative inverse, together with the operation of ordinary matrix multiplication. GL(n, ℂ) are invertible matrices with complex number elements.

The *projective linear group* PGL is the induced action of the general linear group of a vector space **V** on the associated projective space P(**V**). Explicitly, the projective linear group is the quotient group

PGL(**V**) = GL(**V**)/Z(**V**)

where GL(**V**) is the general linear group of **V** and Z(**V**) is the subgroup of all nonzero scalar transformations of **V**. These are quotiented out because they act trivially on the projective space and they form the kernel of the action. The notation "Z" is used because the scalar transformations form the center of the general linear group.

A group homomorphism from D to G is said to be a *Schur cover* of the finite group G if the kernel is contained both in the center and the commutator subgroup of D, and amongst all such homomorphisms, this D has maximal size.

The *Schur multiplier* of G is the kernel of any Schur cover. When the homomorphism is understood, the group D is often called the Schur cover.

Schur's motivation for studying the multiplier was to classify projective representations of a group. A projective representation is much like a group representation except that instead of a homomorphism into the general linear group GL(n, ℂ), one takes a homomorphism into the projective general linear group PGL(n, ℂ). In other words, a projective representation is a representation modulo the center.

4.4. The standard classification of simple groups.

The classification of simple groups lists 26 sporadic simple groups. The number of elements of the sporadic simple groups, together with their Schur multipliers is listed below.

Simple group (p = pariah)	**Order**	**Order of Schur multiplier**
Monster *M*	$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71$	1
Baby monster *B*	$2^{41}.3^{13}.5^{6}.7^{2}.11.13.17.19.23.31.47$	2
Thompson group *Th*	$2^{11}.3^{10}.5^{3}.7^{2}.13.19.31$	1
Lyons group *Ly* (p)	$2^{8}.3^{7}.5^{6}.7.11.31.37.67$	1
Harada-Norton group *HN*	$2^{14}\cdot3^{6}\cdot5^{6}\cdot7\cdot11\cdot19$	1
O'Nan group *O'N* (p)	$2^{9}\cdot3^{4}\cdot5\cdot7^{3}\cdot11\cdot19\cdot31$	3
Suzuki sporadic group *Suz*	$2^{13}\cdot3^{7}\cdot5^{2}\cdot7\cdot11\cdot13$	6
Rudvalis group *Ru* (p)	$2^{14}\cdot3^{3}\cdot5^{3}\cdot7\cdot13\cdot29$	2
Held group *He*	$2^{10}\cdot3^{3}\cdot5^{2}\cdot7^{3}\cdot17$	1
McLaughlin group *MCL*	$2^{7}\cdot3^{6}\cdot5^{3}\cdot7\cdot11$	3
Higman-Sims group *HS*	$2^{9}\cdot3^{2}\cdot5^{3}\cdot7\cdot11$	2
Fischer group Fi_{22}	$2^{17}\cdot3^{9}\cdot5^{2}\cdot7\cdot11\cdot13$	6
Fischer group Fi_{23}	$2^{18}\cdot3^{13}\cdot5^{2}\cdot7\cdot11\cdot13\cdot17\cdot23$	1
Fischer group Fi_{24}	$2^{21}\cdot3^{16}\cdot5^{2}\cdot7^{3}\cdot11\cdot13\cdot17\cdot23\cdot29$	3
Conway group Co_1	$2^{21}\cdot3^{9}\cdot5^{4}\cdot7^{2}\cdot11\cdot13\cdot23$	2
Conway group Co_2	$2^{18}\cdot3^{6}\cdot5^{3}\cdot7\cdot11\cdot23$	1
Conway group Co_3	$2^{10}\cdot3^{7}\cdot5^{3}\cdot7\cdot11\cdot23$	1
Janko group J_1 (p)	$2^{3}\cdot3\cdot5\cdot7\cdot11\cdot19$	1
Janko group J_2	$2^{7}\cdot3^{3}\cdot5^{2}\cdot7$	2
Janko group J_3 (p)	$2^{7}\cdot3^{5}\cdot5\cdot17\cdot19$	3
Janko group J_4 (p)	$2^{21}\cdot3^{3}\cdot5\cdot7\cdot11^{3}\cdot23\cdot29\cdot31\cdot37\cdot43$	1
Mathieu group M_{11}	$2^{4}\cdot3^{2}\cdot5\cdot11$	1
Mathieu group M_{12}	$2^{6}\cdot3^{3}\cdot5\cdot11$	2
Mathieu group M_{22}	$2^{7}\cdot3^{2}\cdot5\cdot7\cdot11$	12
Mathieu group M_{23}	$2^{7}\cdot3^{2}\cdot5\cdot7\cdot11\cdot23$	1
Mathieu group M_{24}	$2^{10}\cdot3^{3}\cdot5\cdot7\cdot11\cdot23$	1

The 20 sporadic groups which are subquotients of the monster are called the *happy family*. The remaining 6 are referred to as *pariahs*. 37 does not divide the order of the monster but divides Ly and J_4, which are therefore pariahs. Four other groups can be shown to be pariahs.

4.5. The E_8 lattice.

For any basis of $\mathbb{R}^n$, the subgroup of all linear combinations with integer coefficients of the basis vectors forms a lattice, which can be viewed as a regular tiling of a space by a primitive cell called the fundamental parallelotope. A *unimodular* lattice is an integral lattice with determinant or hypervolume 1 or −1.

The E_8 lattice is a special lattice in $\mathbb{R}^8$. It can be characterised as the unique positive-definite, even unimodular lattice of rank 8, so that it can be generated by the columns of an 8×8 matrix where the determinant of the fundamental parallelotope of the lattice is ±1. The name derives from the fact that it is the root lattice of the E_8 root system, described in [Ad15], chapter IV.

The *norm* of the E_8 lattice (divided by 2) is a positive definite even unimodular quadratic form in 8 variables, and conversely such a quadratic form can be used to construct a positive-definite, even, unimodular lattice of rank 8. The existence of such a form was first shown by H. Smith in 1867 and the first explicit construction of this quadratic form was given by A. Korkin and G. Zoltarev in 1873. The E_8 lattice is also called the *Gosset lattice* after T. Gosset who was one of the first to study the geometry of the lattice around 1900.

The E_8 lattice is a discrete subgroup of $\mathbb{R}^8$ which spans all of $\mathbb{R}^8$. It can be given explicitly by the set of points $\Gamma_8 \subset \mathbb{R}^8$ such that

- all the coordinates are integers or half-integers but not a mixture of these
- the sum of the eight coordinates is an even integer.

The sum of two lattice points is another lattice point, so that Γ_8 is indeed a subgroup.

An alternative description of the E_8 lattice is the set of all points in $\Gamma'_8 \subset \mathbb{R}^8$ such that

- all the coordinates are integers and the sum of the coordinates is even, or
- all the coordinates are half-integers and the sum of the coordinates is odd.

The lattices Γ_8 and Γ'_8 are isomorphic – we can pass from one to the other by changing the signs of any odd number of coordinates. The lattice Γ_8 is called the *even coordinate system* for E_8 while the lattice Γ_8' is called the *odd coordinate system*. Unless we specify otherwise we will work in the even coordinate system.

Even unimodular lattices occur only in dimensions divisible by 8. In dimension 16 there are two such lattices: $\Gamma_8 \oplus \Gamma_8$, and Γ_{16} constructed analogously to Γ_8. In dimension 24 there are 24 such lattices, called Niemeier lattices, the most important of which is the Leech lattice.

A possible basis for Γ_8 is given by the columns of the upper triangular matrix

$$\Gamma_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Γ_8 is then the integral span of these vectors. All other possible bases are obtained from this one by right multiplication by elements of GL(8, $\mathbb{Z}$).

The shortest nonzero vectors in Γ_8 have norm 2. There are 240 such vectors.

- All half-integer: (can only be ±1/2)
  - All positive or all negative: 2
  - Four positive, four negative: (8×7×6×5)/(4×3×2×1) = 70
  - Two of one, six of the other: 2× (8×7)/(2×1) = 56
- All integer: (can only be 0, ±1)
  - Two ±1, six zeroes: 4×(8×7)/(2×1) = 112

These form a root system of type E_8. The lattice Γ_8 is equal to the E_8 root lattice, which means that it is given by the integral span of the 240 roots. Any choice of 8 simple roots gives a basis for Γ_8.

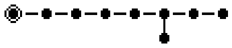
The automorphism group (or symmetry group) of a lattice in $\mathbb{R}^n$ is defined as the subgroup of the orthogonal group O(n) that preserves the lattice. The symmetry group of the E_8 lattice is the Weyl/Coxeter group of type E_8. This is the group generated by reflections in the hyperplanes orthogonal to the 240 roots of the lattice. Its order is given by

The E_8 Weyl group contains a subgroup of order 128·8! consisting of all permutations of the coordinates and all even sign changes. This subgroup is the Weyl group of type D_8. The full E_8 Weyl group is generated by this subgroup and the block diagonal matrix $H_4 \oplus H_4$ where H_4 is the Hadamard matrix

Geometry[edit]

See 5_{21} honeycomb

The E_8 lattice points are the vertices of the 5_{21} honeycomb, which is composed of regular 8-simplex and 8-orthoplex facets. This honeycomb was first studied by Gosset who called it a *9-ic semi-regular figure*[4] (Gosset regarded honeycombs in n dimensions as degenerate n+1 polytopes). In Coxeter's notation,[5] Gosset's honeycomb is denoted by 5_{21} and has the Coxeter-Dynkin diagram:



This honeycomb is highly regular in the sense that its symmetry group (the affine Weyl group) acts transitively on the k-faces for $k \le 6$. All of the k-faces for $k \le 7$ are simplices.

