CHAPTER II

The meaning of branched spaces

2.1. Introduction.

After describing the Euler-Poincaré characteristic, we discuss the Gauss-Bonnet theorem, in which we show how a squared norm may also be implemented in hyperbolic spaces. General stereographic projection is considered.

2.2. The Euler-Poincaré characteristic.

As mentioned in [AH35] and [BLW86] the number of vertex points – edges + areas

$\chi = P - E + A$

as an invariant of a simplical decomposition of a polyhedron – that is, the space is divided up into polygons and simplexes of higher dimension, was first put in an equivalent form by Descartes. χ itself, the Euler-Poincaré characteristic of a manifold, was discovered by Euler [Eu1752]. It is the number of vertices, minus the number of edges, plus the number of faces, etc., as an alternating sum, and describes an invariant of the space – provided the topological shape remains the same – as a sphere or torus, etc. An example is a surface of a cube, which has 8 vertices, 12 edges and 6 faces, so its Euler-Poincaré characteristic is $\chi = 8 - 12 + 6 = 2$, and this is a topological invariant which describes a 2-sphere, in which the vertices, edges and faces can be embedded, and gives the same Euler-Poincaré characteristic for a surface of a tetrahedron (a triangular pyramid) with 4 vertices, 6 edges and 4 faces: $\chi = 4 - 6 + 4 = 2$.

It would be impossible to situate Riemann except in the middle of a long tradition, yet the paper [Ri1851] which defines the *genus*, $g = 1 - \chi/2$ for a surface (this is the number of handles), is often taken as the starting point of our subject. The idea of connectivity given there was then extended to higher dimensions by Betti [Be1871].

The paper of Poincaré on Analysis Situs, and the five supplements to it has been translated into English by John Stillwell [Po10]. The work that developed in topology up to the mid 1930's was vast, particularly in Germany. For a bibliography of this period the reader could consult [ST80]. Of note is Herman Weyl's work on Riemannian surfaces [We47], and Emmy Noether, who further developed the idea of homology groups [No83].

There are 3 main ways to describe the Euler-Poincaré characteristic.

(i) $\chi = \Sigma_i(-1)^i a_i$,
where a_i is the number of i-dimensional faces.

(ii) From discussing the genus, $\chi = \Sigma_i(-1)^i p_i$,
where p_i are the Betti numbers of the space, defining the i-dimensional connectivity, for example as handles. The p_i are as defined by Poincaré, not Betti.

(iii) χ = number of pits – number of passes + number of peaks of a surface as studied by Cayley [Ca1859]. This can be generalised to an n-dimensional manifold by considering a height function of the manifold immersed in $\mathbb{U}^{2n}$.

To generalise and specialise at the same time, the Euler characteristic of a n-surface becomes expandable in two sorts of ways as a hyperintricate polynomial in degree n with variables x, additively as the Euler characteristic

$$\chi^{+} + (-x)^n = \sum_{i=1}^{n} a_i(-x)^i \tag{1}$$

where $a_n = 1$, and multiplicatively as the characteristic $\chi^{\times}$ given by the product

$$\chi^{\times} + (-x)^n = \prod_{i=1}^{n} (b_i - x), \tag{2}$$

in which these two representations are equivalent

$$\chi^{+} = \chi^{\times}. \tag{3}$$

The Euler characteristic is obtained from equations (1) and (2) by putting $x = 1$.

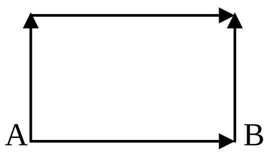
To give an example, the square is related to the polynomial $\{2 - x\}^2$, and the cube has a mapping to $\{2 - x\}^3$, in particular the coefficients in the binomial expansion of $\{2 - x\}^2$ give respectively the number of vertex points, edges and areas of the square, and the coefficients in the binomial expansion of $\{2 - x\}^3$ give the number of vertex points, edges, areas and volumes of the cube. This fits in with a description of the cylinder mapped to $\{1 - x\}\{2 - x\}$, and to the torus, mapped to $\{1 - x\}^2$.

We then introduce *branched spaces* via the polynomial $\{m - x\}^n$, where to begin with m is a whole number for n-dimensional such spaces. For example, the 3-branched cube maps on to the polynomial $\{3 - x\}^3$, and has 27 vertex points, 27 edges, 9 areas and 1 volume.

