CHAPTER IV

Zargonions

4.1. Introduction.

We introduce the theory of zargonions. This includes the theory of novanions developed in *Superexponential algebra* [Ad15], *Elementary universal physics* [Ad18b] and chapter XVII of *Investigations into universal physics* [Ad18c], the last two by Graham Ennis and me.

Zargonions are a generalisation of octonions. Due to their novelty, they are not included as yet in most studies of nonassociative algebras. It is probably the case that the theory was not attempted to investigate them because to do so might have been thought in contradiction with the claimed classification of nonassociative division algebras. A famous theorem published by J. F. Adams in 1960 limits these to subalgebras of the octonions, but this is not so. There exists a 14-vulcannion division algebra, although there is no 16 dimensional 16-vulcannion. The origin of the discrepancy is given in section 19. Not division algebras are the novanions, since just in the case where the scalar component of them is zero is it possible to find two nontrivial novanions which multiplied together result in zero. Since in other cases novanions are division algebras, we describe this type of object as a zargon algebra.

This chapter extends the material in [Ad15], by considering discreteness and zargon rings, the enveloping novanions of classical and nonclassical type, extends the discussion to include zargon brackets, and gives a brief statement of the García classification problem.

In chapter V we use this theory to develop the theory of the classification of simple groups, chapter VII looks at zargonion varieties, chapter IX considers the sphere packing problem from the zargonion point of view and chapters X and XI on analysis and suanalysis consider some theorems involving zargonions. In volume II, chapter VII, on homotopy superstructures we discuss zargonion rotations as knots and twistings, developing these to consider epicycle knots and twisted flows.

4.2. Division algebras.

We repeat the axioms for a field, and successively weaken these axioms, firstly to give them for division algebras.

The axioms for a *field* $\mathbb{F}$, +, ×, which we will denote simply by $\mathbb{F}$, satisfy for a, b, c ∈ $\mathbb{F}$, with a × b being written as ab

additive closure:	$a + b \in \mathbb{F}$	(1)
associativity:	$a + (b + c) = (a + b) + c$	(2)
abelian addition:	$a + b = b + a$	(3)
existence of a zero: there exists a $0 \in \mathbb{F}$ satisfying		
	$a + 0 = a$	(4)
existence of negative elements: there exists a $(-a) \in \mathbb{F}$ with		
	$a + (-a) = 0,$	(5)
which we write introducing subtraction as		
	$a - a = 0$	
multiplicative closure:	$ab \in \mathbb{F}$	(6)
associativity:	$a(bc) = (ab)c$	(7)
commutativity:	$ab = ba$	(8)

existence of a 1: there exists a $1 \in \mathbb{F}$ satisfying

$$a1 = a \tag{9}$$

existence of inverse elements: there exists an $a^{-1} \in \mathbb{F}$ for $a \neq 0$ with

$$a(a^{-1}) = 1, \tag{10}$$

which we write introducing division as

$$a/a = 1$$

distributive law: $$a(b + c) = (ab) + (ac). \tag{11}$$

An associative division algebra satisfies all axioms for a field except the multiplicative commutativity rule (8). A nonassociative division algebra, D, also drops the multiplicative associative rule (7). It may introduce the following axiom:

There do not exist $a, b \neq 0 \in D$ with

$$ab = 0. \tag{12}$$

This rule is unnecessary for associative division algebras, since if $1 \neq 0$ then (12) implies

$$ab \neq c, c \neq 0,$$

and therefore there exists no

$$a(b/c) = 1,$$

but b/c can be chosen to be the multiplicative inverse of a, a contradiction. ☐

4.3. Zargon algebras.

A zargon algebra B drops axiom (12). It may substitute the following rule:

If $a^2, b^2 > 0$ for $a, b \in B$ and a is in $c \in B$, b is in $d \in B$, there do not exist any c, d with

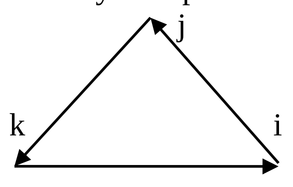
$$cd = 0. \tag{1}$$

We can provide existence proofs for these structures. The complex numbers constitute a field. The quaternions form an associative division algebra. The octonions form a nonassociative division algebra. As we will prove in section 4.9, the 10-novanions, for example, form a zargon algebra. ☐

The consistency of the complex numbers is contingent on the consistency of analysis, given as a Gentzen-type proof in chapter III volume II. The consistency of the quaternions as an associative division algebra is given in reference [Ad15], and we prove in 4.9 the consistency of the 10-novanions as a zargon algebra. The consistency of the octonions now follows from their nature as zargon algebras of a specific type. ☐

4.4. The quaternions.

The quaternions may be represented by a triangle diagram



where the nodes i, j and k satisfy

$$ij = k, jk = i, ki = j, \tag{1}$$

in other words, for a positive sign in the above relations, we are following the arrows. When we are going in a direction opposite to the arrows, we have a negative sign:

$ji = -k,\ kj = -i,\ ik = -j.$ (2)

We have here that 1 commutes with all elements, and also

$1^2 = 1,\ i^2 = j^2 = k^2 = -1.$ (3)

Some matrix representations of the quaternions are given in [Ad15], chapter III, section 3.7. We now introduce the nonassociative exquaternions (which thus cannot be represented by a matrix).

The linearly independent intricate basis elements of chapter III, section 2, satisfy

$1^2 = 1,\ i^2 = -1,\ \alpha^2 = 1,\ \phi^2 = 1,$

$1i = i = i1,\ 1\alpha = \alpha = \alpha 1,\ 1\phi = \phi = \phi 1,$

$i\alpha = -\phi = -\alpha i,\ i\phi = \alpha = -\phi i$ and $\alpha\phi = i = -\phi\alpha.$

This algebra may be modified so that, for instance, all relations are maintained except for one case of the sign, which is altered to

$i\alpha = \phi = -\alpha i.$

If the resulting algebra were associative, then

$\alpha = i\phi = i(i\alpha) = (i^2)\alpha = -\alpha$

and

$-i = \phi\alpha = (i\alpha)\alpha = i\alpha^2 = i,$

so this *extricate* (in contradistinction to *intricate*) algebra is not associative, and thus is not represented by a matrix.

Indeed, we may adopt the relations

$i(i\alpha) = -(i^2)\alpha$

and

$(i\alpha)\alpha = -i(\alpha^2).$

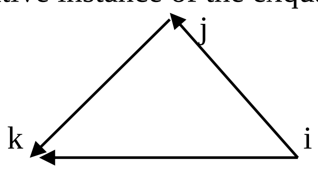
If we apply the quaternion representation of chapter III to the exquaternions, in the sense of an algebra rather than a matrix, then a representation is

$e_1 = i_1,\ e_2 = \alpha_i,\ e_3 = \phi_i,$

and the exquaternion multiplication table on elements becomes

×	***1***	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$
1	1	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	-1	e_3	e_2
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1
$\boldsymbol{e_3}$	e_3	$-e_2$	$-e_1$	-1

A representative instance of the exquaternions can be depicted in the diagram

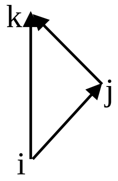


where the direction of the arrows indicates the sign in products. For this particular example

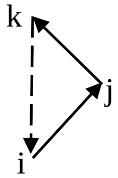
$(j - k)(1 - i) = 0,$ (4)

so that the exquaternions do not form a division algebra. □

We observe that the diagram for the exquaternions put on its side is a Hasse diagram for a lattice. We have seen that under this one orientation the diagram represents an object with no proper division.

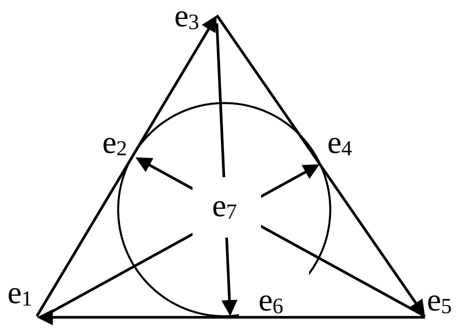


A diagram for the quaternions is the union of two directed graphs with two orientations, for the sum of two arrows and its dual. It is an example of two trees with two operations. Some of the nodes of the two trees are identified. We call the unified structure an amalgam.