The vertex figure of Gosset's honeycomb is the semiregular E_8 polytope (4_{21} in Coxeter's notation) given by the convex hull of the 240 roots of the E_8 lattice.

Each point of the E_8 lattice is surrounded by 2160 8-orthoplexes and 17280 8-simplices. The 2160 deep holes near the origin are exactly the halves of the norm 4 lattice points. The 17520 norm 8 lattice points fall into two classes (two orbits under the action of the E_8 automorphism group): 240 are twice the norm 2 lattice points while 17280 are 3 times the shallow holes surrounding the origin.

A hole in a lattice is a point in the ambient Euclidean space whose distance to the nearest lattice point is a local maximum. (In a lattice defined as a uniform honeycomb these points correspond to the centers of the facets volumes.) A deep hole is one whose distance to the lattice is a global maximum. There are two types of holes in the E_8 lattice:

- *Deep holes* such as the point (1,0,0,0,0,0,0,0) are at a distance of 1 from the nearest lattice points. There are 16 lattice points at this distance which form the vertices of an 8-orthoplex centered at the hole (the Delaunay cell of the hole).

- *Shallow holes* such as the point are at a distance of from the nearest lattice points. There are 9 lattice points at this distance forming the vertices of an 8-simplex centered at the hole.

Sphere packings and kissing numbers[edit]

The E_8 lattice is remarkable in that it gives optimal solutions to the sphere packing problem and the kissing number problem in 8 dimensions.

The sphere packing problem asks what is the densest way to pack (solid) n-dimensional spheres of a fixed radius in $\mathbf{R}^n$ so that no two spheres overlap. Lattice packings are special types of sphere packings where the spheres are centered at the points of a lattice. Placing spheres of radius $1/\sqrt{2}$ at the points of the E_8 lattice gives a lattice packing in $\mathbf{R}^8$ with a density of

It has long been known that this is the maximum density that can be achieved by a lattice packing in 8 dimensions.[6] Furthermore, the E_8 lattice is the unique lattice (up to isometries and rescalings) with this density.[7] Mathematician Maryna Viazovska has recently shown that this density is, in fact, optimal even among irregular packings.[8][9]

The kissing number problem asks what is the maximum number of spheres of a fixed radius that can touch (or "kiss") a central sphere of the same radius. In the E_8 lattice packing mentioned above any given sphere touches 240 neighboring spheres. This is because there are

240 lattice vectors of minimum nonzero norm (the roots of the E_8 lattice). It was shown in 1979 that this is the maximum possible number in 8 dimensions.[10][11]

The sphere packing problem and the kissing number problem are remarkably difficult and optimal solutions are only known in 1, 2, 3, 8, and 24 dimensions (plus dimension 4 for the kissing number problem). The fact that solutions are known in dimensions 8 and 24 follows in part from the special properties of the E_8 lattice and its 24-dimensional cousin, the Leech lattice.

Other constructions[edit]

Hamming code[edit]

The E_8 lattice is very closely related to the (extended) Hamming code $H(8,4)$ and can, in fact, be constructed from it. The Hamming code $H(8,4)$ is a binary code of length 8 and rank 4; that is, it is a 4-dimensional subspace of the finite vector space $(\mathbf{F}_2)^8$. Writing elements of $(\mathbf{F}_2)^8$ as 8-bit integers in hexadecimal, the code $H(8,4)$ can by given explicitly as the set

$$\{00, 0F, 33, 3C, 55, 5A, 66, 69, 96, 99, A5, AA, C3, CC, F0, FF\}.$$

The code $H(8,4)$ is significant partly because it is a Type II self-dual code. It has a minimum Hamming weight 4, meaning that any two codewords differ by at least 4 bits. It is the largest length 8 binary code with this property.

One can construct a lattice Λ from a binary code C of length n by taking the set of all vectors x in $\mathbf{Z}^n$ such that x is congruent (modulo 2) to a codeword of C.[12] It is often convenient to rescale Λ by a factor of 1/√2,

Applying this construction a Type II self-dual code gives an even, unimodular lattice. In particular, applying it to the Hamming code $H(8,4)$ gives an E_8 lattice. It is not entirely trivial, however, to find an explicit isomorphism between this lattice and the lattice Γ_8 defined above.

Applications[edit]

In 1982 Michael Freedman produced a bizarre example of a topological 4-manifold, called the E_8 manifold, whose intersection form is given by the E_8 lattice. This manifold is an example of a topological manifold which admits no smooth structure and is not even triangulable.

In string theory, the heterotic string is a peculiar hybrid of a 26-dimensional bosonic string and a 10-dimensional superstring. In order for the theory to work correctly, the 16 mismatched dimensions must be compactified on an even, unimodular lattice of rank 16. There are two such lattices: $\Gamma_8 \oplus \Gamma_8$ and Γ_{16} (constructed in a fashion analogous to that of Γ_8). These lead to two version of the heterotic string known as the $E_8 \times E_8$ heterotic string and the SO(32) heterotic string.

4.6. The Leech lattice.

The *Leech lattice* is an even unimodular lattice Λ_{24} in 24-dimensional Euclidean space, which is one of the best models for the kissing number problem. It was discovered by John Leech (1967). It may also have been discovered (but unpublished) by Ernst Witt in 1940.

Characterization[edit]

The Leech lattice Λ_{24} is the unique lattice in $\mathbf{E}^{24}$ with the following list of properties:

- It is unimodular; i.e., it can be generated by the columns of a certain 24×24 matrix with determinant 1.
- It is even; i.e., the square of the length of each vector in Λ_{24} is an even integer.
- The length of every non-zero vector in Λ_{24} is at least 2.

The last condition is equivalent to the condition that unit balls centered at the points of Λ_{24} do not overlap. Each is tangent to 196,560 neighbors, and this is known to be the largest number of non-overlapping 24-dimensional unit balls that can simultaneously touch a single unit ball (compare with 6 in dimension 2, as the maximum number of pennies which can touch a central penny; see kissing number). This arrangement of 196560 unit balls centred about another unit ball is so efficient that there is no room to move any of the balls; this configuration, together with its mirror-image, is the *only* 24-dimensional arrangement where 196560 unit balls simultaneously touch another. This property is also true in 1, 2 and 8 dimensions, with 2, 6 and 240 unit balls, respectively, based on the integer lattice, hexagonal tiling and E8 lattice, respectively.

It has no root system and in fact is the first unimodular lattice with no *roots* (vectors of norm less than 4), and therefore has a centre density of 1. By multiplying this value by the volume

of a unit ball in 24 dimensions, , one can derive its absolute density.

Conway (1983) showed that the Leech lattice is isometric to the set of simple roots (or the Dynkin diagram) of the reflection group of the 26-dimensional even Lorentzian unimodular lattice $II_{25,1}$. By comparison, the Dynkin diagrams of $II_{9,1}$ and $II_{17,1}$ are finite.

Applications[edit]

The binary Golay code, independently developed in 1949, is an application in coding theory. More specifically, it is an error-correcting code capable of correcting up to three errors in each 24-bit word, and detecting a fourth. It was used to communicate with the Voyager probes, as it is much more compact than the previously-used Hadamard code.

Quantizers, or analog-to-digital converters, can use lattices to minimise the average root-mean-square error. Most quantizers are based on the one-dimensional integer lattice, but using multi-dimensional lattices reduces the RMS error. The Leech lattice is a good solution to this problem, as the Voronoi cells have a low second moment.

The vertex algebra of the two-dimensional conformal field theory describing bosonic string theory, compactified on the 24-dimensional quotient torus $\mathbf{R}^{24}/\Lambda_{24}$ and orbifolded by a two-element reflection group, provides an explicit construction of the Griess algebra that has the monster group as its automorphism group. This **monster vertex algebra** was also used to prove the monstrous moonshine conjectures.

Constructions[edit]

The Leech lattice can be constructed in a variety of ways. As with all lattices, it can be constructed by taking the integral span of the columns of its generator matrix, a 24×24 matrix with determinant 1.

Leech generator matrix [show]

[1]

Using the binary Golay code[edit]

The Leech lattice can be explicitly constructed as the set of vectors of the form $2^{-3/2}(a_1, a_2, ..., a_{24})$ where the a_i are integers such that

and for each fixed residue class modulo 4, the 24 bit word, whose 1s correspond to the coordinates i such that a_i belongs to this residue class, is a word in the binary Golay code. The Golay code, together with the related Witt design, features in a construction for the 196560 minimal vectors in the Leech lattice.

Using the Lorentzian lattice $II_{25,1}$[edit]

The Leech lattice can also be constructed as where w is the Weyl vector:

in the 26-dimensional even Lorentzian unimodular lattice $II_{25,1}$. The existence of such an integral vector of norm zero relies on the fact that $1^2 + 2^2 + ... + 24^2$ is a perfect square (in fact 70^2); the number 24 is the only integer bigger than 1 with this property. This was conjectured by Édouard Lucas, but the proof came much later, based on elliptic functions.

The vector in this construction is really the Weyl vector of the even sublattice D_{24} of the odd unimodular lattice I^{25}. More generally, if L is any positive definite unimodular lattice of dimension 25 with at least 4 vectors of norm 1, then the Weyl vector of its norm 2 roots has integral length, and there is a similar construction of the Leech lattice using L and this Weyl vector.

