CHAPTER III

Superexponential structures

3.1. Introduction.

We seek to introduce a notation for superexponetiation, sometimes called hyperoperations.

We will extend the operations +, which we will write as ${}^{1}\Uparrow$ and speak as “one up”, $\times$ written as ${}^{2}\Uparrow$ and pronounced “two up”, exponentiation $\uparrow$ as in $a \uparrow b$ more usually written as a^{b}, and written with ${}^{3}\Uparrow$, and a general nth superexponentiation operation ${}^{n}\Uparrow$.

Usual notation	Superexponential notation
$a + b$	$a\ {}^{1}\Uparrow b$
$ab = a \times b$	$a\ {}^{2}\Uparrow b$
$a^{b} = a \uparrow b$	$a\ {}^{3}\Uparrow b$
area of sphere = $4\pi r^{2}$	area of sphere = $4\ {}^{2}\Uparrow \pi\ {}^{2}\Uparrow (r\ {}^{3}\Uparrow 2)$

We note the following points.

The nth superexponentiation operation generates an (n + 1)th operation by induction. Then

$$a + a + a \ldots + a \text{ (m terms)} = am$$

$$(\ldots((a \uparrow a) \uparrow a) \ldots) \text{ (m terms)} = a\ {}^{4}\Uparrow m,$$

so that, for instance for +, given by ${}^{1}\Uparrow$

$$(\ldots((a\ {}^{1}\Uparrow a)\ {}^{1}\Uparrow a) \ldots) \text{ (m terms)} = a\ {}^{2}\Uparrow m,$$

a general case being

$$(\ldots((a\ {}^{n}\Uparrow a)\ {}^{n}\Uparrow a) \ldots) \text{ (m terms)} = a\ {}^{n+1}\Uparrow m.$$

3.2. $\mathbb{D}$w superexponential structures.

For n > 2 in the simplified version the exponential operations for superexponential $\mathbb{D}$w algebras we specify as satisfying the rules for a field and

$$(a^{i\lambda})\Uparrow^{n} b = (a^{\lambda})\Uparrow^{n} ib,$$

$$(a^{i\lambda})\Uparrow^{n} ib = (a^{iw\lambda})\Uparrow^{n} b,$$

and for left nesting

$$(a^{\lambda})\ {}^{n}\Uparrow b = a^{(\lambda < n\text{-}1 \Uparrow b)},$$

$$(a^{i\lambda})\ {}^{n}\Uparrow b = a^{i(\lambda < n\text{-}1 \Uparrow b)},$$

$$(a^{\lambda})\ {}^{n}\Uparrow ib = a^{i(\lambda < n\text{-}1 \Uparrow b)},$$

$$(a^{i\lambda})\ {}^{n}\Uparrow ib = a^{iw'(\lambda < n\text{-}1 \Uparrow b)},$$

where in general $w \neq w'$, and these may be complex numbers. ☐

For matrices in the simplified intricate representation we employ the following assumptions.

(1) The binomial theorem applies. This means an intricate expression in $\mathcal{JAF}$ format

$$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^{(h + j\mathcal{J})}$$

is evaluated as

$$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^{h}.(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^{j\mathcal{J}},$$

where a, b, c, d, h and j $\in \mathbb{U}$.

(2) We emphasise that the upper component enclosed in brackets, $(h + j\mathcal{J})$, is formed by converting to intricate $\mathcal{JAF}$ format specifically for $\mathcal{J}$, and the lower term in $\mathcal{JAF}$ format, being $(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})$, includes the *same* term $\mathcal{J}$.