Non-oriented manifolds were introduced by Möbius [Mö1887], and first systematically classified by von Dyck [Dy1882], [Dy1885], [Dy1890]. The value of χ for these is obtained by putting the coefficients b_i negative in formula (2). An important idea is the representation of branched Möbius strips. The standard Möbius strip is given by

$$(-1 - x)(2 - x)$$

in which we are denoting the twist by the presence of a minus sign: $(-1 - x)$. The standard cylinder is represented by



in which vertices on both edges of A and B are identified and A and B are identified. Thus its Euler characteristic of $(1 - x)(2 - x)$ tells us that the cylinder has one area, 3 edges and two vertices, as is obtained under this identification. For the Möbius strip, the vector at B is in the opposite direction, so A and B under vector identification cancel. Because of the twist, there is now only one edge. The rule is, under gluing add the edge vectors and subtract one of them. Then $(-1 - x)(2 - x)$ gives one area, one edge and two points. The generalisation to branched spaces is clear, so that all positive and negative values are admissible.

A Klein bottle is now represented by $\chi = -(1 + x)(1 - x)$, and $\chi = -(2 + x)(2 - x)$ by a pair of oppositely oriented edges derived from the diagram above.

We can consider rational, algebraic, transcendental and complex numbers of hypervolumes, volumes, areas, edges and points. More abstractly, we can consider matrices and more general objects. Where the numbers are ladder numbers and the coefficients are ordinal infinites, we describe the branched space as an *explosion*, and the case where the coefficients are infinitesimals, as an *implosion*.

The spaces we have so far been considering are not the most general. Firstly we have considered so far only one variable, x. This may be expanded to a variety in a number of variables. The generalisation we consider in chapter IV is a superexponential variety.

We introduce as examples a conceptual model in the 1-dimensional case of what is meant by branched lines and points, describing this by what is known as generalized 'Dedekind cuts', and in the 2-dimensional case provide a model of a branched square.

We discuss branched deformation retracts, branched orientation, n-branched surgery with h-handles and h-crosscaps, and that $\partial\partial = 0$ can fail for k-*explosions*.

2.3. The familiar square, cylinder, torus and cube.

The number of points, number of edges and area of a square, each with sign given by the Euler characteristic χ, are related by

$$\chi = P - E + A$$

for P the number of vertex points, E the number of edges and A the number of areas, and these are given in sequence by the coefficients of

$$\{2 - x\}^2 = 4 - 4x + x^2,$$

so P = 4, -E = -4 and A = 1.

For a cylinder, formed when two opposite edges and two opposite points of a square are identified, the values of P, -E and A are given in sequence by the coefficients of

$$\{1 - x\}\{2 - x\} = 2 - 3x + x^2.$$

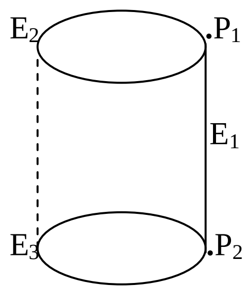


Figure. The familiar torus with one vertex point, two edges and one area is obtained from the familiar cylinder shown, by gluing the top and bottom edges E_2 and E_3 together, so P_1 and P_2 coincide.

For a torus, where I identify equally oriented edges E_2 and E_3 and points P_1 and P_2 above, P, -E and A are given in sequence by the coefficients of

$$\{1 - x\}^2 = 1 - 2x + x^2.$$

For a cube, the volume, and the values of P, -E and A are given by the coefficients of

$$\{2 - x\}^3 = 8 - 12x + 6x^2 - x^3.$$

For the cube with two opposite faces identified, and the two sets of 4 points of those square faces identified, these are given by the coefficients of

$$\{1 - x\}\{2 - x\}^2 = 4 - 8x + 5x^2 - x^3,$$

with two sets of two opposite faces identified as

$$\{1 - x\}^2\{2 - x\} = 2 - 5x + 4x^2 - x^3,$$

etc., and for a 4-dimensional hypercube, by the coefficients of

$$\{2 - x\}^4.$$

2.4. Branched spaces.

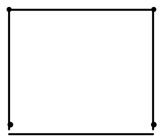
I now generalise this idea to *branched* spaces. A 3-branched space, for example a 3-branched square, has P, -E and A given by the coefficients of $\{3 - x\}^2$. It therefore has:

9 points 6 sides and 1 area.