Based on other lattices[edit]

Conway & Sloane (1982) described another 23 constructions for the Leech lattice, each based on a Niemeier lattice. It can also be constructed by using three copies of the E8 lattice, in the same way that the binary Golay code can be constructed using three copies of the extended Hamming code, H_8. This construction is known as the *Turyn* construction of the Leech lattice.

As a laminated lattice[edit]

Starting with a single point, Λ_0, one can stack copies of the lattice Λ_n to form an $(n + 1)$-dimensional lattice, Λ_{n+1}, without reducing the minimal distance between points. Λ_1 corresponds to the integer lattice, Λ_2 is to the hexagonal lattice, and Λ_3 is the face-centered cubic packing. Conway & Sloane (1982b) showed that the Leech lattice is the unique laminated lattice in 24 dimensions.

As a complex lattice[edit]

The Leech lattice is also a 12-dimensional lattice over the Eisenstein integers. This is known as the *complex Leech lattice*, and is isomorphic to the 24-dimensional real Leech lattice. In the complex construction of the Leech lattice, the binary Golay code is replaced with the ternary Golay code, and the Mathieu group M_{24} is replaced with the Mathieu group M_{12}. The E_6 lattice, E_8 lattice and Coxeter–Todd lattice also have constructions as complex lattices, over either the Eisenstein or Gaussian integers.

Using the icosian ring[edit]

The Leech lattice can also be constructed using the ring of icosians. The icosian ring is abstractly isomorphic to the E8 lattice, three copies of which can be used to construct the Leech lattice using the Turyn construction.

Witt's construction[edit]

In 1972 Witt gave the following construction, which he said he found in 1940 January 28.

Suppose that H is an n by n Hadamard matrix, where n=4ab. Then the matrix defines a bilinear form in 2n dimensions, whose kernel has n dimensions. The quotient by this kernel is a nonsinguar bilinear form taking values in (1/2)**Z**. It has 3 sublattices of index 2 that are integral bilinear forms. Witt obtained the Leech lattice as one of these three sublattices by taking a=2, b=3, and taking H to be the 24 by 24 matrix (indexed by $\mathbf{Z}/23\mathbf{Z} \cup \infty$) with entries X($m$+$n$) where X(∞)=1, X(0)=−1, X(n)=is the quadratic residue symbol mod 23 for nonzero n. This matrix H is a Paley matrix with some insignificant sign changes.

Using a Paley matrix[edit]

Chapman (2001) described a construction using a skew Hadamard matrix of Paley type. The

Niemeier lattice with root system can be made into a module for the ring of integers of the field . Multiplying this Niemeier lattice by a non-principal ideal of the ring of integers gives the Leech lattice.

Using octonions[edit]

If *L* is the set of octonions with coordinates on the lattice. Then the Leech lattice is the set of triplets such that:

where

Symmetries[edit]

The Leech lattice is highly symmetrical. Its automorphism group is the Conway group Co_0, which is of order 8 315 553 613 086 720 000. The center of Co_0 has two elements, and the quotient of Co_0 by this center is the Conway group Co_1, a finite simple group. Many other sporadic groups, such as the remaining Conway groups and Mathieu groups, can be constructed as the stabilizers of various configurations of vectors in the Leech lattice.

Despite having such a high *rotational* symmetry group, the Leech lattice does not possess any hyperplanes of reflection symmetry. In other words, the Leech lattice is chiral.

The automorphism group was first described by John Conway. The 398034000 vectors of norm 8 fall into 8292375 'crosses' of 48 vectors. Each cross contains 24 mutually orthogonal vectors and their negatives, and thus describe the vertices of a 24-dimensional orthoplex. Each of these crosses can be taken to be the coordinate system of the lattice, and has the same symmetry of the Golay code, namely $2^{12} \times |M_{24}|$. Hence the full automorphism group of the Leech lattice has order 8292375 × 4096 × 244823040, or 8 315 553 613 086 720 000.

Geometry[edit]

Conway, Parker & Sloane (1982) showed that the covering radius of the Leech lattice is ; in other words, if we put a closed ball of this radius around each lattice point, then these just cover Euclidean space. The points at distance at least from all lattice points are called the ***deep holes*** of the Leech lattice. There are 23 orbits of them under the automorphism group of the Leech lattice, and these orbits correspond to the 23 Niemeier lattices other than the Leech lattice: the set of vertices of deep hole is isometric to the affine Dynkin diagram of the corresponding Niemeier lattice.

The Leech lattice has a density of . Cohn & Kumar (2009) showed that it gives the densest lattice packing of balls in 24-dimensional space. Henry Cohn, Abhinav Kumar, and Stephen D. Miller et al. (2016) improved this by showing that it is the densest sphere packing, even among non-lattice packings.

The 196560 minimal vectors are of three different varieties, known as *shapes*:

- 1104 vectors of shape $(4^2,0^{22})$, for all permutations and sign choices;
- 97152 vectors of shape $(2^8,0^{16})$, where the '2's correspond to octads in the Golay code, and there is an even number of minus signs;
- 98304 vectors of shape $(-3,1^{23})$, where the changes of signs come from the Golay code, and the '3' can appear in any position.

The ternary Golay code, binary Golay code and Leech lattice give very efficient 24-dimensional spherical codes of 729, 4096 and 196560 points, respectively. Spherical codes are higher-dimensional analogues of Tammes problem, which arose as an attempt to explain the distribution of pores on pollen grains. These are distributed as to maximise the minimal angle between them. In two dimensions, the problem is trivial, but in three dimensions and higher it is not. An example of a spherical code in three dimensions is the set of the 12

vertices of the regular icosahedron.

History[edit]

Many of the cross-sections of the Leech lattice, including the Coxeter–Todd lattice and Barnes–Wall lattice, in 12 and 16 dimensions, were found much earlier than the Leech lattice. O'Connor & Pall (1944) discovered a related odd unimodular lattice in 24 dimensions, now called the odd Leech lattice, one of whose two even neighbors is the Leech lattice. The Leech lattice was discovered in 1965 by John Leech (1967, 2.31, p. 262), by improving some earlier sphere packings he found (Leech 1964).

Conway (1968) calculated the order of the automorphism group of the Leech lattice, and, working with John G. Thompson, discovered three new sporadic groups as a by-product: the Conway groups, Co_1, Co_2, Co_3. They also showed that four other (then) recently announced sporadic groups, namely, Higman-Sims, Suzuki, McLaughlin, and the Janko group J_2 could be found inside the Conway groups using the geometry of the Leech lattice. (Ronan, p. 155)

> Bei dem Versuch, eine Form aus einer solchen Klasse wirklich anzugeben, fand ich mehr als 10 verschiedene Klassen in Γ_{24}
>
> *Witt (1941, p. 324)*

Witt (1941, p. 324), has a single rather cryptic sentence mentioning that he found more than 10 even unimodular lattices in 24 dimensions without giving further details. Witt (1998, p. 328–329) stated that he found 9 of these lattices earlier in 1938, and found two more, the

Niemeier lattice with A_1^{24} root system and the Leech lattice (and also the odd Leech lattice), in 1940.

4.7. Novanion algebras.

We first give the axioms for a field, and successively weaken these axioms to give them for division algebras and novanion algebras.

The axioms for a *field* $\mathbb{F}$, +, ×, which we will denote simply by $\mathbb{F}$, satisfy for a, b, c ∈ $\mathbb{F}$, with a × b being written as ab

additive closure: $a + b \in \mathbb{F}$ (1)

associativity: $a + (b + c) = (a + b) + c$ (2)

abelian addition: $a + b = b + a$ (3)

existence of a zero: there exists a $0 \in \mathbb{F}$ satisfying

$$a + 0 = a \quad (4)$$

existence of negative elements: there exists a $(-a) \in \mathbb{F}$ with

$$a + (-a) = 0, \quad (5)$$

which we write introducing subtraction as

$$a - a = 0$$

multiplicative closure: $ab \in \mathbb{F}$ (6)

associativity: $a(bc) = (ab)c$ (7)

commutativity: $ab = ba$ (8)

existence of a 1: there exists a $1 \in \mathbb{F}$ satisfying

$$a1 = a \quad (9)$$

existence of inverse elements: there exists an $a^{-1} \in \mathbb{F}$ for $a \neq 0$ with

$$a(a^{-1}) = 1, \quad (10)$$

which we write introducing division as

$$a/a = 1$$

distributive law: $a(b + c) = (ab) + (ac)$. (11)

An associative division algebra satisfies all axioms for a field except the multiplicative commutativity rule (8). A nonassociative division algebra, D, also drops the multiplicative associative rule (7). It may introduce the following axiom:

There do not exist $a, b \neq 0 \in D$ with

$$ab = 0. \quad (12)$$

This rule is unnecessary for associative division algebras, since if $1 \neq 0$ then (12) implies

$$ab \neq c, c \neq 0,$$

and therefore there exists no

$$a(b/c) = 1,$$

but b/c can be chosen to be the multiplicative inverse of a, a contradiction. ☐

A novanion algebra B drops axiom (12). It may substitute the following rule:

If $a^2, b^2 > 0$ for $a, b \in B$ and a is in c, b is in $d \in B$, there do not exist any c, d with

$$cd = 0. \quad (13)$$

We can provide existence proofs for these structures. The complex numbers constitute a field. The quaternions form an associative division algebra. The octonions form a nonassociative division algebra. As we will prove in section 4.11, the 10-novanions, for example, form a novanion algebra. ☐