This is because for intricate i, α, ϕ

$$a^{p1 + (qi + r\alpha + s\phi)t} \neq a^{p1}.a^{qti}.a^{rt\alpha}.a^{st\phi},$$

but with $\mathcal{J}^2 = (qi + r\alpha + s\phi)^2 = (-q^2 + r^2 + s^2) = \pm 1$ or 0,

$$a^{p1 + t\mathcal{J}} = a^{p1}.a^{t\mathcal{J}}.$$

(3) We form the 'lower algebra' evaluation of $\mathcal{JAF}$ exponentials:

$$\mathcal{J}^{\mathcal{J}} = \mathcal{J},\ \mathcal{A}^{\mathcal{J}} = \mathcal{A} \text{ and } \mathcal{F}^{\mathcal{J}} = \mathcal{F}.$$

Once chosen, this evaluation is unique, including for intricate terms like

$$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})\uparrow[(f + g\mathcal{J})\uparrow(h + k\mathcal{J})]. \ \square$$

We can generalise these features, not only hyperintricately. A matrix w_{jk} with j, k = 1 to 4 can be defined so that

$$[\Sigma_i\, a_i\, J_i]\uparrow[\Sigma_i\, a'_j\, J_j] = \Pi_k[(\Sigma_i\, a_i\, J_i)\uparrow(w_{jk}a'_jJ_j)]$$

where in the intricate case J_i and J_j vary over 1, $\mathcal{J}$, $\mathcal{A}$, $\mathcal{F}$. We expect when j = 1 that $w_{jk} = 1$. The w_{jk} may be expressed and related dependently by supervariety relations.

This can be extended further to a general format where ↑ is replaced by the superexponential operator ${}^n\Uparrow$ or $\Uparrow^n$, and matrix operations are replaced by matrix superoperations. $\square$

3.3. Supernorms and branching.

Topologically, we wish to evaluate the size, or supernorm, of an explosion, and the number of ways the structure branches.

Definition 3.3.1. For a real number j, a *supernorm* of a $\mathbb{D}$w superexponential structure $e\ {}^n\Uparrow j$, or respectively $e\Uparrow^n j$, consists of its real value.

Definition 3.3.2. For a real number j and intricate basis element $\mathcal{J}$, let the component of a superexponential structure be evaluated as $e\ {}^n\Uparrow j\mathcal{J}$ or respectively $e\Uparrow^n j\mathcal{J}$. Then the *left*, or respectively *right, branch number* is the number of distinct values of this evaluation.

3.4. The number Λ.

To take an example, for the exponential operation we can form

$$a^1 = a,$$

but in a conventional exponential algebra, for a ≠ 1 there is no

$$1^b = a.$$

What we do is introduce an irreducible number Λ so that

$$1^{\Lambda a} = a.$$

This number has been introduced culturally in an analogous way to our introduction of division by multizeros in the zero algebras of *Superexponential algebra*, volume I, chapter III. The idea can be extended to superexponential operations.

An interesting feature of Λ is that it is greater than all transnatural ordinals, and thus gives a computable model for transcendence, and therefore in a proper sense, for real numbers.

3.5. Superexponential derivatives.

For addition in a field we have met what we will call a neutral element 0, satisfying

$$a + 0 = a = 0 + a,$$

and for multiplication a neutral element 1 with

$$a \times 1 = a = 1 \times a.$$

For exponentiation, this is not commutative, so the left neutral element v and the right neutral element w differ:

$$a \uparrow w = a \uparrow 1 = a$$

but

$$v \uparrow a = ((a \uparrow (1/a)) \uparrow a) = a,$$

so that the left neutral element v is $(a \uparrow (1/a))$, and the right neutral element w is 1.

There is the question of the value of the expression 0^0. For a field, 0^{-1} is not defined, and therefore neither is $0^1.0^{-1} = 0^0$, but for a zero algebra

$$(a0)^0 = (a0)^1(a0)^{-1} = 1.$$

A left neutral element, v_{an}, under a superexponential operation $<n\Uparrow$ satisfies

$$v_{an}\ {}^{n}\Uparrow\ a = a,$$

where v_{an} is in general dependent on a, and a right neutral element for the superexponential operation $<n\Uparrow$ given by w_{an} has

$$a\ {}^{n}\Uparrow\ w_{an} = a,$$

with similar cases for $\Uparrow n>$.