The reader will with difficulty develop a visual model for this topology, but the idea is as consistent as $\{3 - x\}^2$, and I provide a model example at the end of this chapter. Recall that imaginary numbers were first thort of as not describing the 'real' world.

The question then arises, how do I compute the number of points etc., of a general branched simplex?

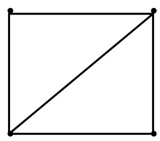
Consider a familiar square and a 3-branched square



?

4 points, 4 edges, 1 area 9 points, 6 edges, 1 area

I can triangulate the familiar square by forming a diagonal



I now have 4 points, 5 edges and 2 areas.

So I have for the untriangulated familiar square topology

$$\chi = 4 - 4 + 1 = 1,$$

for the cylinder

$$\chi = 2 - 3 + 1 = 0$$

and for the torus

$$\chi = 1 - 2 + 1 = 0.$$

These values of χ are invariant under a change of triangulation that maintains the topological shape. Can I assume the same for branched simplexification?

If I do, then the branched Euler characteristics are

$\{3 - x\}^2$: $\chi = 9 - 6 + 1 = 4$,
$\{1 - x\}\{2 - x\}$: $\chi = 3 - 4 + 1 = 0$,
$\{2 - x\}\{3 - x\}$: $\chi = 6 - 5 + 1 = 2$,
and $\{3 - x\}^3$: $\chi = 27 - 27 + 9 - 1 = 8$, etc.

Suppose for $\{3 - x\}^3$ I add one edge, but keep the number of points constant. Then I must create an extra area to keep χ the same. I can always add points and increment the number of edges correspondingly. Inductively, for any dimension I can add a hyper-area and add a hyper-edge whilst retaining χ invariant.

I note that for $\{3 + x\}^2$, (with a *plus* sign) if I add a hyper-area I must subtract a hyper-edge to retain χ invariance, likewise for $\{k + x\}^n$, k a complex number.

For complex hyper-volumes, take the example of adding a *semi-point*, say half a point, then the addition of the corresponding compensating semi-edge must be adjusted to leave χ invariant.

2.5. Models for branched lines and areas.

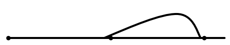
I now provide model examples of branched spaces, firstly in one dimension.

To begin with, consider {1 – x}, which represents a circle. If the circle consists of real numbers, then a ‘Dedekind cut’ – a removal of one point – leaves the resulting ‘manifold’ in one piece.

If I consider {2 – x}, representing a real line with two end points, each end point of which is connected in only one way with the rest of the interval – in other words the line is a *closed* interval, then removal of an interior point leaves the resulting manifold in two pieces.

Now look at {3 – x}. I consider three end points, each end point of which is connected in only one way with the rest of the interval, so by analogy with the previous case I will call this interval again closed. Then a Dedekind cut – the removal of one interior point – leaves the resulting manifold in *three* pieces. Thus a branched line represented by {n – x} with n a variable, under removal of a unique interior point, divides the line into n pieces. Normally, if the point were not unique, there would be more than n ends. An alternative is that the branched line is considered *affine*, so that always the removal of the first selected point (so that the axiom of choice is restricted to a first selection) divides the line into n pieces and there are n ends. These are extended meanings of line or ‘edge’.

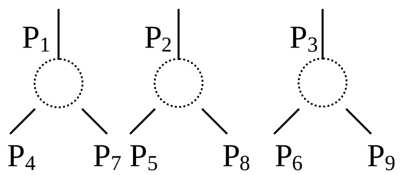
The sequence of points in this interval can be reconnected in its interior, for example:



where I have shown three such points. All such points can be reconnected in this way.

If there are no interior reconnections, so that all the points are connected in an expanding tree, I call the resulting analogue of a real number interval an *explosion*.

The next model example, of a 3-branched square, was first developed by Doly García. All sets of interior edges except for one are reconnected, or the space is affine. I represent 3 sets of ‘3 vertex points and one edge’ as follows:



I then connect vertex point P_1 with an ‘edge’ to simultaneously P_2 and P_3, then P_4 with an edge to P_5 and P_6, and P_7 to P_8 and P_9, making 9 points, 6 edges and 1 area. The closed end points are here connected as a 2-branch.

Further we note that removing an edge from the 3-branched square reduces the dimension by one – the area dimension disappears. Reversibly, in the process of adding an edge, the number of areas is increased by one, thus retaining the Euler characteristic.