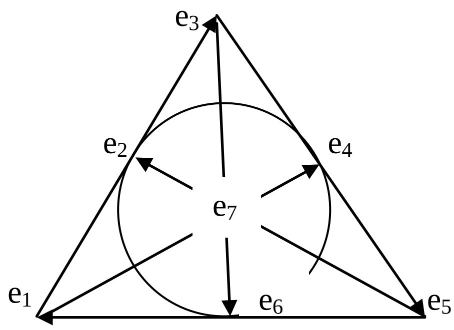
The consistency of the complex numbers is contingent on the consistency of analysis, given as a Gentzen-type proof in chapter III volume III. The consistency of the quaternions as an associative division algebra is given in reference to [Ad15], and we will prove in 4.11 the consistency of the 10-novanions as a novanion algebra. The consistency of the octonions now follows from their nature as novanion algebras of a specific type. ☐

4.8. The classification of novanions.

J. F. Adams proved in 1960 that the only possible division algebras are at maximum those of the octonions, which are nonassociative, where quaternions and complex numbers are subalgebras of these. However, there exist nonassociative algebras, the n-novanions, with nonzero division, but when just the scalar part is zero two nonzero novanions may have a zero product. I obtain here the proof for n = 10, and extend it for n greater than 10. Novanions have a dimension of $n = 1 + 3^f \prod_{i \in \mathbb{N}} (3^{g_i} - 2)_i$, for which $f \in \mathbb{N}_{\cup 0}$, $g_i \in \mathbb{N}$. Thus the n-novanions comprise a new type of quasi division algebra. We then introduce novanion rings. These do not have division, and thus can represent discrete structures.

4.9. The nonassociative octonion division algebra.

The algebra of the octonions, also called the Cayley numbers, was discovered by J. Graves in 1843, and is given by the Fano plane for their basis elements 1, e_1, e_2, e_3, e_4, e_5, e_6 and e_7 in the figure below.



Typical cyclic identities are

$$e_7e_5 = e_2 \tag{1}$$
$$e_5e_6 = e_1$$

and

$$e_4e_2 = e_6.$$

Note what we have said here. The inner triple $e_2e_4e_6$ acts like a quaternion, but the outer triple $e_1e_3e_5$ does not. Nevertheless, we will need to allocate later a list ordered as right triple $(e_2e_4e_6)$ + central triple $(e_1e_3e_5)$ + one (e_7). Octonions form a division algebra, in particular

$$e_c^2 = -1, \tag{2}$$
$$e_ae_b = -e_be_a \ (a \neq b),$$

and the inverse of

$$a1 + \sum_{n=1}^{7} b_n e_n$$

is

$$(a1 - \sum_{n=1}^{7} b_n e_n)/(a^2 + \sum_{n=1}^{7} b_n^2). \tag{3}$$

The octonions, $\mathbb{O}$, are also generated by the Cayley-Dickson construction [Ba01]. This builds up algebras from the complex numbers, to the quaternions, to the octonions, to the sixteen dimensional sedenions, etc.

Define a T-algebra to be an algebra equipped with conjugation, a linear map T satisfying

$a^{TT} = a,$ (4)

$(ab)^T = b^T a^T.$ (5)

Starting from any T-algebra, the Cayley-Dickson construction gives a new algebra

$(a, b)(c, d) = (ac – db^T, ad^T + cb),$ (6)

with conjugation defined by

$(a, b)^T = (a^T, -b).$ (7)

Let us look at how this works in practice. The complex numbers have multiplication

$(a, b)(c, d) = (ac – bd, ad + bc).$

The complex conjugate

$(a, b)^T = (a^T, –b) = (a, –b),$

since the scalar part a is a real number with $a^T = a$.

The complex conjugate satisfies

$(a, b)^T(a, b) = (aa + bb, ab – ba) = (a^2 + b^2, 0).$

We define $a^2 + b^2$ as the *norm* of the complex number. It is a non-negative real number.

For the noncommutative quaternions equations (6) and (7) hold. The order of the terms in (6) is important. We now have

$(a, b)^T(a, b) = (a^T, – b)(a, b) = (a^Ta + b^Tb, ba^T – ba^T)$
$= (a^Ta + b^Tb, 0),$

where $a^Ta + b^Tb$ is the norm.

The T-algebra generates the basis element multiplication table $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$ of the octonions

×	***1***	***e₁***	***e₂***	***e₃***	***e₄***	***e₅***	***e₆***	***e₇***
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	–1	e_3	$–e_2$	e_5	$–e_4$	$–e_7$	e_6
e_2	e_2	$–e_3$	–1	e_1	e_6	e_7	$–e_4$	$–e_5$
e_3	e_3	e_2	$–e_1$	–1	e_7	$–e_6$	e_5	$–e_4$
e_4	e_4	$–e_5$	$–e_6$	$–e_7$	–1	e_1	e_2	e_3
e_5	e_5	e_4	$–e_7$	e_6	$–e_1$	–1	$–e_3$	e_2
e_6	e_6	e_7	e_4	$–e_5$	$–e_2$	e_3	–1	$–e_1$
e_7	e_7	$–e_6$	e_5	e_4	$–e_3$	$–e_2$	e_1	–1

We can generate for each

$\mathbb{O} \times \mathbb{O} \to \mathbb{O}: e_i \times e_j \to e_k$

a Cayley-Dickson construction of

$e_i \times (-e_j) \to -e_k,$

so that each of the 7 non-scalar basis elements in a row of the table can be multiplied by -1 to provide $2^7 = 128$ possible Cayley-Dickson constructions.

The Fano plane has 7 non-scalar basis elements. The number of non-scalar quaternionic triplets is 7, each of which, since exquaternions are excluded, operates under a forward or a reversed orientation – again 2^7 possibilities. The following Fano triplets map bijectively to the standard Cayley-Dickson construction for the octonions

$(e_1, e_2, e_3), (e_3, e_4, e_5), (e_1, e_4, e_6), (e_4, e_6, e_2), (e_1, e_7, e_6), (e_4, e_7, e_3), (e_5, e_7, e_2).$

Thus distinct possibilities for the Fano plane match distinct instances of the Cayley-Dickson construction. □

4.10. Eigenvalues.

Our intention now is to introduce a generalisation of quaternions and octonions called n-novanions, which will be involved in the mathematics we wish to describe. In order to prove the consistency of the n-novanion algebra, we will need to introduce some further ideas on matrices. Like the octonions, the n-novanions cannot be described directly by matrices, since matrices are multiplicatively associative, meaning (AB)C = A(BC), but novanions are not. Nevertheless we can characterise some of the properties of n-novanions by a set of linear equations. These can be described by a matrix equation. An important technique in analysing the matrix equations is to map the equation to a scalar equation involving complex numbers. These scalars are called eigenvalues, and satisfy a matrix equation like (1) below.

Within the field of complex numbers $\mathbb{C}$, the complex conjugate of c = a + ib is c* = a – ib. For the corresponding matrix C with entries $c_{jk} = a_{jk} + ib_{jk}$, the conjugate $C^* = a_{jk} - ib_{jk}$. The transpose of a matrix C is denoted by C^T and has entries c_{kj}. The transpose is a contravariant (order reversing) operation:

$$(CD)^T = D^T C^T.$$

A matrix is defined as antisymmetric if $C^T = -C$.

The following proof is derived from chapter 11 of [Uh01].

Theorem 4.10.1. *All eigenvalues of a real antisymmetric matrix* $E = -E^T$ *are pure imaginary*.

Proof. Consider the case of the eigenvalue λ and possibly complex column vector called an eigenvector $\mathbf{x} \neq \mathbf{0}$. According to the Fundamental Theorem of Algebra $\lambda \in \mathbb{C}$. Hence

$$E\mathbf{x} = \lambda\mathbf{x}. \qquad (1)$$

If we take the complex conjugate of both sides of the eigenvalue-eigenvector equation (1), we obtain

$$(E\mathbf{x})^* = (\lambda\mathbf{x})^* = \lambda^*\mathbf{x}^*.$$

Transposing yields

$$(E\mathbf{x})^{*T} = x^{*T}E^{*T} = \lambda^*\mathbf{x}^{*T}.$$

Define the norm as

$$||E\mathbf{x}||^2 = (\lambda^*)\lambda(\mathbf{x}^{*T})\mathbf{x}, \qquad (2)$$

where for the set of real numbers $\mathbb{R}$

$$||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}} \in \mathbb{R}.$$

Since $E^T = -E$ and $E^{*T} = E^T$ for a real antisymmetric matrix E, we can write (2) as

$$\begin{aligned} ||E\mathbf{x}||^2 &= \mathbf{x}^{*T}E^{*T}\,E^T\mathbf{x}, \\ &= \mathbf{x}^TE^TE\mathbf{x} \\ &= -\mathbf{x}^TE^2\mathbf{x} \\ &= -\mathbf{x}^T\lambda^2\mathbf{x}, \end{aligned}$$

because $E^2\mathbf{x} = E(E\mathbf{x}) = E(\lambda\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, so

$$||E\mathbf{x}||^2 = -\lambda^2\mathbf{x}^T\mathbf{x}. \qquad (3)$$

Now $\mathbf{x}^T\mathbf{x} \neq 0$, and thus $\lambda^*\lambda = -\lambda^2$, by comparing (2) and (3). Thus a real antisymmetric matrix $E = -E^T$ can only have imaginary eigenvalues λ. ☐

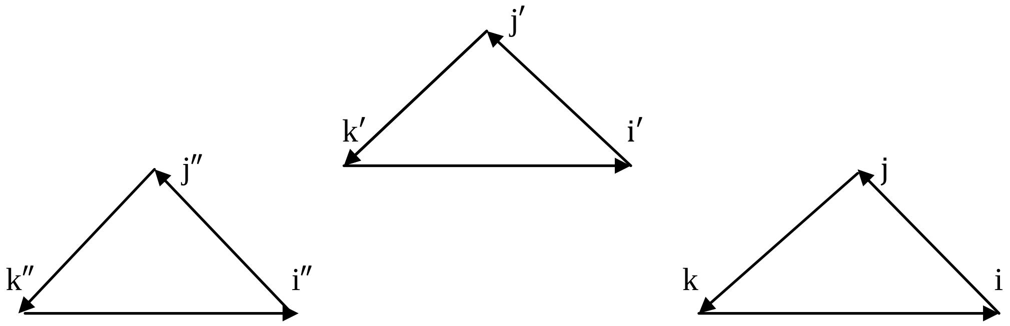
4.11. The 10-novanions.