4.5. The nonassociative octonion division algebra.

We will see next that there exist nonassociative algebras, which cannot be represented by matrices, the octonions. The algebra of the octonions, also called the Cayley numbers, was discovered by J. Graves in 1843, and is given by the Fano plane for their basis elements 1, e_1, e_2, e_3, e_4, e_5, e_6 and e_7 in the figure below.



Typical cyclic identities are

$$e_7e_5 = e_2 \qquad (1)$$

$$e_5e_6 = e_1$$

and

$$e_4e_2 = e_6.$$

Note what we have said here. The inner triple $e_2e_4e_6$ acts like a quaternion, but the outer triple $e_1e_3e_5$ does not. Nevertheless, we will need to allocate later a list ordered as right triple ($e_2e_4e_6$) + central triple ($e_1e_3e_5$) + one (e_7). Octonions form a division algebra, in particular

$$e_c^2 = -1, \qquad (2)$$

$$e_ae_b = -e_be_a \ (a \neq b),$$

and the inverse of

$$a1 + \sum_{n=1}^{7} b_n e_n$$

is

$$(a1 - \sum_{n=1}^{7} b_n e_n)/(a^2 + \sum_{n=1}^{7} b_n^2). \qquad (3)$$

The octonions, $\mathbb{O}$, are also generated by the Cayley-Dickson construction [Ba01]. This builds up algebras from the complex numbers, to the quaternions, to the octonions, to the sixteen dimensional sedenions, etc.

Define a T-algebra to be an algebra equipped with conjugation, a linear map T satisfying

$$a^{TT} = a, \tag{4}$$

$$(ab)^T = b^T a^T. \tag{5}$$

Starting from any T-algebra, the Cayley-Dickson construction gives a new algebra

$$(a, b)(c, d) = (ac - db^T, ad^T + cb), \tag{6}$$

with conjugation defined by

$$(a, b)^T = (a^T, -b). \tag{7}$$

Let us look at how this works in practice. The complex numbers have multiplication

$$(a, b)(c, d) = (ac - bd, ad + bc).$$

The complex conjugate

$$(a, b)^T = (a^T, -b) = (a, -b),$$

since the scalar part a is a real number with $a^T = a$.

The complex conjugate satisfies

$$(a, b)^T(a, b) = (aa + bb, ab - ba) = (a^2 + b^2, 0).$$

We define $a^2 + b^2$ as the *norm* of the complex number. It is a non-negative real number.

For the noncommutative quaternions equations (6) and (7) hold. The order of the terms in (6) is important. We now have

$$(a, b)^T(a, b) = (a^T, -b)(a, b) = (a^Ta + b^Tb, ba^T - ba^T)$$
$$= (a^Ta + b^Tb, 0),$$

where $a^Ta + b^Tb$ is the norm.

The T-algebra generates the basis element multiplication table $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$ of the octonions

×	***1***	***e₁***	***e₂***	***e₃***	***e₄***	***e₅***	***e₆***	***e₇***
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

We can generate for each

$$\mathbb{O} \times \mathbb{O} \to \mathbb{O}: e_i \times e_j \to e_k$$

a Cayley-Dickson construction of

$$e_i \times (-e_j) \to -e_k,$$

so that each of the 7 non-scalar basis elements in a row of the table can be multiplied by -1 to provide $2^7 = 128$ possible Cayley-Dickson constructions.

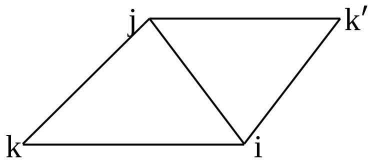
The Fano plane has 7 non-scalar basis elements. The number of non-scalar quaternionic triplets is 7, each of which, since exquaternions are excluded, operates under a forward or a reversed orientation – again 2^7 possibilities. The following Fano triplets map bijectively to the standard Cayley-Dickson construction for the octonions

$(e_1, e_2, e_3), (e_3, e_4, e_5), (e_1, e_4, e_6), (e_4, e_6, e_2), (e_1, e_7, e_6), (e_4, e_7, e_3), (e_5, e_7, e_2).$

Thus distinct possibilities for the Fano plane match distinct instances of the Cayley-Dickson construction. □

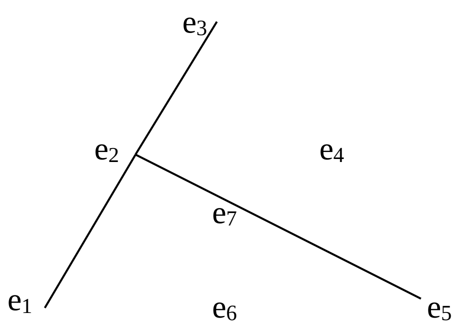
4.6. T-junctions, fans and vulcannions.

We note that for two adjoined quaternion triangles, whatever the orientation of each, if we display these as follows



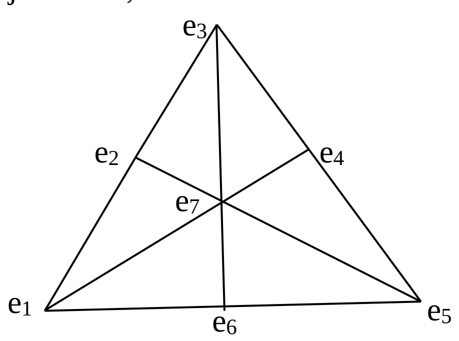
(1)

then ignoring sign, if ij = k, we cannot have ij = k′. In what follows we will be looking at combining diagrams which include parts of the Fano plane diagram at the beginning of section 4.5. A question arises: if no two triangles can adjoin, how can a larger object be constructed? The answer is provided by T-junctions.



(2)

At the junction node e_2 shown, no two triangles adjoin. It is then possible to construct a figure composed of three T-junctions,

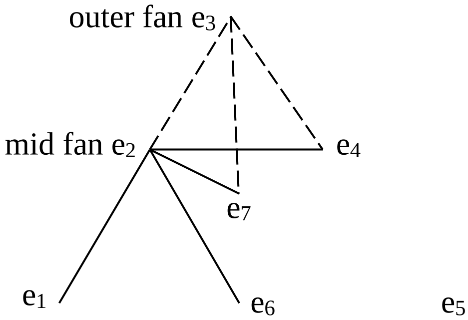


(3)

with junction nodes e_2, e_4 and e_6 forming a junction triangle.

To reinterpret the above discussion, the two adjoining parts of the Fano plane we will analyse we call an outer fan and a mid fan.

The number of lines emanating from the outer fan at e_3 is three of which one goes to e_7, and the number of lines from the mid fan e_2 is four. The two fans cover the Fano octonion plane, in the sense that by an extension by multiplication of node elements in the fans to quaternion lines and triangles in this Fano plane, all node elements of the diagram are covered.



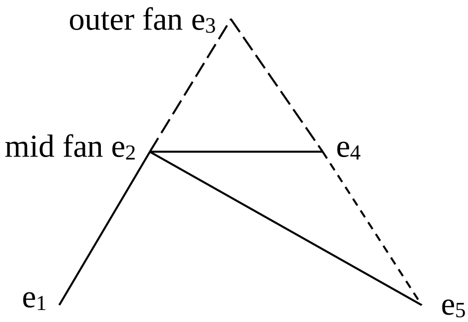
(4)

We note the relation for these lines

(two + one) + four = seven,

and seven is the total number of nodes in the diagram. Further, from each line in, say, the outer fan there is a line in the mid fan so that the result is a diagram, formed either as a line or a triangle, both types of which correspond to a quaternion.