2.6. Deformation retracts and orientation.

Our definition of the Euler characteristic, χ, of a familiar m-dimensional hypercube, given as the sum of the coefficients of $(2 - x)^m$, corresponds with its assignation as a deformation retract with ends two (m – 1)-dimensional copies of a hypercube.

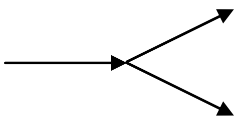
It is possible to amalgamate these two copies by gluing at both ends an opposite retract, with glued retracts corresponding to two orientation types – with the same or reversed orientation.

The question then arises whether this definition in terms of a deformation retract is extendable to branched spaces. It is. We treat this as a global phenomenon, where a localization of this is visible at the ends of the retract. Since our philosophy is that the retract is built out of objects which are not necessarily real numbers, the question is evident as to how the localization is manifest in the interior of the retract. In terms of connectivity a branched retract is isomorphic to the inverse operation of what we previously called a generalized Dedekind cut.

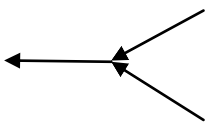
Consider $(n - x)^m$ for a branched space. We will call n the *branched root* and m the *branched degree*. We will show the branched root for an n-edge differs in general from the number of orientations of the n-edge.

To give an example, consider the García diagram for a 3-branched square given above. Its 1-dimensional subobjects are the 3-edges of which we have displayed 3.

The orientation of each 3-edge may be represented by



which can be subject to a threefold rotation or a reflection. The opposite orientation, which is a dual map, may be represented by



again with a threefold rotational symmetry, or combined with a reflection about the horizontal axis.

If we consider the reflections as equivalent orientations then the total number orientations for a 3-edge is 6. The three 3-edges in the García diagram are free to have each of the six possibilities.

If we select a set of these, then the orientation of a new connection between P_1, P_2 and P_3 etc. to flow as a continuation in the same direction, is fixed. Thus P_1 is connected to three possibilities: P_2, P_5 and P_8, then to a further 3 possibilities: P_3, P_6 and P_9, making 9 possible connections with P_1.

The number of connections with P_2 is then reduced, since, say, P_1 has been already selected with P_2, making two possibilities with P_5 and P_8, and again 2 possibilities with, say, P_6 and P_9.

Finally, P_3 has only one set of connections available. Thus the number of orientations for a 3-branched square is $3x6 + 3^2 + 2^2 + 1^2 = 32$.

For a line segment, the number of orientations corresponds with its number of end-points. We have seen this is not the case for an n-edge. Thus what in former considerations was isomorphic has become distinct.

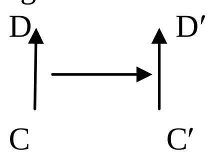
2.7. Branched handles, crosscaps and surgery of branched spaces.

In order to understand how we can extend the idea of gluing handles and Möbius strips to holes in 2-branched spaces (manifolds), to encompass n-branched spaces, we need a formulation that is first of all compatible with our previous considerations relating χ from $(2 – x)^3$, topologically a ball, to χ for $(1 – x)^3$, a handlebody – which can be pictured as a torus in classical 3-space swept out and reconnected along a fourth dimension, and likewise a disk, $(2 – x)^2$, to a handle, $(1 – x)^2$.

In our model, an n-branched object will be called *closed* when its boundary (of say vertices) is present, and *open* when it is absent.

There are two basic modes of construction we can perform. The first is, having been provided with a ready-made n-object with boundary, to identify parts of this boundary, possibly via other objects. The second is to perform surgery to remove a number of n-object copies and then glue other derived n-objects. To do this we need a concept of the interior of an n-object, and in order to introduce this, it will be useful to describe the abutment of n-objects to create an extended n-object. We discuss to begin with the first of these ideas, then for a 3-branched object we are interested in surgery involving u = 1 and 2.

To generate a 2-branched torus from a 2-branched square

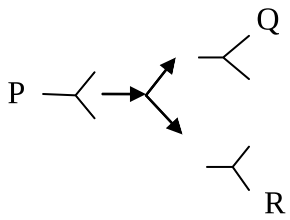


identify C and C′, D and D′ along the entirety of the retracts CD and C′D′, and then identify at the C, C′, D and D′ boundaries the oriented 2-edges CC′ and DD′.

To generate a Möbius strip, identify C and D′, D and C′ along the entirety of the retracts CD and C′D′.