We have seen a novanion algebra B drops rule 4.7.(12). It may substitute the following rule:

If a^2, $b^2 > 0$ for a, b ∊ B and a is in c, b is in d ∊ B, there do not exist any c, d with

$$cd = 0. \quad (1)$$

We now introduce the 10-novanions, represented by the set of triangle diagrams



where in general each triangle is a quaternion without 1.

The primed variables ( ), (′) and (″) act as holders of information concerning an algebra for them. When the variables all contain a common instance, for example (k), (k′) and (k″), then the algebra is that of the quaternions, in which we have a cyclic algebra

$$kk' = k'' = - k'k. \quad (2)$$

When the variables contain different instances, such as k and i′, then the product contains the primed variable that does not belong to the first two elements, but the primed part commutes.

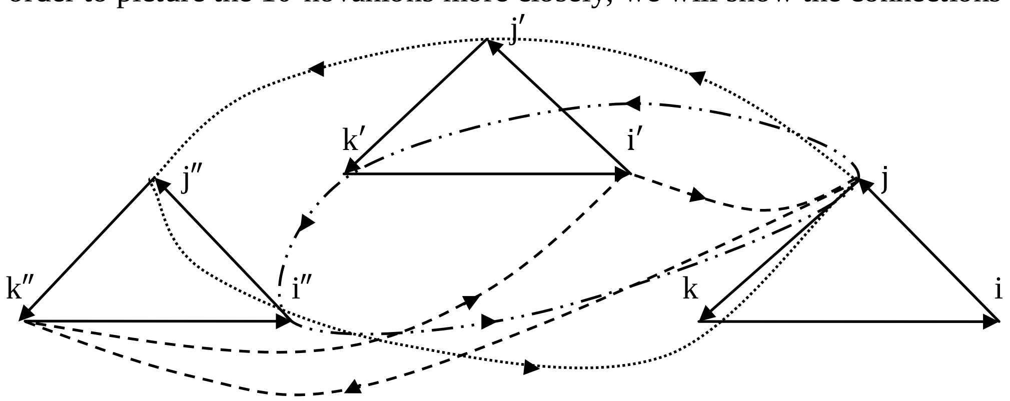
On top of this is the fact that the variables, say k and i, satisfy a quaternion algebra, so say

$$ki = j = -ik \quad (3)$$

and consequently

$$ki' = j'' = -i'k. \quad (4)$$

In order to picture the 10-novanions more closely, we will show the connections from node j



Our claim is that the inverse of

$$a1 + \sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} b_n^m e_n$$

is

$$(a1 - (\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} b_n^m e_n))/(a^2 + (\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} (b_n^m)^2)), \quad (5)$$

and this constitutes a type of division algebra with no divisors of zero provided a1 ≠ 0 – the 10 dimensional 10-novanions.

We see that the n-novanions are nonassociative, since they have more than 4 basis elements; more explicitly

$$(j''k')j' = ij' = k'' \neq j''(k'j') = -j''i' = k.$$

We wish to enquire under what conditions there exist two 10-novanion numbers multiplied together giving zero:

$$(a1 + bi + cj + dk + b'i' + c'j' + d'k' + b''i'' + c''j'' + d''k'') \times (p1 + qi + rj + tk + q'i' + r'j' + t'k' + q''i'' + r''j'' + t''k'') = 0. \qquad (6)$$

Their product is

real part:

$$ap - bq - cr - dt - b'q' - c'r' - d't' - b''q'' - c''r'' - d''t'' = 0, \qquad (7)$$

i part:

$$bp + aq - dr + ct - b''q' - d''r' + c''t' + b'q'' - d'r'' + c't'' = 0, \qquad (8)$$

j part:

$$cp + dq + ar - bt + d''q' - c''r' - b''t' + d'q'' + c'r'' - b't'' = 0, \qquad (9)$$

k part:

$$dp - cq + br + at - c''q' + b''r' - d''t' - c'q'' + b'r'' + d't'' = 0, \qquad (10)$$

i′ part:

$$b'p + b''q - d''r + c''t + aq' - d'r' + c't' - bq'' - dr'' + ct'' = 0, \qquad (11)$$

j′ part:

$$c'p + d''q + c''r - b''t + d'q' + ar' - b't' + dq'' - cr'' - bt'' = 0, \qquad (12)$$

k′ part:

$$d'p - c''q + b''r + d''t - c'q' + b'r' + at' - cq'' + br'' - dt'' = 0, \qquad (13)$$

i″ part:

$$b''p - b'q - d'r + c't + bq' - dr' + ct' + aq'' - d''r'' + c''t'' = 0, \qquad (14)$$

j″ part:

$$c''p + d'q - c'r - b't + dq' + cr' - bt' + d''q'' + ar'' - b''t'' = 0, \qquad (15)$$

k″ part:

$$d''p - c'q + b'r - d't - cq' + br' + dt' - c''q'' + b''r'' + at'' = 0. \qquad (16)$$

Alternative definition 4.11.1. D is a (possibly nonassociative) division algebra whenever for any element a in D and any nonzero element b in D there exists just one element x in D with a = bx and only one element y in D with a = yb.

If a = 0, the 10-novanions contain possibilities for two nonzero 10-novanions giving a product which is zero. We give an example due to Doly García, showing that the 10-novanions satisfying a = 0 do not form a division algebra of standard type

$$(i + i' + i'')(j + j' - 2j'') = 0. \qquad (17)$$

Thus the 10-novanions are not a division algebra given by the condition for equation 8.7.(12), nor do they satisfy definition 8.11.1 since for an arbitrary real number g

$$(i + i' + i'')[(j + j' - 2j'')g] = 0. \qquad (18)$$

From now on we will assume $a \neq 0$. By a symmetrical argument applied also to the following reasoning, we need to assume with this that $p \neq 0$.

Equations (7) to (16) form a matrix E + aI, where E is an antisymmetric matrix and I is the unit diagonal, multiplied on the right by the eigenvector (p, q, r, t, q′, r′, t′, q″, r″, t″). We have already given a proof that the eigenvalues of a real antisymmetric matrix are entirely imaginary, so these correspond to -a, which is real, whereas we are now excluding the only possibility for this, a = 0. So 10-novanions form a novanion algebra satisfying the conditions of equation (1). □

4.12. n-novanions.

The octonions given by a Fano plane under suitable orientations may be given three copies in primed variables ( ) , (′) and (″), and by an analogous procedure this constitutes a 1 + (3 × 7) = 22 dimensional division algebra.

Extending these ideas further to multiple occurrences of the three or seven primed variables, say (′), (′)′ and (′)″, we obtain in general an $n = 1 + 3^f 7^g$ dimensional novanion algebra, the n-novanions, in which if a common variable, k, is employed, then the lowest value within brackets of say (k), (k′)′ and (k″)″ is evaluated. ☐

4.13. Discreteness and novanion rings.

We relate our discussion on quaternions to Eli Cartan's *The theory of spinors* [Ca66] where he claims a bijection between spinors and the relativistic (Dirac) equation of the electron which maps to quaternions. We will find these spinors do not map to quaternions and we cannot adjoin spinors to quaternions in a matrix formalism, because the dimension of this combination is greater than 4, so by a result [Ad15], the algebra cannot be a matrix one, and is nonassociative.

Cartan introduces spinors via the observable coefficients of vectors with zero norm

$$x_1^2 + x_2^2 + x_3^2 = 0 \tag{1}$$

satisfied by two numbers h_0, h_1 given by

$$x_1 = h_0^2 - h_1^2$$
$$x_2 = i(h_0^2 + h_1^2)$$
$$x_3 = -2h_0h_1,$$

with possible solutions

$$h_0 = \pm\sqrt{\frac{x_1 - ix_2}{2}}, \tag{2}$$

$$h_1 = \pm\sqrt{\frac{-x_1 - ix_2}{2}}. \tag{3}$$

Now choose a vector dependent only on the coordinates x_1 and x_2, say $x_1 - ix_2$. If this is rotated about the axis x_3 to a vector $e^{i\theta}(x_1 - ix_2)$, then h_0 is transformed to $e^{i\theta/2}h_0$, and likewise h_1 is transformed to $e^{i\theta/2}h_1$. Then when $\theta = 2\pi$, a complete rotation, x_3 is transformed to $-x_3$. Thus the spinors h_0 and h_1 are *locally fermionic*.