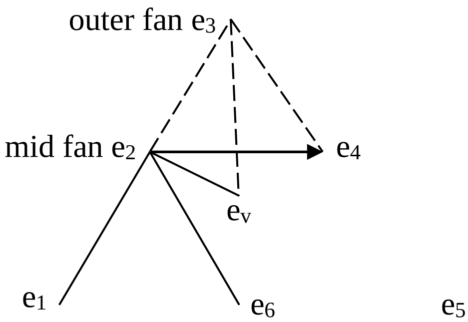
Before we start our analysis for the case of $v' = 6 + v$ nodes for a general odd Vulcan number v, let us see what happens in the case which corresponds with Vulcan number $v' = 5$, so that there are five nodes, where in this case no complete set of T-junctions is available.



(5)

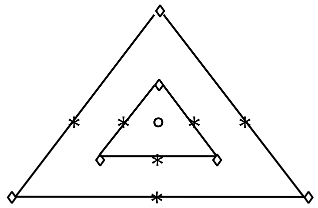
Then connecting the nodes e_4 and e_5, we see that we have two adjoining triangles, which we have seen at the start of this section is inconsistent.

Let us now consider what happens with two adjoining fans for a figure with 6 + v nodes, where the outer fan consists of 2 + v nodes and the mid fan consists of 3 + v nodes. The two fans generate all nodes in this diagram except the node e_v as quaternion lines or triangles.



(6)

If e_v is a collection of nodes, and this collection corresponds to a (v + 1)-vulcannion, then we claim that in the case v = 7 the resulting structure is once again a vulcannion. We have seen that when e_v corresponds to the Vulcan number v = 1, so that the number of complex parts is 1 (a complex number) then the generated vulcannion is an octonion with 7 nodes, and when the number of parts of e_v is 7, so e_v is an octonion, this generates a 14-vulcannion. At this point, we do not assume there are any junction node triangles, but picture the situation in the further diagram below



(7)

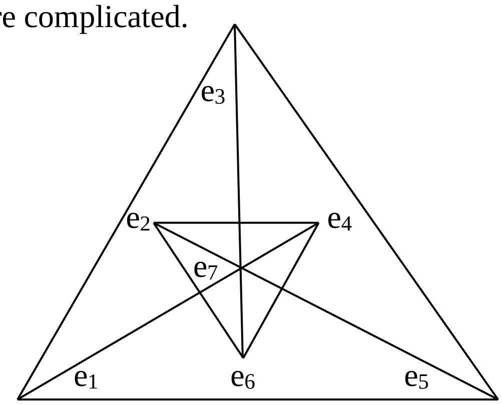
where the ∘ is a node that belongs to all octonions, whereas ∗ and ⋄ indicate nodes that belong to only one octonion. It can be seen immediately that there are layers of octonions forming this representation of the previous figure (6). Then we can divide the diagram into octonions in the following way. First consider all ∗'s together with ∘ as an octonion, then for all ⋄'s together, then in the inner triangle ∗'s and ⋄'s together, then the same for the outer triangle, and then both finally ∗'s in the outer triangle with ⋄'s in the inner triangle, and then swapping round, say, the triangles for this. These are all combinations. There is however a difference between the octonion, or 8-vulcannion as they could be called, and a 14-vulcannion, in that in this configuration all junction nodes are part of an octonion T-junction diagram, and so there is no junction triangle. The diagram (7) has 13 nodes, and thus constitutes a 14-vulcannion.

A further question is whether a 20-vulcannion has a junction triangle or none. In fact, there is an alternating sequence as the Vulcan generator ascends between junction triangles existing and the next vulcannion in the sequence with none.

It is interesting that we can form from the arrow e_2 to e_4, the direction of all other arrows in the fan diagram. This situation does not happen for the n-novanions, as we will eventually see. The value of v generates by repetition vulcannions of dimensions given by the sequence $v' = 6 + v$, so the sequence begins with 14-vulcannions, then 20-vulcannions, followed by 26-vulcannions, etc. The existence of this sequence given by the number 6 can by shown from diagram (6), where we know by induction that node e_v is a vulcannion. The existence of 26-vulcannions is interesting. There are also 26-novannions.

It is clear that if e_v has 5 nodes, there are no consistent interconnections there, and so no vulcannions generate $v' = 6 + 5v$.

We now need to prove that there are no vulcannions in a sequence with Vulcan number $v = 3$. This is slightly more complicated.



(8)

We will need to prove first that there are no nonstandard octonion diagrams of type (8) with junction nodes (e_2, e_4 and e_6) and (e_1, e_2 and e_3) forming junction triangles. There are two cases to consider. For the first, let $e_2e_7e_5$ and $e_3e_7e_6$ be T-junctions, with junction at e_7. Then

e_2 and e_3 do not connect, but they must. For the second, let $e_2e_7e_3$ and $e_2e_7e_6$ be triangles, but then these triangles adjoin, which is impossible.

The proof then proceeds by noting that e_v in diagram (6) is a quaternion, that diagram (7) with an extra triangle on the outside now constitutes this scenario, but that it has no inner nodes unlike the nodes in other triangles which form octonions. For these other triangles, if we consider them fixed, there exist pairs one of which is like $e_2e_4e_6$ in diagram (3) which is a triangle, and one like $e_1e_3e_5$, which is not a proper triangle. Thus the 'dangling' triangle in the scenario now connects to a diagram like (8), which is impossible.

It can be demonstrated by induction that there is no v = 3 + 6k object. On considering the fact that the fan diagram (6) must connect, this object will be a type (7) diagram with a dangling triangle, and thus there will exist within it a diagram of type (8) which cannot exist.

4.7. The classification of novanions.

So the proof of J. F. Adams in 1960 that the only possible division algebras are at maximum those of the octonions, which are nonassociative, where quaternions and complex numbers are subalgebras of these, is wrong.

However, there exist nonassociative algebras, the n-novanions, with nonzero division, but when just the scalar part is zero two nonzero novanions may have a zero product. I obtain here the proof for n = 10, and extend it for n greater than 10. The classical novanions have a dimension of $n = 1 + 3^f \prod_{i \in \mathbb{N}}(3^{g_i} - 2)_i$, for which $f \in \mathbb{N}_{\cup 0}$, $g_i \in \mathbb{N}$. Thus the n-novanions comprise a new type of quasi division algebra, which we have called a zargon algebra.

We will classify these algebras, and then introduce zargon rings. These do not have division, and thus can represent discrete structures.

4.8. Eigenvalues.

Our intention now is to introduce a generalisation of quaternions and octonions called n-novanions, which will be involved in the mathematics we wish to describe. In order to prove the consistency of the n-novanion algebra, we will need to introduce some further ideas on matrices.

The representations discussed in this section are not diagonal or antidiagonal, but classical representations by row and column.

A nonsquare matrix A with u rows and v columns can be multiplied by another nonsquare matrix X with v rows (the same as the number of columns in A) and w columns. The element in the jth row and kth column in the result is found by multiplying each element from the jth row with the corresponding element from the kth column, adding the result together, just as for n × n matrices.

For example, if u = v = 2 and w = 1, with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

then

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix},$$

where x with one column is called a *column vector*.

If we start off with a set of independent linear equations, for example

$a_{11}x_1 + a_{12}x_2 = y_1$
$a_{21}x_1 + a_{22}x_2 = y_2$,

these can be represented by the matrix equation

$$Ax = y, \tag{1}$$

where x and y are column vectors.

We can change the basis from x to the linearly independent basis x′ so that

$x_1' = p_{11}x_1 + p_{12}x_2$
$x_2' = p_{21}x_1 + p_{22}x_2$,

so we can define a new matrix P in which

$$x' = Px. \tag{2}$$

If we apply the same transformation to y, we change to a basis in which y′ is represented as

$$y' = Py. \tag{3}$$

From (1) in (3)

$y' = PAx$,

then from (2)

$y' = PAP^{-1}x'$,

and this corresponds to a transformation of A under change of basis to PAP^{-1}. □

By definition, square matrices B and A are *similar*, when there exists a non-singular matrix P where

$B = PAP^{-1}$.