For the formation of the 3-branched square we have inserted and connected three more 3-edges from those at the vertices of P, Q and R shown below, where the arrows are the retract.

There are six 3-edges.



Now identify the three 3-edges P, Q and R, corresponding to the initial retract, which are to be amalgamated at the retraction of their vertices and to a common 3-edge. If we allocate these vertices in the order they are connected by the remaining three 3-edges, and then amalgamate these remaining three 3-edges, this is the 3-branched torus.

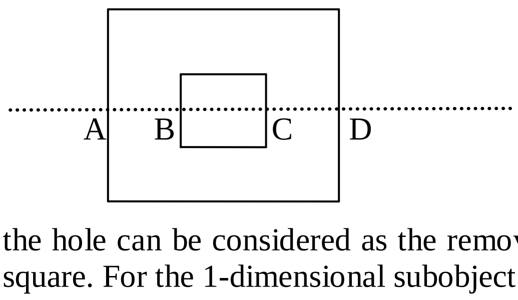
If the 3-edges, P, Q and R, are amalgamated at their vertices in an order that is different than the initial retract, this is a 3-branched Möbius strip. The amalgamation can be cyclic, in which case there is one boundary, or a swap, in which case there are two boundaries, one corresponding to the swap and one corresponding to the identity.

In general the amalgamation is given by the group of permutations on n objects, called the symmetric group, and the number of boundaries is equal to the number of cycles, including individual retract identities.

The 3-branched retract we have been considering has an (m – 1)-dimensional 3-object on the left and (3^m – 1) 3-objects on the right. To form an abutment of these (3^m – 1) 3-objects on the right, for each of these amalgamate a 3-object on the left associated with (3^m – 1) 3-objects on the right. Then for k such iterated abutments, there will exist $(3^m - 1)^{k+1}$ 3-objects each of dimension (m – 1) on the right.

It is possible to form h – 1 further copies of this abutted object and amalgamate the retracted part of the boundaries of the h versions. *Object* A will leave the left hand of these retracts unamalgamated.

In order to deal with surgery, we first need to explore its simplest instances. For the square with a hole



the hole can be considered as the removal of a subobject of the same type as the containing square. For the 1-dimensional subobject given by the horizontal line we can also consider this as three retracts (synonymous in this case with two abutments) given by

A —— B C—— D

the (point) retract AB, the surgery subobject BC, and the retract CD. The subobject classifier here is defined as Boolean. Extensions beyond the Boolean are given in chapter XII. Then as probabilities AB and CD map to τ, or *certain*, and BC to υ as *impossible*.

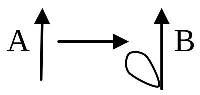
There are possible three types of horizontal line in the above diagram, as above the hole, where the standard retract holds, intersecting with the hole, as given, and below the hole. Correspondingly there are 3 vertical lines, under the designation of the squares as Cartesian products.

For a 3-square, consider 2 further abutments. Let surgery be performed, represented along a *horizontal* 3-edge, as an allocation of a τ or υ classifier, and a τ classifier above and below. To allow this existence of the unimpeded retract above and below, form two further *vertical* abutments of the already abutted object. Then the new object has an interior hole which is classified by υ as four 3-branched squares, two for each of the horizontal and vertical assignments.

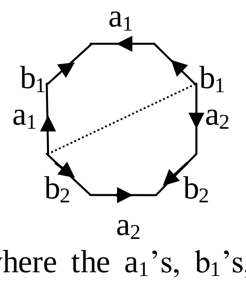
That there are *two* such assignments, horizontal and vertical, follows from the *two* pairs of three 3-edges for each 3-branched square.

The classification of derived objects can be developed further. We have mentioned only object A. Interior holes as already described can be glued to objects of type A. For a normal handle, it would not seem reasonable to glue h copies of a cylinder to a single hole. The conceptual model of n-branched spaces liberates us from that constraint.

A construction which generates from the 2-branched square a 2-branched torus with an extra handle is shown below.