Then even in the nonassociative and noncommutative case we define the superexponential operation as satisfying a contravariant (order reversing) operation on a left inverse element $a_L^{n\sim}$

$$a\ {}^{n}\Uparrow\ a_L^{n\sim} = v_{an},$$

and for a right inverse $a_R^{n\sim}$

$$a_R^{n\sim}\ {}^{n}\Uparrow\ a = w_{an}. \ \square$$

The difference operator acting on a function f(x) for fields is the expression

$$\frac{f(x + \delta) - f(x)}{\delta},$$

and the commutative and associative differentiable operator for fields is defined by

$$\lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta},$$

where we define this to be the evaluation firstly of the numerator divided by the denominator, then all terms varying with $\delta \neq 0$ are suppressed. “Evaluation” here includes equating all δ / δ to 1, and “suppress”, which follows evaluate, includes setting all terms containing δ in positive powers to zero. If terms with δ in negative powers are set to zero, the derivative then specifies its convergent part.

For a superexponential operator $<n\Uparrow$, with the limit tending to the left neutral element its analogue is

$$\lim_{\delta \to v_{\delta\, n\text{-}1}} [f(x <n{-}1\Uparrow \delta) <n{-}1\Uparrow f(x)_L^{n-1\sim}] <n\Uparrow\delta_L^{n\sim}, \tag{1}$$

and for the limit tending to the right neutral element, the derivative is

$$\lim_{\delta \to w_{\delta\, n\text{-}1}} \delta_R^{n\sim}\Uparrow n>[f(x)_R^{n-1\sim}\Uparrow n{-}1>f(x\Uparrow n{-}1> \delta)]. \tag{2}$$

For example we will look at the function $f(x) = x^x$ in the case n = 3. Then

$$f(x\delta) \,/\, f(x) = \frac{(x\delta)^{x\delta}}{x^x} = \frac{x^{x\delta}\delta^{x\delta}}{x^x}.$$

It is clear that in the limit $\delta \to 1$ this evaluates to 1. Thus in this example the left derivative given by equation (1) for $f(x) = x^x$ is 1. For the right derivative of equation (2) we have

$$\lim_{\delta \to 1} (1^{1/\delta}) = 1,$$

and thus in this case the left and right superexponential derivatives are the same.

A similar argument for the function $f(x) = x$ or $f(x) = 1$ gives a superexponential $n = 3$ derivative of 1. Thus in these cases of superexponentiation the derivative is trivial.

As a second example, let us now choose $n = 4$. For the neutral elements

$$v_{\delta 3} = (\delta \uparrow (1/\delta)),$$

and

$$w_{\delta 3} = 1,$$

whereas if $\delta = 1$

$$v_{\delta 4} = 1,$$

it is less than 1 if the absolute value of δ is less than 1 and is greater than 1 if the absolute value of δ is greater than 1, and

$$w_{\delta 4} = 1.$$

Then

$$\delta\ {}^{2}\!\Uparrow 1 = \delta \times 1 = \delta = 1 \times \delta = 1 \Uparrow^{2} \delta,$$

giving

$$\delta_L^{2\sim} = (1/\,\delta) = \delta_R^{2\sim},$$

and $\delta_L^{3\sim}$ satisfies

$$\delta\ {}^{3}\!\Uparrow \delta_L^{3\sim} = \delta \uparrow \delta_L^{3\sim} = v_{\delta 3} = (\delta \uparrow (1/\delta)),$$

and in a similar way

$$\delta_R^{3\sim} \Uparrow^{3} \delta = w_{\delta 3} = 1,$$

with

$$\delta_R^{3\sim} = 1 \uparrow (1/\delta).$$

Thus for instance the differential of equation (1) is evaluated in the case $n = 4$ as

$$\lim_{\delta \to \delta \uparrow (1/\delta)} \big(f(x{\uparrow}\delta) \uparrow -f(x)\big) {<}4{\Uparrow}\delta_L^{4\sim},$$

and this is non-trivial. ☐

Matrices $A = a_{ik}$ and $B = b_{ik}$ satisfy

$$A + B = a_{ik} + b_{ik}$$

and

$$AB = \textstyle\sum_j a_{ij} b_{jk}.$$

As in chapter XVII of *Superexponential algrbra*, similarly we will say for n-superexponential operations that they satisfy

$$A <\!m\!\Uparrow B = a_{ik} <\!m\!\Uparrow b_{ik},$$

for $m < n$ and

$$A <\!n\!\Uparrow B = <\!n{-}1\!\Uparrow_j(a_{ij} <\!n\!\Uparrow b_{jk})$$

where $<\!n{-}1\!\Uparrow_j$ indicates that the operation $<\!n{-}1\!\Uparrow$ combines a_{ij} and b_{jk} in sequence over all values of j. ☐

It is now possible to consider matrix superexponential differentiation.