Our discussion shows that two $n \times n$ matrices A and B are similar if and only if they represent the same linear transformation, in particular a linear transformation represented by a matrix A relative to a basis x_i is represented by a matrix PAP^{-1} in a new basis x'_i where P is non-singular. The algebra of matrices applies to the case of diagonal matrices.

An important technique in analysing the matrix equations is to map the equation to a scalar equation involving complex numbers. These scalars are called eigenvalues, and satisfy a matrix equation like (1) below.

A vector x with finite basis is an *eigenvector* and λ is an *eigenvalue* of the $n \times n$ matrix A if and only if for all linearly independent x

$Ax = \lambda x$.

Thus for the unique diagonal matrix I

$IAx = Ax = \lambda Ix$

and

$(A - \lambda I)x = 0$

has dependent rows, and since a determinant changes sign under a swap of rows (but stays the same when a linear combination of rows is added to a row), its determinant is zero

$\det(A - \lambda I) = 0$. □

Similar matrices have the same eigenvalues, since

$$\begin{aligned}\det(A - \lambda I) &= \det(PP^{-1}(A - \lambda I)) \\ &= \det(P)\det(A - \lambda I)\det(P^{-1}) \\ &= \det(PAP^{-1} - \lambda PIP^{-1}) \\ &= \det(PAP^{-1} - \lambda I).\ \square\end{aligned}$$

By the fundamental theorem of algebra, if λ is a complex but not otherwise intricate number, the independent solutions for λ of the equation

$$\det(A - \lambda I) = 0,$$

in number at most the dimension, n, of A, are unique. □

Like the octonions, the n-novanions cannot be described directly by matrices, since matrices are multiplicatively associative, meaning (AB)C = A(BC), but novanions are not. However, we can characterise some of the properties of n-novanions by a set of linear equations. These can be described by a matrix equation.

Within the field of complex numbers $\mathbb{C}$, the complex conjugate of $c = a + ib$ is $c^* = a - ib$. For the corresponding matrix C with entries $c_{jk} = a_{jk} + ib_{jk}$, the conjugate $C^* = a_{jk} - ib_{jk}$. The transpose of a matrix C is denoted by C^T and has entries c_{kj}. The transpose is a contravariant (order reversing) operation:

$$(CD)^T = D^TC^T.$$

A matrix is defined as antisymmetric if $C^T = -C$.

The following proof is derived from chapter 11 of [Uh01].

Theorem 4.6.1. *All eigenvalues of a real antisymmetric matrix* $E = -E^T$ *are pure imaginary*.

Proof. Consider the case of the eigenvalue λ and possibly complex column vector called an eigenvector $\mathbf{x} \neq \mathbf{0}$. According to the Fundamental Theorem of Algebra $\lambda \in \mathbb{C}$. Hence

$$E\mathbf{x} = \lambda\mathbf{x}. \tag{4}$$

If we take the complex conjugate of both sides of the eigenvalue-eigenvector equation (4), we obtain

$$(E\mathbf{x})^* = (\lambda\mathbf{x})^* = \lambda^*\mathbf{x}^*.$$

Transposing yields

$$(E\mathbf{x})^{*T} = x^{*T}E^{*T} = \lambda^*\mathbf{x}^{*T}.$$

Define the norm as

$$||E\mathbf{x}||^2 = (\lambda^*)\lambda(\mathbf{x}^{*T})\mathbf{x}, \tag{5}$$

where for the set of real numbers $\mathbb{R}$

$$||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}} \in \mathbb{R}.$$

Since $E^T = -E$ and $E^{*T} = E^T$ for a real antisymmetric matrix E, we can write (5) as

$$\begin{aligned} ||E\mathbf{x}||^2 &= \mathbf{x}^{*T}E^{*T}\, E^T\mathbf{x}, \\ &= \mathbf{x}^TE^TE\mathbf{x} \\ &= -\mathbf{x}^TE^2\mathbf{x} \\ &= -\mathbf{x}^T\lambda^2\mathbf{x}, \end{aligned}$$

because $E^2\mathbf{x} = E(E\mathbf{x}) = E(\lambda\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, so

$$||E\mathbf{x}||^2 = -\lambda^2\mathbf{x}^T\mathbf{x}. \tag{6}$$

Now $\mathbf{x}^T\mathbf{x} \neq 0$, and thus $\lambda^*\lambda = -\lambda^2$, by comparing (5) and (6). Thus a real antisymmetric matrix $E = -E^T$ can only have imaginary eigenvalues λ. □

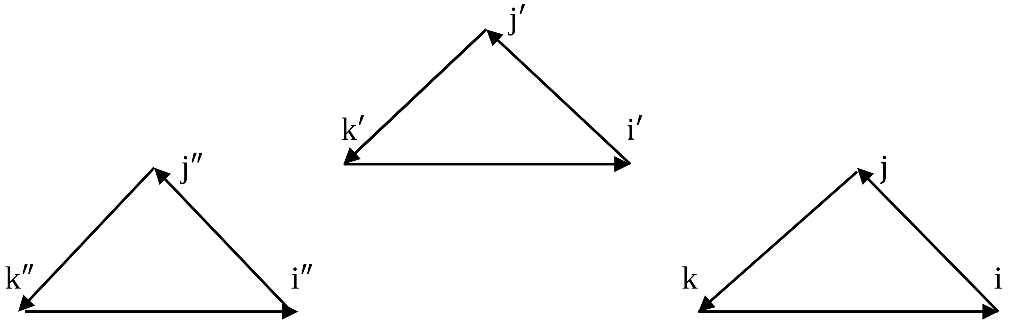
4.9. The 10-novanions.

We have seen a zargon algebra B drops rule 4.2.(12). It may substitute the following rule:

If $a^2, b^2 > 0$ for $a, b \in B$ and a is in c, b is in $d \in B$, there do not exist any c, d with

$$cd = 0. \tag{1}$$

We now introduce the 10-novanions, represented by the set of triangle diagrams



where in general each triangle is a quaternion without 1.

The primed variables ( ), (′) and (″) act as holders of information concerning an algebra for them. When the variables all contain a common instance, for example (k), (k′) and (k″), then the algebra is that of the quaternions, in which we have a cyclic algebra

$$kk' = k'' = - k'k. \tag{2}$$

When the variables contain different instances, such as k and i′, then the product contains the primed variable that does not belong to the first two elements, but the primed part commutes.

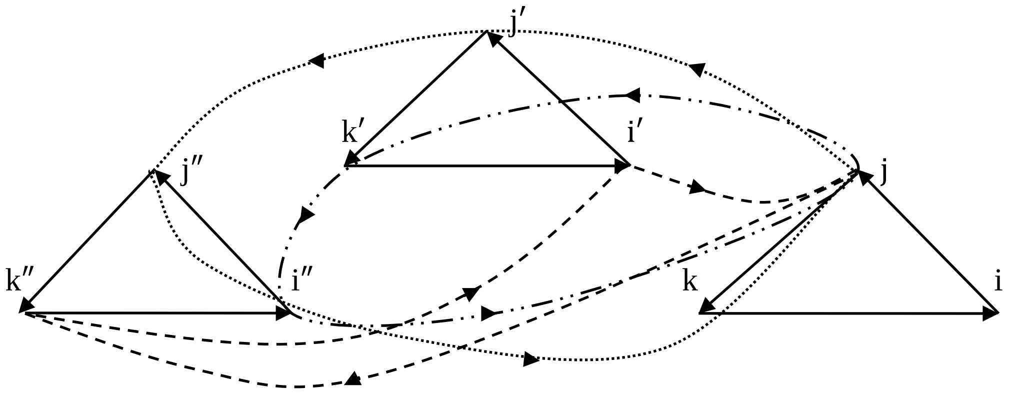
On top of this is the fact that the variables, say k and i, satisfy a quaternion algebra, so say

$$ki = j = -ik \tag{3}$$

and consequently

$$ki' = j'' = -i'k. \tag{4}$$

In order to picture the 10-novanions more closely, we will show the connections from node j



Our claim is that the inverse of

$$a1 + \textstyle\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} b_n^m e_n$$

is

$$(a1 - (\textstyle\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} b_n^m e_n))/(a^2 + (\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} (b_n^m)^2)), \tag{5}$$

and this constitutes a type of division algebra with no divisors of zero provided a1 ≠ 0 – the 10 dimensional 10-novanions.