He considers in the notation of [Ad15] the matrix basis

$$\phi_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad -i_i = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so that

$$\phi_1^2 = (-i_i)^2 = \alpha_1^2 = 1_1,\ \phi_1(-i_i) = -(-i_i\phi_1),\ -i_i\alpha_1 = -\alpha_1(-i_i),\ \alpha_1\phi_1 = -\phi_1\alpha_1,$$

which he states are related to the quaternions on multiplication by -1_i

$$I_1 = -1_i\phi_1,\ I_2 = -1_i(-i_i),\ I_3 = -1_i\alpha_1. \tag{4}$$

This is closely related to the basis already given by us in [Ad15], 1_1, α_i, i_1 and ϕ_i, which matches (4) except for minus signs whose square is 1 (it is only possible to change the basis left- or right-handedness by this means). From it Cartan is able to deduce the quaternion algebra

$$I_1^2 = I_2^2 = I_3^2 = -1,\ I_1I_2 = -I_2I_1,\ I_2I_3 = -I_3I_2,\ I_3I_1 = -I_1I_3, \tag{5}$$

which he relates to the Dirac equation. ☐

However we must note that the transformations derived in (2) and (3) involve complex numbers, which are commutative, whereas equation (5) involves quaternions which are noncommutative. Thus there is no equation as given by Cartan of this type. ☐

In this system we can choose to embed this local quaternion structure in a global manifold which is oriented, that is say, an ix vector moving through 2π radians in a jy and kz circle returns to itself with the ix vector pointing in the same direction. When this happens, we say the observable derived from the quaternion is *globally bosonic*. We can also implement nonoriented global manifolds in which an ix vector moving through 2π radians in a jy and kz circle returns to itself with the ix vector pointing in the opposite direction. We then say the quaternion is *globally fermionic*.

Thus we are reduced to considering globally bosonic or globally fermionic quaternion structures. We will see that the globally bosonic and globally fermionic idea can be extended to the octonions and n-novanions. Our escape route from contradiction is that the extended quaternion object given by the n-novanions is not representable by a matrix, and a conclusion is that it is nonassociative. ☐

Since the n-novanions contain the quaternions or the octonions, and the octonions also contain quaternions, it follows that the arguments of [Ad15] chapter III on Wedderburn's little theorem apply also to such subalgebras, and therefore
Theorem 4.13.1. *Any finite novanion division ring is commutative.*

Thus novanion rings cannot in general be finite, although they may be discrete. Discrete rings without division can be bounded from below by finite elements ignoring the sign, for example by zero, and otherwise be infinite. If a ring is bounded from above and below by the mapping of, say, plus infinity to the integer novanion part 1 and minus infinity to -1, then an infinite example can be given over real components.

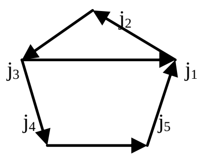
If novanion rings are given a lower and an upper bound in each of their n coordinates, x_0, x_1, ... , x_{n-1} as $\pm a_0$, $\pm a_1$, ... , $\pm a_{n-1}$, then the pair $(+a_i, + a_j)$ and the pair $(-a_i, -a_j)$ can be glued in two ways as a_i to $-a_i$ and a_j to $-a_j$, or as a_i to $-a_j$ and a_j to $-a_i$. The first defines a global bosonic structure and the second a global fermionic structure.

We will label bosonic structures by b and fermionic structures by f. If we look at just the novanionic imaginary components of the novanion, then for say a quaternion, there are configurations {b, b, b}, {f, b, b}, {f, f, b} and {f, f, f}. ☐

4.14. The search for other novanion algebras.

Are there other novanion algebras of a type not already covered? This question has been stimulated by a first-year Sussex University student's identification of novanions with strings in physics, [Ad18], where the allocation n = 10 is the same number as the dimensionality of the heterotic string, and for which we wish to investigate the bosonic allocation $1 + 5^2 = 26$.

In the pentagonal diagram shown next, an initial attempt depicts only one out of five subtriangles.



The pentagon can be enumerated cyclically, so that

$$j_1j_2 = j_3,\ j_2j_3 = j_4,\ j_3j_4 = j_5,\ j_4j_5 = j_1,\ j_5j_1 = j_2, \qquad (1)$$

and jumping a vertex we evaluate the closest triangle

$$j_3j_1 = j_2,\ j_4j_2 = j_3,\ j_5j_3= j_4,\ j_1j_4 = j_5,\ j_2j_5 = j_1, \qquad (2)$$

where on inverting the orientation, we get a minus sign.

This latter fact implies we have an inbuilt norm and inverse; the inverse of

$$a1 + \sum_{n=1}^{5} b_n j_n$$

is

$$a1 - \sum_{n=1}^{5} b_n j_n/(a^2 + \sum_{n=1}^{5} b_n^2), \qquad (3)$$

which is nonassociative, as is demonstrated by

$$(j_3j_1)j_4 = -j_3 \neq j_3(j_1j_4) = -j_4.$$

The question arises as to whether this constitutes a novanion algebra, which would now be extended from previous considerations to include the dimensions

$$n = 1 + 3^f5^g7^h.$$

The possibility of the existence of the division algebra violating equation

$$(a1 + bj_1 + cj_2 + dj_3 + ej_4 + fj_5) \times (p1 + qj_1 + rj_2 + tj_3 + uj_4 + vj_5) = 0 \qquad (4)$$

will now be investigated. Under the constraints (1) and (2) we obtain the set of equations

real part:

$$ap - bq - cr - dt - eu - fv = 0, \qquad (5)$$

j_1 part:

$$bp + aq - fr + 0 - fu + (c + e)v = 0, \qquad (6)$$

j_2 part:

$$cp + (f + d)q + ar - bt + 0 - bv = 0, \qquad (7)$$

j_3 part:

$$dp - cq + (b + e)r + at - cu + 0 = 0, \qquad (8)$$

j_4 part:

$$ep + 0 - dr + (c + f)t + au - dv = 0, \qquad (9)$$

j_5 part:

$$fp - eq + 0 - et + (d + b)u + av = 0, \qquad (10)$$

from which it follows that the E type matrix is not antisymmetric, but it may be represented as the sum of two matrices F and G, where F has all pure imaginary eigenvalues:

$$F = \begin{bmatrix} 0 & -b & -c & -d & -e & -f \\ b & 0 & -f & c & -f & e \\ c & f & 0 & -b & d & -b \\ d & -c & b & 0 & -c & e \\ e & f & -d & c & 0 & -d \\ f & -e & b & -e & d & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & c \\ 0 & d & 0 & 0 & -d & 0 \\ 0 & 0 & e & 0 & 0 & -e \\ 0 & -f & 0 & f & 0 & 0 \\ 0 & 0 & -b & 0 & b & 0 \end{bmatrix},$$

and we do not have pure imaginary eigenvalues for $F + G - \lambda I$.

Indeed a general matrix H may be represented as a sum of a symmetric part H_{sym} and an antisymmetric part H_{anti}. Then H_{sym} and H_{anti} are linearly independent over real coefficients, meaning there exist no real numbers c and d satisfying

$$cH_{sym} + dH_{anti} = 0.$$

By a demonstration analogous to that in section 4.10, and proved directly in [Uh01], the eigenvalues of a symmetric matrix are real. Further, a complex number with real coefficients h_1 and h_2 is linearly independent between h_1 and h_2i over real coefficients. It now follows that the eigenvalue equation

$$H = hI \tag{11}$$

which has a unique set of n solutions is satisfied by

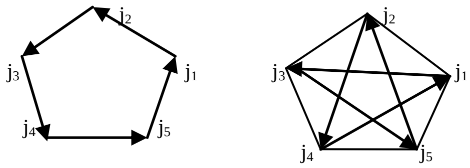
$$H_{sym} = h_1I$$

with n solutions and

$$H_{anti} = h_2I,$$

also with n solutions. These possible solutions are the only ones, since the solution set of (11) is unique. So if H_{sym} is not the zero matrix, a real h_1 exists. This implies there is no novanion algebra available.

We find a similar type of situation for the pentagonal diagrams



since the diagram for j_1, j_2, j_3 is unoriented and therefore does not constitute a quaternion, or in fact a division algebra, so we have failed on modifying equations (5) to (10) to come up with other solutions, where for pentagonal diagrams we can show these pentagons consist in the general case of combined oriented and unoriented quaternion diagrams. □

However, the identification relates to the number 1 + 25, and we now wish to probe the allocation 25 = 7 + 9 + 9, where 7 is the number of non-real basis elements of the octonions, and 9 is the number for the 10-novanions. There is an analogy here. The octonion non-real basis elements of 7 may be represented as 1 + 3 + 3, where 3 is the number of such basis elements for the quaternions, and 1 for the complex numbers. We are forced for a number of reasons to decompose such an allocation into triplets, basically to retain the cyclic algebra for the quaternions.

The allocation will be as follows, where we subscript 3 and 1 to distinguish them

3_a, 3_b, 3_c (i)
3_d, 3_e, 3_f (ii)
3_g, 3_h, 1_u (iii)

where allocations (i) and (ii) are internally similar to 10-novanions, and allocation (iii) is internally an octonion. We will explain why we use the word 'similar' later.

There are a number of possible configurations.

We want an algebra linking between (i), (ii) and (iii). Vertical allocations are present. We will choose next from straight lines going from left to right, for example the diagonal going upwards from 3_g, 3_e to 3_c. This is similar to a 10-novanion algebra. The descending line from 3_a, 3_e to 1_u is an octonion algebra. We then incorporate the algebra taking for example 3_d, 3_b to 1_u, an octonion algebra, or 3_g, 3_b to 3_f, this is similar to a 10-novanion algebra.