3.6. Nonassociative superexponential representations.

To construct a method of looking at superexponential operations so that their nonassociative features can be dealt with in a more familiar way, in order to do this note that (a $^n\Uparrow$ b) $^n\Uparrow$ d and c $^n\Uparrow$ (e $^n\Uparrow$ f) are examples of the more symmetrical (a $^n\Uparrow$ b) $^n\Uparrow$ (e $^n\Uparrow$ f) = c $^n\Uparrow$ d. We describe this representation as canonical form. If we treat in the first case a $^n\Uparrow$ b as a mapping $G_a \times G_b \rightarrow G_c$ and the second case as $G_e \times G_f \rightarrow G_d$, then the total mapping is $G_c \times G_d \rightarrow G_h$. If the number of elements of G_x is $n(G_x)$, the mappings in the last case can be represented by a superexponential table of $n(G_c)$ by $n(G_d) = n(G_c)n(G_d)$ elements.

This superexponential object can be extended. Firstly, all operations on variables may be expressed in terms of their $^n\Uparrow$ operators in the above form. Then treating this object as a new variable, it can be extended to a new canonical form involving the $\Uparrow^n$ operator. Finally, each occurrence of the variable n in this construction can be applied to the operators $^{n-1}\Uparrow$ and $\Uparrow^{n-1}$, and inductively.

We can then apply superexponential algorithms to determine the structure of supervariety objects. Thus the variable x_k and the operators $^k\Uparrow$ and $\Uparrow^k$ for k = {1, 2, ... , n} can be used to define superexponential functions

$$k \rightarrow f(k)$$

which determine the values of the x_k under $^k\Uparrow$ and $\Uparrow^k$.

Of course, we do not have the standard identity as we do in group theory, for example $1 \uparrow a = 1 = a \uparrow 0$, but apart from this we do have a group multiplication table. If we adjoin to the elements a, b etc. the irreducible number Λ then we do have

$$a \uparrow 1 = a = 1 \uparrow \Lambda a, \qquad (1)$$

so in this circumstance we have introduced an operation for $\uparrow$ extended to Λ operations when this acts on 1.

By the strict transfer principle $1 \uparrow a\boldsymbol{\Omega}_{\mathbb{M}t} = 1$. We have introduced ladder algebra so far only for fields, so we might wish to find another proof involving exponentiation. But $1 \uparrow c = 1$ for $c \in \mathbb{M}_{t+1}$ where c is greater than any number in $\mathbb{M}_t$, so the result follows for $\mathbb{M}_t$. Since $1 \uparrow \Lambda a$ is greater than this, the introduction of Λ creates a higher infinity than can be obtained from ladder algebra. We will assume the Λ algebra satisfies the additive and multiplicative axioms for a field including Λ as a symbol within it. Thus we adjoin Λ to the algebra in the same sort of way we introduce i as a symbol for complex numbers. The axioms for exponentiation now include equation (1), but are otherwise standard. It may happen that expressions like $b \uparrow \Lambda a$ cannot be reduced to a simpler type of expression, and remain irreducible.

The superexponential table in general corresponds to 4^m parenthesis arrangements. Writing (identity) for the superexponential identity object, a $^n\Uparrow$ b can be given as four types: a $^n\Uparrow$ b, (identity) $^n\Uparrow$ b, a $^n\Uparrow$ (identity′) and (identity) $^n\Uparrow$ (identity′). This evaluation of a $^n\Uparrow$ b can be represented by a 2×2 matrix, and more generally expressions can be represented by $2^m \times 2^m$ matrices, which can be given a hyperintricate representation.