We see by a result of [Ad15], chapter II, section 8, that the n-novanions are nonassociative, since they have more than 4 basis elements; more explicitly

$$(j''k')j' = ij' = k'' \neq j''(k'j') = -j''i' = k.$$

We wish to enquire under what conditions there exist two 10-novanion numbers multiplied together giving zero:

$$(a1 + bi + cj + dk + b'i' + c'j' + d'k' + b''i'' + c''j'' + d''k'') \times$$
$$(p1 + qi + rj + tk + q'i' + r'j' + t'k' + q''i'' + r''j'' + t''k'') = 0. \quad (6)$$

Their product is

real part:

$$ap - bq - cr - dt - b'q' - c'r' - d't' - b''q'' - c''r'' - d''t'' = 0, \quad (7)$$

i part:

$$bp + aq - dr + ct - b''q' - d''r' + c''t' + b'q'' - d'r'' + c't'' = 0, \quad (8)$$

j part:

$$cp + dq + ar - bt + d''q' - c''r' - b''t' + d'q'' + c'r'' - b't'' = 0, \quad (9)$$

k part:

$$dp - cq + br + at - c''q' + b''r' - d''t' - c'q'' + b'r'' + d't'' = 0, \quad (10)$$

i′ part:

$$b'p + b''q - d''r + c''t + aq' - d'r' + c't' - bq'' - dr'' + ct'' = 0, \quad (11)$$

j′ part:

$$c'p + d''q + c''r - b''t + d'q' + ar' - b't' + dq'' - cr'' - bt'' = 0, \quad (12)$$

k′ part:

$$d'p - c''q + b''r + d''t - c'q' + b'r' + at' - cq'' + br'' - dt'' = 0, \quad (13)$$

i″ part:

$$b''p - b'q - d'r + c't + bq' - dr' + ct' + aq'' - d''r'' + c''t'' = 0, \quad (14)$$

j″ part:

$$c''p + d'q - c'r - b't + dq' + cr' - bt' + d''q'' + ar'' - b''t'' = 0, \quad (15)$$

k″ part:

$$d''p - c'q + b'r - d't - cq' + br' + dt' - c''q'' + b''r'' + at'' = 0. \quad (16)$$

Alternative definition 4.9.1. D is a (possibly nonassociative) division algebra whenever for any element a in D and any nonzero element b in D there exists just one element x in D with a = bx and only one element y in D with a = yb.

If a = 0, the 10-novanions contain possibilities for two nonzero 10-novanions giving a product which is zero. We give the following example due to Doly García, showing that the 10-novanions satisfying a = 0 do not form a division algebra of standard type

$$(i + i' + i'')(j + j' - 2j'') = 0. \quad (17)$$

Thus the 10-novanions are not a division algebra given by the condition for equation 4.2.(12), nor do they satisfy definition 4.9.1 since for an arbitrary real number g

$$(i + i' + i'')[(j + j' - 2j'')g] = 0. \quad (18)$$

From now on we will assume a ≠ 0. By a symmetrical argument applied also to the following reasoning, we need to assume with this that p ≠ 0.

Equations (7) to (16) form a matrix E + aI, where E is an antisymmetric matrix and I is the unit diagonal, multiplied on the right by the eigenvector (p, q, r, t, q′, r′, t′, q″, r″, t″). We have already given a proof that the eigenvalues of a real antisymmetric matrix are entirely imaginary, so these correspond to -a, which is real, whereas we are now excluding the only possibility for this, a = 0.

So 10-novanions form a zargon algebra satisfying the conditions of equation (1). ☐

4.10. n-novanions.

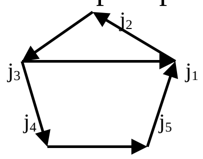
The octonions given by a Fano plane under suitable orientations may be given three copies in primed variables ( ) , (′) and (″), and by an analogous procedure this constitutes a $1 + (3 \times 7) = 22$ dimensional division algebra.

Extending these ideas further to multiple occurrences of the three or seven primed variables, say (′), (′)′ and (′)″, we obtain in general an $n = 1 + 3^f7^g$ dimensional novanion algebra, the n-novanions, in which if a common variable, k, is employed, then the lowest value within brackets of say (k), (k′)′ and (k″)″ is evaluated. □

4.11. The search for other novanion algebras.

Are there other novanion algebras of a type not already covered? This question has been stimulated by a first-year Sussex University student's identification of novanions with strings in physics, [Ad18], where the allocation n = 10 is the same number as the dimensionality of the heterotic string, and for which we wish to investigate the bosonic allocation $1 + 5^2 = 26$.

We provide an alternative argument to that already presented in section 4.6. In the pentagonal diagram shown next, an initial attempt depicts only one out of five subtriangles.



The pentagon can be enumerated cyclically, so that

$$j_1j_2 = j_3,\ j_2j_3 = j_4,\ j_3j_4 = j_5,\ j_4j_5 = j_1,\ j_5j_1 = j_2, \tag{1}$$

and jumping a vertex we evaluate the closest triangle

$$j_3j_1 = j_2,\ j_4j_2 = j_3,\ j_5j_3= j_4,\ j_1j_4 = j_5,\ j_2j_5 = j_1, \tag{2}$$

where on inverting the orientation, we get a minus sign.

This latter fact implies we have an inbuilt norm and inverse; the inverse of

$$a1 + \textstyle\sum_{n=1}^{5} b_n j_n$$

is

$$a1 - \textstyle\sum_{n=1}^{5} b_n j_n/(a^2 + \sum_{n=1}^{5} b_n^2), \tag{3}$$

which is nonassociative, as is demonstrated by

$$(j_3j_1)j_4 = -j_3 \neq j_3(j_1j_4) = -j_4.$$

The question arises as to whether this constitutes a novanion algebra, which would now be extended from previous considerations to include the dimensions

$$n = 1 + 3^f5^g7^h.$$

The possibility of the existence of the division algebra violating equation

$$(a1 + bj_1 + cj_2 + dj_3 + ej_4 + fj_5) \times (p1 + qj_1 + rj_2 + tj_3 + uj_4 + vj_5) = 0 \tag{4}$$

will now be investigated. Under the constraints (1) and (2) we obtain the set of equations

<u>real part</u>:

$$ap - bq - cr - dt - eu - fv = 0, \tag{5}$$

<u>j_1 part</u>:

$$bp + aq - fr + 0 - fu + (c + e)v = 0, \tag{6}$$

j_2 part:

$$cp + (f + d)q + ar - bt + 0 - bv = 0, \tag{7}$$

j_3 part:

$$dp - cq + (b + e)r + at - cu + 0 = 0, \tag{8}$$

j_4 part:

$$ep + 0 - dr + (c + f)t + au - dv = 0, \tag{9}$$

j_5 part:

$$fp - eq + 0 - et + (d + b)u + av = 0, \tag{10}$$

from which it follows that the E type matrix is not antisymmetric, but it may be represented as the sum of two matrices F and G, where F has all pure imaginary eigenvalues:

$$F = \begin{bmatrix} 0 & -b & -c & -d & -e & -f \\ b & 0 & -f & c & -f & e \\ c & f & 0 & -b & d & -b \\ d & -c & b & 0 & -c & e \\ e & f & -d & c & 0 & -d \\ f & -e & b & -e & d & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & c \\ 0 & d & 0 & 0 & -d & 0 \\ 0 & 0 & e & 0 & 0 & -e \\ 0 & -f & 0 & f & 0 & 0 \\ 0 & 0 & -b & 0 & b & 0 \end{bmatrix},$$

and we do not have pure imaginary eigenvalues for $F + G - \lambda I$.