We have used the words 'similar to a 10-novanion algebra', and we now explain why. If we look at allocation (iii), this is part of the 3_g 3_h 1_u octonion, where 3_h and 1_u are linked.

Although 3_g is indeed a quaternion, we have already mentioned that 3_h is not. Therefore the vertical allocation given by 3_a 3_d 3_g is a 10-novanion, since it is made of genuine quaternions, but the vertical allocation 3_b 3_e 3_h is not. 3_b, 3_e and 3_h occur in octonion representations. If we were to state that the central triple 3_b, 3_e and 3_h algebras were quaternions, we would have an inconsistency. Therefore for these allocations as part of a 'similar to 10-novanion' structure, we decide that the octonion structure overrides the 10-novanion one. Since there is only one special 1_u part for the octonions, this part of the allocation is unique. The similar 10-novanion structure is now not a closed algebra within the 10-novanions; part of it belongs to the octonions. The corresponding situation just for 10-novanions with no octonionic overlap but with novanionic overlap is described by the octonionic allocation already discussed.

Since there is no other mixing of allocations, the result is as consistent as the 10-novanions and the octonions. This can be checked with equations like 4.11.(6) to (15), for which it is clear eigenvalues are pure imaginary. Finally a calculation like 4.11.(16) shows that this is a novanion algebra. ☐

The existence of 10-, 26- and 80-novanions (the latter obtained by an array cube of items like (i) to (iii) – all configurations lie in planes of the cube, and an m-cube gives rise to a $(3^m \pm 1)$-novanion) implies that results derived for division algebras have a different extension for n-novanions. ☐

4.15. Sedenions and 64-novanions.

The 16-dimensional sedenions are formed by the Cayley-Dickson construction [Ba01]. Since they are not alternative, they do not form a division algebra. That is, we do not have

$$x(xy) = (xx)y$$

and

$$(yx)x = y(xx)$$

for all x and y in the algebra, the proof using basis elements. Every associative algebra is alternative, but so too are some strictly non-associative algebras such as the octonions.

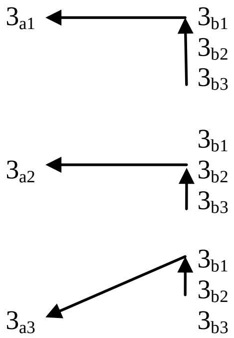
Whether there exist other 16-dimensional algebras not generated by the Cayley-Dickson construction was answered in the negative by J. F. Adams in 1960.

Irrespective of this result, the Cayley-Dickson construction generating a 64-dimensional algebra shows that this is not a division algebra, since in particular this contains the sedenions as a subalgebra. However, a 64-novanion has $63 = 3^2 \times 7$ non-scalar basis elements, and we will see that novanion algebras of this type are consistent. A 64-novanion is given by the cube with slices

$3_a, 3_b, 1_p$	$3'_a, 3'_b, 1'_p$	$3''_a, 3''_b, 1''_p$
$3_c, 3_d, 1_q$	$3'_c, 3'_d, 1'_q$	$3''_c, 3''_d, 1''_q$
$3_e, 3_f, 1_r$	$3'_e, 3'_f, 1'_r$	$3''_e, 3''_f, 1''_r$.

To evaluate a typical slice algebra, if we take the leftmost array above, we know that 3_b is not a quaternion, so we will build an override structure for the composition of two elements in 3_b. Within this slice 3_b belongs to three octonionic arrangements, those given by 3_a, 3_b, 1_p, or 3_c, 3_b, 1_r, or 3_e, 3_b, 1_q, so we need to select an override on the nonquaternion 3_b, so that when two elements are multiplied within it, just one allocation to an octonionic structure is selected.

We will need to look at this typical example in detail, so denote the three elements of 3_b by 3_{b1}, 3_{b2} and 3_{b3}. We will display the 3 elements of 3_b combining in pairs to form arrows with the following typical structures. We will choose at first, arbitrarily, a link to the 3_a, 3_b, 1_p octonionic structure. Of course, two arrows shown below combine to give an oriented quaternion triple, for which reversal of arrows leads to a minus value.



The central triples 3_d and 3_f have similar structures, mapping to separate values in 3_c and 3_e respectively. We have stated the 1_p, 1_q and 1_r elements combined with 3_b give on composition with one element of 3_b the octonion structures (3_a, 3_b, 1_p), (3_c, 3_b, 1_r) and (3_e, 3_b, 1_q). Because 3_{b1} links to 3_{a1}, we have to ensure that the link 1_p to 3_{b1} does not also link to 3_{a1}, but this can be arranged.

Alternative structures can be considered. For example, if 3_{b3}, 3_{b1} links to 3_{a1} as before, we could also have 3_{b3}, 3_{b2} linking to 3_{c1} and 3_{b2}, 3_{b1} linking to 3_{e1}. □

4.16. Further investigations.

The exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 are related to the existence of division algebras limited in number to those embedded within the octonions [Wi09], [CSM95].

We have seen in chapter IV of [Ad15] that for matrices A, B and C the Lie bracket

$$[AB] = AB - BA$$

satisfies the Jacobi identity

$$[[AB]C] + [[BC]A] + [[CA]B] = 0. \qquad (1)$$

For octonions, we do not have a matrix algebra, but we might wish to form Lie brackets from them satisfying (1). From section 8.7, for the octonions the only nonquaternion triple which is scanned from one set to another is $e_1e_3e_5$ and

$$[[e_1\, e_3]\, e_5] + [[e_3\, e_5]\, e_1] + [[e_5\, e_1]\, e_3] = 2[e_2\, e_5] + 2[e_4\, e_1] + 2[e_6\, e_3]$$
$$= -12e_7, \qquad (2)$$

so this does not satisfy (1). However, there exist other nonquaternion triples, for example $e_1e_7e_5$, and this satisfies

$$[[e_1\, e_7]\, e_5] + [[e_7\, e_5]\, e_1] + [[e_5\, e_1]\, e_7] = 2[e_4\, e_5] - 2[e_2\, e_1] + 2[e_6\, e_7]$$
$$= -12e_3. \qquad (3)$$

We need to find out whether a factor of 12 is present in all special calculations. We have

$$[[e_3\, e_7]\, e_1] + [[e_7\, e_1]\, e_3] + [[e_1\, e_3]\, e_7] = 2[e_6\, e_1] - 2[e_4\, e_3] - 2[e_2\, e_7]$$
$$= 12e_5 \qquad (4)$$

and

$$[[e_5\, e_7]\, e_3] + [[e_7\, e_3]\, e_5] + [[e_3\, e_5]\, e_7] = 2[e_2\, e_3] - 2[e_6\, e_5] - 2[e_4\, e_7]$$
$$= 12e_1. \qquad (5)$$

In order to create a viable Lie bracket we note that we have constructed the octonions from the quaternions by the Cayley-Dickson construction in section 9. Thus, if we apply an inverse Cayley-Dickson construction to retrieve a pair of quaternions from the octonions, since the quaternions are representable by matrices, on each item of this pair we can create a Lie bracket satisfying (1). The information we might wish to keep in these Lie brackets could also be formed from the sum, difference, matrix product, and the product of a matrix by an inverse matrix of the pair, or any combination of these.

A better solution is to take Lie brackets including equations (2) to (5) (mod 12). We can also use a subgroup, that is, (mod 2), (mod 3), (mod 4) or (mod 6). □

The 10-novanions we have considered contain the quaternions as a subalgebra. Even when an override structure is imposed, its three elements constitute a quaternion. For the octonion structures we have considered, these also contain a quaternion, and this still applies if a nonstandard override is applied. Thus the n-novanions all contain quaternion subalgebras. □

Finally in this section we mention that, just as quaternions may be given globally bosonic or globally fermionic structures, as in section 4.13, so too can octonions and n-novanions. The analogous situation in the novanion algebra is that combined bosonic and fermionic structures may reside in the same n-novanion. □

4.17. The García classification problem.

Doly García has given a factorisation of 10-novnions so that

$$(i + i' + i'')(j + j' - 2j'') = 0.$$

For n-novanions a classification problem is to find all factorisations of novanions such that for real a_k, b_k, the nonscalar basis elements i_k give

$$(a_1i_1 + \dots a_ni_n)(b_1i_1 + \dots b_ni_n) = 0.$$

In novanion physics these may be the generators of the universe at time t = 0. □

4.18. The E_8 and novanion group constructions from octonions.

[To be modified.]

The E_8 lattice is closely related to the nonassociative algebra of real octonions **O**. It is possible to define the concept of an integral octonion analogous to that of an integral quaternion. The integral octonions naturally form a lattice inside **O**. This lattice is just a rescaled E_8 lattice. (The minimum norm in the integral octonion lattice is 1 rather than 2). Embedded in the octonions in this manner the E_8 lattice takes on the structure of a nonassociative ring.

Fixing a basis (1, *i*, *j*, *k*, ℓ, ℓ*i*, ℓ*j*, ℓ*k*) of unit octonions, one can define the integral octonions as a maximal order containing this basis. (One must, of course, extend the definitions of *order* and *ring* to include the nonassociative case). This amounts to finding the largest subring of **O** containing the units on which the expressions x^*x (the norm of *x*) and $x + x^*$ (twice the real part of *x*) are integer-valued. There are actually seven such maximal orders, one corresponding to each of the seven imaginary units. However, all seven maximal orders are isomorphic. One such maximal order is generated by the octonions *i*, *j*, and ½ (*i* + *j* + *k* + ℓ).