There is a programme in mathematics to replace it by generalised transformations operating on generalised objects, such as groups, known as category theory. Just as we can introduce a differential structure on polynomials, but the reverse operation, integration, over unspecified limits introduces an arbitrary constant, so the transformational description of objects given in category theory loses some of the information on objects, that is, we no longer keep the information on what the transformation is from. Our point of view is that transformations are transformations of states. The transformations then behave properly when the number of states is full, they may also behave properly when the number of states is empty, but when the number of states remaining is less than empty this may introduce problems, and the transformational structure may break down.

In category theory a good example, that is an example which has all the features of the general case, becomes a ‘universal’, described by a set of maps. Sets, which are particular types of mathematical objects, can be given a more general categorical description in terms of mappings, called a topos. A general idea is that sets can be given an ordering, for instance as a partially ordered set, or poset. Dual maps reversing the direction of mapping arrows can be given. Equalisers, a fancy name for the zeros of a function, where this function can be described as the difference between two functions, when these two functions become equal, can be used to define the values of supervarieties. It can be shown that good examples which we have called universals exist for mappings in the special case of set theory, given by the Yoneda mapping. The noncommutative properties of ordered groups can be expressed in more general mapping terms as ‘adjoint functors’, where a functor is a mapping of categories. Integration and differentiation of these ordered groups can be introduced as ‘Kan extensions’, etc.

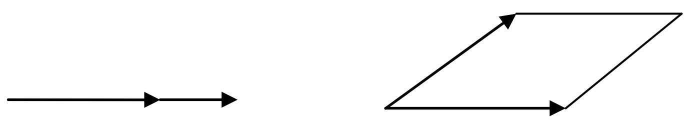
This above discussion means that superexponential operations can be expressed in terms of Λ category theory, which drops the associative axiom, and for computations introduces an operation in the canonical form described. Superexponentiated objects of arbitrary degree can be represented by a binary object. This binary object is a mapping of mappings (a functor). There exists an interpretation system in which Λ category theory is represented by graphs, indeed it defines them, since there is a mapping from canonical form in ${}^{n}\upuparrows$ or $\upuparrows^{n}$ to ${}^{n-1}\upuparrows$ or respectively $\upuparrows^{n-1}$, so that an iteration of such maps reaches multiplicative and additive theory describing the metric structure of a graph. So all the features of ‘abstract nonsense’ (category theory) can be used. $\square$

3.7. Superexponential substructures and singularities.

For an associative structure represented by an $m \times m$ matrix M, we have seen that there exist extensions to supervarieties. Looking at their additive and multiplicative parts, we note that there exist matrices K derived from M and determinants, or hypervolumes, L of M, satisfying

$$MK = L.$$

When $L = 0$, a singularity occurs. This may be interpreted as the hypervolume of the matrix M defined by its row or column vectors contains linear dependencies between these vectors, which we can show in the diagrams



linearly dependent linearly independent

so that in the case of 2-space, only the linearly independent vectors define a nonzero area, and therefore a nonzero determinant, expressed as saying that the matrix is nonsingular.

There are two distinct types of instance when the hypervolume is zero. The first occurs when the linearly dependent vectors define a sum which is the zero vector. This corresponds to the normal interpretation of a singularity. The second is when the space defined by the vectors is of lower dimension than the matrix M, but not the zero vector. This interpretation gives a structure to the singularity not available to the first type.

For an n-dimensional space, there may be more than a decrement of one dimension to get a set of linearly independent vectors in that space. This can be found, since a 1-space, or scalar, is trivially linearly independent unless it is zero.

The extension to supervarieties is that a singularity occurs when there is a linear dependency between its subobjects.

We note that in chapter II we have described the Euler characteristic for branched spaces as a polynomial. The natural extension is to define a superexponential Euler characteristic by a supervariety, and this forms a superbranched space.

Thus we have two models for superbranced spaces, the first is in topological terms as a branched space with a superexponential Euler characteristic, and the second is of a space where the supervariety defines a metric, or measure of distance, on the space.

When we defined explosions, we had the space branching everywhere. This is extended to superbranched spaces. What happens is that superbranched spaces may have singularities. The metric, or distance, in this space is then zero at the singularity. We have the option now of defining the singularity so that it is a metrical subobject where a linear dependency between its subobjects has been found, and where for a lower dimension the subobjects are independent, or otherwise topologically it is a superbranched space of lower dimension. □