Indeed a general matrix H may be represented as a sum of a symmetric part H_{sym} and an antisymmetric part H_{anti}. Then H_{sym} and H_{anti} are linearly independent over real coefficients, meaning there exist no real numbers c and d satisfying

$$cH_{sym} + dH_{anti} = 0.$$

By a demonstration similar to that in section 4.6, which is proved directly in [Uh01], the eigenvalues of a symmetric matrix are real. Further, a complex number with real coefficients h_1 and h_2 is linearly independent between h_1 and h_2i over real coefficients. It now follows that the eigenvalue equation

$$H = hI \tag{11}$$

which has a unique set of n solutions is satisfied by

$$H_{sym} = h_1I$$

with n solutions and

$$H_{anti} = h_2I,$$

also with n solutions. These possible solutions are the only ones, since the solution set of (11) is unique. So if H_{sym} is not the zero matrix, a real h_1 exists. This implies there is no novanion algebra available. □

The identification relates to the number 1 + 25, and we now wish to probe the allocation 25 = 7 + 9 + 9, where 7 is the number of non-real basis elements of the octonions, and 9 is the number for the 10-novanions. There is an analogy here. The octonion non-real basis elements of 7 may be represented as 1 + 3 + 3, where 3 is the number of such basis elements for the quaternions, and 1 for the complex numbers. We are forced for a number of reasons to decompose such an allocation into triplets, to retain the cyclic algebra for the quaternions.

The allocation will be as follows, where we subscript 3 and 1 to distinguish them

$3_a, 3_b, 3_c$ (i)

$3_d, 3_e, 3_f$ (ii)

$3_g, 3_h, 1_u$ (iii)

where allocations (i) and (ii) are internally similar to 10-novanions, and allocation (iii) is internally an octonion. We will explain why we use the word ‘similar’ later.

There are a number of possible configurations.

We want an algebra linking between (i), (ii) and (iii). Vertical allocations are present. We will choose next from straight lines going from left to right, for example the diagonal going upwards from 3_g, 3_e to 3_c. This is similar to a 10-novanion algebra. The descending line from 3_a, 3_e to 1_u is an octonion algebra. We then incorporate the algebra taking for example 3_d, 3_b to 1_u, an octonion algebra, or 3_g, 3_b to 3_f, this is similar to a 10-novanion algebra.

We have used the words ‘similar to a 10-novanion algebra’, and we now explain why. If we look at allocation (iii), this is part of the 3_g 3_h 1_u octonion, where 3_h and 1_u are linked.

Although 3_g is indeed a quaternion, we have already mentioned that 3_h is not. Therefore the vertical allocation given by 3_a 3_d 3_g is a 10-novanion, since it is made of genuine quaternions, but the vertical allocation 3_b 3_e 3_h is not. 3_b, 3_e and 3_h occur in octonion representations. If we were to state that the central triple 3_b, 3_e and 3_h algebras were quaternions, we would have an inconsistency. Therefore for these allocations as part of a ‘similar to 10-novanion’ structure, we decide that the octonion structure overrides the 10-novanion one. Since there is only one special 1_u part for the octonions, this part of the allocation is unique. The similar 10-novanion structure is now not a closed algebra within the 10-novanions; part of it belongs to the octonions. The corresponding situation just for 10-novanions with no octonionic overlap but with novanionic overlap is described by the octonionic allocation already discussed.

Since there is no other mixing of allocations, the result is as consistent as the 10-novanions and the octonions. This can be checked with equations like 4.9.(6) to (15), for which it is clear eigenvalues are pure imaginary. Finally a calculation like 4.9.(16) shows that this is a novanion algebra. □

The existence of 10-, 26- and 80-novanions (the latter obtained by an array cube of items like (i) to (iii) – all configurations lie in planes of the cube, and an m-cube gives rise to a $(3^m \pm 1)$-novanion) implies that results derived for division algebras have a different extension for n-novanions. □

4.12. The 64-novanions.

The 16-dimensional sedenions are formed by the Cayley-Dickson construction [Ba01]. Since they are not alternative, they do not form a division algebra. That is, we do not have

$$x(xy) = (xx)y$$

and

$$(yx)x = y(xx)$$

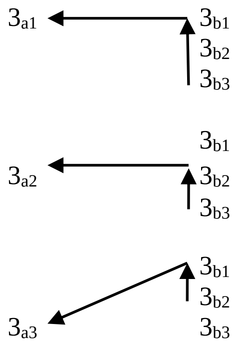
for all x and y in the algebra, the proof using basis elements. Every associative algebra is alternative, but so too are some strictly non-associative algebras such as the octonions.

The Cayley-Dickson construction generating a 64-dimensional algebra shows that this is not a division algebra, since in particular this contains the sedenions as a subalgebra. However, a 64-novanion has $63 = 3^2 \times 7$ non-scalar basis elements, and we will see that novanion algebras of this type are consistent. A 64-novanion is given by the cube with slices

$$\begin{array}{lll} 3_a, 3_b, 1_p & 3'_a, 3'_b, 1'_p & 3''_a, 3''_b, 1''_p \\ 3_c, 3_d, 1_q & 3'_c, 3'_d, 1'_q & 3''_c, 3''_d, 1''_q \\ 3_e, 3_f, 1_r & 3'_e, 3'_f, 1'_r & 3''_e, 3''_f, 1''_r. \end{array} \qquad (1)$$

To evaluate a typical slice algebra, if we take the leftmost array above, we know that 3_b is not a quaternion, so we will build an override structure for the composition of two elements in 3_b. Within this slice 3_b belongs to three octonionic arrangements, those given by 3_a, 3_b, 1_p, or 3_c, 3_b, 1_r, or 3_e, 3_b, 1_q, so we need to select an override on the nonquaternion 3_b, so that when two elements are multiplied within it, just one allocation to an octonionic structure is selected.

We will need to look at this typical example in detail, so denote the three elements of 3_b by 3_{b1}, 3_{b2} and 3_{b3}. We will display the 3 elements of 3_b combining in pairs to form arrows with the following typical structures. We will choose at first, arbitrarily, a link to the 3_a, 3_b, 1_p octonionic structure. Of course, two arrows shown below combine to give an oriented quaternion triple, for which reversal of arrows leads to a minus value.



(2)

The central triples 3_d and 3_f have similar structures, mapping to separate values in 3_c and 3_e respectively. We have stated the 1_p, 1_q and 1_r elements combined with 3_b give on composition with one element of 3_b the octonion structures (3_a, 3_b, 1_p), (3_c, 3_b, 1_r) and (3_e, 3_b, 1_q). Because 3_{b1} links to 3_{a1}, we have to ensure that the link 1_p to 3_{b1} does not also link to 3_{a1}, but this can be arranged.

Alternative structures can be considered. For example, if 3_{b3}, 3_{b1} links to 3_{a1} as before, we could also have 3_{b3}, 3_{b2} linking to 3_{c1} and 3_{b2}, 3_{b1} linking to 3_{e1}. □

4.13. The García classification problem.

Doly García has given a factorisation of 10-novnions so that

$$(i + i' + i'')(j + j' - 2j'') = 0.$$

For n-novanions a classification problem is to find all factorisations of novanions such that for real a_k, b_k, the nonscalar basis elements i_k give

$$(a_1 i_1 + \dots a_n i_n)(b_1 i_1 + \dots b_n i_n) = 0.$$

In novanion physics these may be the generators of the universe at time t = 0. □

4.14. The generation and classification of Zargon algebras.

All novanions are embedded within an enveloping zargon algebra. This algebra is infinite dimensional, and we will construct a direct demonstration of this fact.

We start from a modification of diagram (1) of 4.12, the triple and singleton elements of which, taken together form a node. Then the 3-cube of this diagram contains 3 × 3 × 3 = 27 nodes. We will now always consider that a 3n-cube always contains a singleton element at its corner. This is the anchor node. From the anchor node, along each pair of axes we will consider that anchor nodes form a sheet of singleton elements, and that, further, interior nodes always consist of triples.