A detailed account of the integral octonions and their relation to the E_8 lattice can be found in Conway and Smith (2003).

Example definition of integral octonions[edit]

Consider octonion multiplication defined by triads: 137, 267, 457, 125, 243, 416, 356. Then integral octonions form vectors:

1) , i=0, 1, ..., 7

2) , indexes abc run through the seven triads 124, 235, 346, 457, 561, 672, 713

3) , indexes pqrs run through the seven tetrads 3567, 1467, 1257, 1236, 2347, 1345, 2456.

Imaginary octonions in this set, namely 14 from 1) and 7*16=112 from 3), form the roots of the Lie algebra . Along with the remaining 2+112 vectors we obtain 240 vectors that form roots of Lie algebra . See the Koca work on this subject.[13]

If we look at octonions, they have a multiplicative identity 1, and seven units e_1, e_2, ... , e_7. If we ignore their additive structure for the moment, we also have plus or minus these elements, and the plus and minus are distinguishable in the multiplicative group structure. The total number of elements of these types is now 16.

If we go over to a ring structure, additively it contains zero. We have also seen that the octonions with Lie brackets given by [A, B] = AB – BA, where for example this can be a matrix representation, although octonions are not matrices, since they are not associative, nevertheless they can be given the above Lie algebra structure (mod 12), satisfying the Jacobi identities.

The elements contain zero, for example $6e_7 \times 2e_7 = 0$ (mod 12). We therefore need to create a multiplicative group without zero. Note we can proceed not only (mod 12), but as we have mentioned, (mod 6), (mod 4), (mod 3) and (mod 2). If we take the case (mod 6), since for example if $3 \times 2 = 0$, then $3^{-1} \times 3 \times 2 = 0$, so 2 = 0 and has no inverse. The only occurrence of multiplicative terms ≠ 0 for an abelian group with result zero is 3×2 (mod 6). Then if we specify a modification of the group so that $3 \times 2 = 1$, since by prime factorisation this is the only pair ≠ 0 that gives zero, we can define this new allocation in which $3^{-1} = 2$ and $2^{-1} = 3$, and this retains the structure of the Lie algebra, and the multiplicative group defined in this way does not contain zero, which we have excluded from it. A more general way of stating this is that we replace 0 (mod 6) in the Lie algebra by ±1 in its group.

To consider (mod 12), the products which are zero (mod 12) contain powers of factors 3 and 2, where 3 and 3^2 occur (mod 12) and so do 2, 2^2 and 2^3. To analyse this, first look at (mod 9) where the only product of terms ≠ 0 that has a product 0 (mod 9) is 3×3. So a Lie algebra (mod 9) can give rise to a multiplicative group where $3 \times 3 = 1$, in which 3 is its own inverse. Likewise for (mod 8) = (mod 2^3), the only terms ≠ 0 with abelian product zero are 4×2. The element 4 is now set with inverse 2, and on replacing zero in the (mod 8) Lie algebra by ± 1 as an element in its group, we again obtain a multiplicative group. For (mod 12) we consider elements (mod 9×8) = (mod 72). Then there exists a corresponding group in which elements are classified multiplicatively by their (mod 9) or (mod 8) counterparts where 9 is irreducible in terms of 8. We could choose for example an element (mod 72) as $3^m \times 2^n \times$(prime < 72) where 3^m is in (mod 9), 2^n is in (mod 8) and the prime is considered (mod 72). Then from the Lie algebra (mod 12) derived from (mod 72), on setting zero to ± 1 the mapping to the group is obtained.

If 9 is reducible in terms of 8, then consider again the case where 1 is identified with zero (mod 12), so that the multiplicative algebra at this stage contains 11 elements. Then $3 \times 4 = 1$, so $3 = 4^{-1} = 2^{-2}$. We have $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $5 = 2^4$, $10 = 2^5$, $9 = 2^6$, $7 = 2^7$, $3 = 2^8$, $6 = 2^9$ and $1 = 2^{10}$. To check consistency, for example $2^6 = 9 = 3 \times 3 = 2^8 \times 2^8 = 2^{16} = 2^{10} \times 2^6 = 2^6$. Then 2 is represented by the cyclic permutation (1 2 4 8 5 10 9 7 3 6), and the reduction of 3 in terms of the generator 2 gives a multiplicative group on 10 elements. If we were to choose $3 \times 4 = -1$, then $3 = -2^{-2} = 2^8$, so $2^{10} = -1$, which is the mapping $2 \to 2i$ from the previous case.

Consider the original elements e_7, e_6, ... , 1, –1, $–e_1$, ... , $–e_7$, 16 elements in all, and zero. If we allow new elements $0e_r = 0$, $1e_r = e_r$, $2e_r$, ..., $11e_r$, with $12e_r = 0e_r$, so that this exists in a (mod 12) arithmetic, then the Lie algebra derived from the octonions in this way has first a set of 11 elements to choose from (that is, not 0). Then if we create new elements by adding a second set, because the addition is $ae_r + be_s$ with $e_r \neq e_s$, but $ae_r + a(–e_r) = 0$, the paired e_r and $(–e_r)$ elements (mod 12) have 12 values (including $0e_r + 0(–e_r) = 0$) that we must exclude, but $–[0e_r + 0(–e_r)] = 0e_r + 0(–e_r) = 0$ is the same case. Thus we have 12 – 6 elements that are not zero. Further, we have values

(1) $1e_7 + 0(–e_7) = 1e_7$

(2) $2e_7 + 1(–e_7) = 1e^7$

... ...

(11) $11e_7 + 10(-e_7) = 1e_7$

(12) $12e_7$ [which is $0(e_7)$] $+ 11(-e_7) = 1e_7$,

which we must also collect together as one item, but case (1) (mod 12) is the same as case (12), etc, with case (6) the same as case (7). Thus there are 12/2 = 6 distinct cases for $1e_7$ and the total number of exclusions for differences $2e_7$, $3e_7$, ..., $11e_7$ is the same. Thus for the pair e_7, $(–e_7)$ we obtain $12^2 – 12 \times 6 = 12 \times 6$ possibilities. There are 8 values e_7, e_6, ... e_1, 1 that we can account for in a similar way. Then the total number of distinct possibilities is $(12 \times 6)^8$, but we must exclude the cases where we have a sum of all the e_r equal to zero. For e_7 this is 12 cases, as we have seen, and likewise for e_6, ... e_1, 1. Thus we exclude $(12)^8$ items, and the total number of elements of the octonion algebra is

$$(12 \times 6)^8 – 12^8 = 12^8(6^8 – 1).$$

This is the order of the Lie algebra we have derived from the octonions. We obtain the order of the Lie group from it. ☐

4.19. Novanion simple group constructions. [To be modified.]

Our objective now is to convert novanion algebras, with novanion brackets (mod m) like a Lie bracket, to novanion algebras, and then map these novanion algebras to groups. These groups, being multiplicatively invertible, do not contain zero, so we map $0 \to \pm 1$, which retains the novanion algebra structure. The resulting novanion group thus does not have the same structure as the multiplicative part of the ring mapped directly to the novanion algebra.

For the 10-novanions consider the elements e_9, e_8, ... , 1, –1, –e_1, ... , –e_9, 20 elements in all, and zero. Using elements $0e_r = 0$, $1e_r = e_r$, $2e_r$, ..., $11e_r$ again, then in the case when we are not using ± 1, the extended Lie group derived from the 10-novanions has in a calculation similar to one given for the octonions, $(12\times 6)^9$ elements. If we assume the scalar part is not zero, the available scalar values are ±1, ±2, ..., ±11, amounting to 22 components, but we have 1 = -11 (mod 12), etc., so effectively there are 11 components. We know that when the scalar part is not zero, we cannot get a zero result on multiplying 10-novanions together, provided we ignore the (mod 12) restriction. Thus the order of the 10-novanion group derived from this algebra is $(12\times 6)^9\times 11$ elements. If we obtain 0 (mod 12) in any multiplication, then as before we will map this result to ± 1 without changing the Lie-type bracket algebra.

For the 26-novanions or indeed any n-novanion algebra, the case is similar with $(12\times 6)^n\times 11$ elements. We know that overrides derived from the octonion structure operate for all 10-novanion components in a 26-novanion, thus external to the 10-novanion or an octonion or their subalgebras, there are no other proper subentities in this algebra, which implies that the groups defined by a quotient of a 10-novanion group or an octonion group with the 26-novanion group are simple.

Note that we have demonstrated in section 12 that both bosonic and fermionic structures can be implemented in novanions. Thus any argument which says that a bosonic 26-novanion cannot contain a 10-novanion is incorrect. Since the bosonic 10-novanion is a maximal proper subgroup of the 26-novanions, the 26-novanions do not generate a simple group directly, only via quotients. ☐

Note that 26 – 10 = 16, but we have no indication that this is two copies of E_8. Indeed, it cannot be, since the structure is not closed, encroaching in overrides onto the 10-novanion. So we show once again that the novanion groups do not lead to the compactification assumed in string theory of the identification of the 16 dimensions with $E_8 \times E_8$. ☐

These considerations may be extended without limit, for example to the 80-novanions, or more generally the $(3^n – 1)$-novanions, n > 0, which are those hypercube novanions with one corner containing not a triplet, but a singlet. In particular, for the 80-novanions there exists a 28-novanion slice group which has no containing groups other than the 80-novanion group itself, so there is an analogous structure to that for the 26-novanion group. The remaining dimensions are 79 – 27 = 52, and this generates a simple group via the quotient with the 28-novanion group. ☐