We note that for any (2 + 6k)-vulcannion, we can go along from an anchor node, and inside a 3-cube find two triples along a diagonal line. Then, for this vulcannion, we can choose a cube of sufficiently high dimension so that there is the anchor node and 6k triples in its interior. Thus any vulcannion of dimension (2 + 6k), can be embedded in a cube of dimension equal or greater than 3k, and certainly in an enveloping cube of infinite dimension 3**Ω**. This is the enveloping zargonion.

We need to prove next that any novanion can be embedded in the enveloping cube. Firstly notice that any cube with slices, such as we used in 4.12.(1), is embedded in the 3**Ω**-cube. Further, any structure built up from the nodes diagonally opposite the anchor node, in any combination of axes, is a triple, and thus all combinations of triples can be included, and this includes in combinations with their opposite nodes.

Finally, all novanion overrides can be incorporated, since at each finite stage their number is finite, and thus can be embedded in the enveloping zargonion 3**Ω**-cube. This exhausts all possibilities for positive dimensional zargonions.

4.15. Zargon algebras of negative dimension.

We saw in section 6 that there are no vulcannions with Vulcan number v = 5, and that vulcannions can be generated for positive v in the sequence 6 + v, starting with v = 1. The question now arises as to whether there exist vulcannions for negative v. We will see that this is the case.

In the diagram below we depict a line with Zargon number, z, in integers including zero. For a (z + 1)-zargonion, z = 0 corresponds with the scalar zargonion. Positive z's after this correspond to integers containing, say, Vulcan numbers v = z. The first negative dimensional vulcannion corresponds to (v = 1) – 6 = –5, so its negative Vulcan number is v = –5. Does this vulcannion exist, and if so, how can it be interpreted?

-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 (1)

Zargon number, z

If we take Zargon number z = 0 as corresponding to the real numbers, z = 1 the complex numbers, z = 3 the quaternions and z = 7 the octonions, with the vulcannions corresponding to z = 1 and 7 to begin its sequence, then scalars have the property that a scalar, s, satisfies

$$s^2 = 1 \qquad (2)$$

whereas other positive components have square –1. We interpret negative z as occurring when all basis elements are scalars. For negative dimensional scalars to be distinct, they must anticommute

$$s_1s_2 = -s_2s_1. \qquad (3)$$

We have seen for v = +5 there is no vulcannion because of the existence of adjoining triangle diagrams. Since there are no scalars with square –1, for negative dimensional vulcannions all scalars must be in adjoining scalar triangle diagrams. Since no vulcannion exists with Vulcan number 3, there is no triangle for squares –1 with dimension 3 – 6 = –3. Thus this situation is consistent.

Further, since there exist triangle diagrams where every scalar triangle adjoins every other triangle, which is known as a barycentric subdivision in topology, and this occurs for every dimensions less than –2, this can be interpreted that there exist zargonions in negative dimensions for all such dimensions. Further, $z = -1$ satisfies (2), and $z = -2$ satisfies (2) and (3), so zargonions exist for all negative dimensions.

Can we incorporate the positive dimensional scalar in this algebra? If this commutes with other negative dimensional scalars, then $(1 + s_1)(1 - s_1) = 0$, so this is not a zargon algebra.

Note that we have a number of different types of negative dimensional algebra, for example (mod n) we could have $s_m^2 = s_{(m+1)}$, which is a cyclic algebra. We can also have algebras defined by other groups.

In physics, we say the scalar part is time and other positive zargonion basis elements represent space. Thus the above discussion can be reformulated that there exist consistent negative dimensional universes containing an arbitrary number of time components, but no space components, and these time components anticommute.

4.16. Zargon brackets.

The exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 are related to the existence of division algebras limited in number to those embedded within the octonions [Wi09], [CSM95].

We have seen in chapter IV of [Ad15] that for matrices A, B and C the Lie bracket

$$[AB] = AB - BA$$

satisfies the Jacobi identity

$$[[AB]C] + [[BC]A] + [[CA]B] = 0. \qquad (1)$$

For octonions, we do not have a matrix algebra, but we might wish to form Lie brackets from them satisfying (1). From section 4.5, for the octonions the only nonquaternion triple which is scanned from one set to another is $e_1e_3e_5$ and

$$\begin{aligned}[[e_1\, e_3]\, e_5] + [[e_3\, e_5]\, e_1] + [[e_5\, e_1]\, e_3] &= 2[e_2\, e_5] + 2[e_4\, e_1] + 2[e_6\, e_3] \\ &= -12e_7, \end{aligned} \qquad (2)$$

so this does not satisfy (1). However, there exist other nonquaternion triples, for example $e_1e_7e_5$, and this satisfies

$$\begin{aligned}[[e_1\, e_7]\, e_5] + [[e_7\, e_5]\, e_1] + [[e_5\, e_1]\, e_7] &= 2[e_4\, e_5] - 2[e_2\, e_1] + 2[e_6\, e_7] \\ &= -12e_3. \end{aligned} \qquad (3)$$

We need to find out whether a factor of 12 is present in all special calculations. We have

$$\begin{aligned}[[e_3\, e_7]\, e_1] + [[e_7\, e_1]\, e_3] + [[e_1\, e_3]\, e_7] &= 2[e_6\, e_1] - 2[e_4\, e_3] - 2[e_2\, e_7] \\ &= 12e_5 \end{aligned} \qquad (4)$$

and

$$\begin{aligned}[[e_5\, e_7]\, e_3] + [[e_7\, e_3]\, e_5] + [[e_3\, e_5]\, e_7] &= 2[e_2\, e_3] - 2[e_6\, e_5] - 2[e_4\, e_7] \\ &= 12e_1. \end{aligned} \qquad (5)$$

In order to create a viable Lie bracket a possible solution is to take Lie brackets including equations (2) to (5) (mod 12). We can also use a quotient subgroup, that is, (mod 2), (mod 3), (mod 4) or (mod 6) applied to the (mod 12) group.

Where novanions do not contain octonions, all nodes mutually connect as quaternions, which are representable by matrices, and equation (1) applies. Where novanions have octonion components, which can occur as overrides, then at least equations (2) to (5) hold. This also applies to vulcannions, which are made of quaternion T-junctions, possibly with a dangling quaternion triangle, the latter implying the existence of equations of type (2) to (5). □

The question can be raised as to whether there is any proper extension of Lie brackets so that these Zargon brackets are viable generalisations without the (mod 12) restriction. We might note that quaternions, being representable by matrices, satisfy a Lie bracket algebra. Quaternions have three quaternionic components which are not scalars, and Lie brackets consist of 3 terms. An idea might be that for Vulcan number v, the number of terms in a Vulcan bracket might also be v. This could be tried directly for the octonions, with Vulcan number 7, and indeed for the octonions we can supply a proof using the observations on octonion brackets just made, and this follows from the observation that of the four equations (2) to (5), the last three give $-12e_3$, $12e_5$ and $12e_1$, whereas the Lie bracket in (2) combines just these three e_3, e_5 and e_1 terms to give $-12e_7$, and when applied in combination all terms cancel to zero, but we would prefer a direct method that is applicable to the general case. □

We relate our discussion on quaternions to Eli Cartan's *The theory of spinors* [Ca66] where he claims a bijection between spinors and the relativistic (Dirac) equation of the electron which maps to quaternions. We will find these spinors do not map to quaternions and we cannot adjoin spinors to quaternions in a matrix formalism, because the dimension of this combination is greater than 4, so by a result [Ad15], the algebra cannot be a matrix one, and is nonassociative.

Cartan introduces spinors via the observable coefficients of vectors with zero norm

$$x_1^2 + x_2^2 + x_3^2 = 0 \tag{1}$$

satisfied by two numbers h_0, h_1 given by

$$x_1 = h_0^2 - h_1^2$$
$$x_2 = i(h_0^2 + h_1^2)$$
$$x_3 = -2h_0h_1,$$

with possible solutions

$$h_0 = \pm\sqrt{\frac{x_1 - ix_2}{2}}, \tag{2}$$

$$h_1 = \pm\sqrt{\frac{-x_1 - ix_2}{2}}. \tag{3}$$

Now choose a vector dependent only on the coordinates x_1 and x_2, say $x_1 - ix_2$. If this is rotated about the axis x_3 to a vector $e^{i\theta}(x_1 - ix_2)$, then h_0 is transformed to $e^{i\theta/2}h_0$, and likewise h_1 is transformed to $e^{i\theta/2}h_1$. Then when $\theta = 2\pi$, a complete rotation, x_3 is transformed to $-x_3$. Thus the spinors h_0 and h_1 are *locally fermionic*.

He considers in the notation of [Ad15] the matrix basis

$$\phi_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad -i_i = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\phi_1^2 = (-i_i)^2 = \alpha_1^2 = 1_1,\ \phi_1(-i_i) = -(-i_i\phi_1),\ -i_i\alpha_1 = -\alpha_1(-i_i),\ \alpha_1\phi_1 = -\phi_1\alpha_1,$$

which he states are related to the quaternions on multiplication by -1_i

$$I_1 = -1_i\phi_1,\ I_2 = -1_i(-i_i),\ I_3 = -1_i\alpha_1. \tag{4}$$

This is closely related to the basis already given by us in [Ad15], 1_1, α_i, i_1 and ϕ_i, which matches (4) except for minus signs whose square is 1 (it is only possible to change the basis left- or right-handedness by this means). From it Cartan is able to deduce the quaternion algebra

$$I_1^2 = I_2^2 = I_3^2 = -1,\ I_1I_2 = -I_2I_1,\ I_2I_3 = -I_3I_2,\ I_3I_1 = -I_1I_3, \tag{5}$$

which he relates to the Dirac equation. □

However we must note that the transformations derived in (2) and (3) involve complex numbers, which are commutative, whereas equation (5) involves quaternions which are noncommutative. Thus there is no equation as given by Cartan of this type. ☐

Just as quaternions may be glued under rotations to give an even number of turns at right angles to the axis of rotation, which we call globally bosonic structure, or can be glued under rotations to give an odd number of twists, which we call a globally fermionic structure, so too can octonions and n-novanions. The analogous situation in the novanion algebra is that combined bosonic and fermionic structures may reside in the same n-novanion. ☐

Since the n-novanions contain the quaternions or the octonions, and the vulcannions also contain quaternions, it follows that the arguments of [Ad15] chapter III on Wedderburn's little theorem apply also to such subalgebras, and therefore

Theorem 4.17.1. *Any finite positive dimensional zargon division ring is commutative*.

Thus these division rings cannot in general be finite, although they may be discrete. Discrete rings without division can be bounded from below by finite elements ignoring the sign, for example by zero, and otherwise be infinite. If a ring is bounded from above and below by the mapping of, say, plus infinity to an integer zargonion part 1 and minus infinity to -1, then an infinite example can be given over real components. ☐

4.18. Tharlonions and tribbles.

In our discussion of intricate numbers in chapter III, section 3, we introduced the idea of an intricate number J with three possible values

$$J^2 = -1, \tag{1}$$

$$J^2 = 0 \tag{2}$$

and

$$J^2 = 1, \tag{3}$$

where we gave intricate representations of these in a standard parameterised form.

The zargon algebras we have met satisfy the property, both for vulcannions and novanions, that as well as a time-like, or scalar, component they possess space-like, or imaginary, components of type (1).

We will extend these ideas to introduce space-like components satisfying (2) embedded in what we will call tribbles, and components satisfying equation (3) embedded in what we call tharl algebras.

For a tribble

$$t = a1 + b_1j_1 + \ldots + b_kj_k, \tag{4}$$

its conjugate is

$$t^* = a1 - b_1j_1 - \ldots - b_kj_k, \tag{5}$$

for which

$$j_m^2 = 0 \tag{6}$$

and

$$j_mj_n = -j_nj_m, \tag{7}$$

with $1 \leq m$ and $n \leq k$, so this implies

$$tt^* = a^2, \tag{8}$$

and the inverse when it exists satisfies

$$t(t^{-1}) = t\left(\frac{t^*}{a^2}\right) = 1$$

so that

$$t^{-1} = \frac{t^*}{a^2} \qquad (9)$$

Thus a tribble algebra satisfies the conditions of a zargon algebra, provided $a \neq 0$. ☐

For a tharlonion with components

$$T = a1 + b_1j_1 + \ldots + b_kj_k, \qquad (10)$$

its conjugate is

$$T^* = a1 - b_1j_1 - \ldots - b_kj_k, \qquad (11)$$

with

$$j_m^2 = 1 \qquad (12)$$

and

$$j_mj_n = -j_nj_m, \qquad (13)$$

which implies

$$TT^* = a^2 - b_1^2 - \ldots - b_k^2. \qquad (14)$$

Since a^2, b_1^2, …, b_k^2 are real, it follows that unless $a = 0$ or all $b_1, \ldots, b_k = 0$, there exist values of TT^* with

$$TT^* = 0, \qquad (15)$$

and thus in general there is no

$$(T^{-1}) = T\left(\frac{T^*}{TT^*}\right) = 1, \qquad (16)$$

so that any tharl algebra in which (15) is permissible is not a zargon algebra, although we say that tharl vulcannions and tharl novannions, defined so that equation (3) replaces (1), are tharl rings, which ignore the division property of a field in their definitions. ☐

We note that in section 4.15 we came across tharl rings, in studying an instance of zargon rings with negative dimension, where our interpretation, which is merely a choice of names, was that what we have called here space components, was there described as scalar time components with special noncommutative properties.

As mentioned in the works *Investigations into universal physics* [Ad18a] and *Elementary universal physics* [Ad18b] by Graham Ennis and me, the form (14) describes a relativistic line element in k dimensions. This line element may be considered to have submanifolds with curved topologies. This is a feature of a possible interpretation system describing gravitation in general relativity, but we think that if this approach to physics is a good one that zargon relativity is the correct model.

Condition (16) for no inverse T^{-1} only holds for a field. We have introduced the idea in *Superexponential algebra*, volume I, of division by multizeros, which occurs in a different mathematical context to a field, that of a zero algebra. These do not satisfy $-(-1) = 1$. The extension of our discussion to zero algebras is developed in volume II of *Number, space and logic*. ☐

4.19. The origin of the discrepancy of our results with K theory.

The work of J.F. Adams *On the nonexistence of elements of Hopf invariant one* has as a result the claim that there are no division algebras of dimension higher than the octonions. In the discovery of the n-vulcannnions, we have refuted this claim. What is the origin of this discrepancy?

Jack Adams gave two proofs of his result. The first uses Steenrod squares, and the second, developed with M.F. Atiyah, uses K theory. K theory has many applications in mathematics, including Bott periodicity, which defines homotopy for Lie groups. In particular, we seem to be challenging the homotopy classification of n-dimensional spheres. Our reasoning appears to be correct, so this is a serious breach of the consensus on current understanding.

The work on Hopf invariant one uses Steenrod squares, which have the properties claimed by José Adem in his thesis. The K theory uses group theory up to the then-known study of the classification of Lie groups. In particular, although the results of José Adem and the K theory results use group theory up to the sympletic groups, which contain matrices with quaternion coefficients, these theories do not relate to the further classification of simple groups, then under study as a separate area of research, which led up to the discovery, for example, of the monster simple group. Does the group theory of the Adem relations and of K theory need to be extended to include group theory for the sporadic simple groups?

In fact, a major thesis of our work, is that it does. We will see in chapter V on zargon groups that zargonions are intimately connected with the classification of simple groups, including but not limited to the monster.

It thus appears that the results using the Adem relations and K theory to prove that there do not exist division algebras beyond the octonions are entirely circular. If we extend the theory, we cannot reach this conclusion.