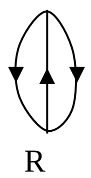


We have obtained the 2-branched square, by inserting and connecting two more 2-edges from those at the vertices of A and B, where the horizontal arrow is the retract, and produced a torus from this, except for the loop shown at the bottom vertex of B, by identifying the four 2-edges in pairs with the A retracted to B pair matched. The loop may be detached at the base, but is reconnected under the identification of the base vertex of B with the base vertex of A. This now forms a hole in the torus, which can be glued to the hole of a copy of that torus with a hole. This torus with one handle, that is, a sphere with two handles, can be represented by the diagram below



where the a_1's, b_1's, a_2's and b_2's are identified by gluing in matched directions and the identified hole is given by the dashed line.

Analogously, for a 3-branched torus with one 3-branched handle, we need the equivalent of a loop. This is shown below as a self-attached 3-edge,



so the 3-branched loop is all reconnected at R.

To the 3-branched torus, now attach a 3-branched loop at R. The loop can be detached at R, but is reconnected under the identification of vertices for the 3-branched torus, so it forms a 3-branched hole. To form a 3-branched h-handle, identify by gluing onto the 3-branched hole h copies of this 3-branched torus with hole.

Thus a 3-branched torus with hole can be identified by gluing with a 3-branched Möbius strip. This is the crosscap construction for 2-branched spaces to produce non-orientable manifolds.

The constructions we have mentioned can be extended in a natural manner to n-branched spaces. If we were to follow Steenrod, the n-branched m-sphere identifies the boundary of an n-branched m-hypercube to a point [St51].

2.8. Explosion boundaries.

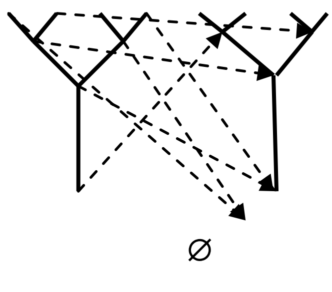
The analogue of a real line is an *explosion*, for which we include next a discussion in terms of explosion analysis. Note that, say, a 3-explosion has an infinity of ends, which themselves may be uncountable and be reassembled to form a manifold in the usual sense. Thus we open ourselves to the possibility of a triple boundary $\partial\partial\partial = 0$, more generally of a k-branched explosion with boundary a k – 1 branched explosion, so $\partial^k = 0$. Here is an example of $\partial^k = 0$ and $\partial^{k-1} \neq 0$, $k > 2$.

Let И be a k-explosion, k > 2. Consider a real interval (2-explosion), R, within it. Let there be a metric on this real line, and let the total length of the interval be t. For each point p_i of R, select a further 2-explosion not belonging to R except at p_i, with length $u(p_i)$ from all p_i. Let the end point of this line be at q_i. Then the boundary of R is the end-points of R together with all q_i.

For each p_i, $p_j \in$R with distance interval t_{ij}, consider an *induced metric* on q_i, q_j with length also t_{ij}. Then the boundary of R includes the q_i, and the q_i have induced the structure of a real line, which itself will have two boundary points, the boundary of which is zero. Thus if this is the only line selected $\partial\partial\partial = 0$, but $\partial\partial \neq 0$.

2.9. Trees and amalgams.

The retract structure we have developed may be described in the finite case by finite trees [Se00]. Reconnecting nodes to other nodes may then be represented by a mapping of trees, which is itself a graph, where the diagram shows such a mapping from tree T to T, some arrows being to the empty set.



This mapping may itself be a multifunction, that is, described by a tree mapping. We can also discuss trees with a finite, or an infinite countable or uncountable number of nodes.

2.10. The general polynomial.

If Π indicates multiplication from i = 1 to m, the branched spaces given by the coefficients of an mth degree polynomial are represented by

$$\prod_{i=1}^{m}\{n_i - x\},$$

This assignation is consistent with the idea of a topos in which its morphisms lie in a category, in particular when the union of an element and a negative element is the initial object. Further examples are Grothendieck groups [Ro84].

We now extend the idea of the branched representation where we had x, n and m at most as complex numbers, to x, n and m *matrices*.

We have detailed in other papers [Ad12b], [Ad12c] the hyperintricate representation of matrices, in which the complex numbers occur as subobjects of intricate numbers – representable by 2 x 2 real matrices. In this formalism, the branched representation now becomes expandable in terms of a hyperintricate polynomial.

Thus having described the branched Euler characteristic in terms of a polynomial, questions of polynomial representations arise, even hyperintricately. In particular, we now derive a polynomial isomorphism, g, between the additive part of the characteristic

$$\chi_E^{+} = \sum_{i=1}^{k} a_i x^i$$

with $a_k = 1$, and the multiplicative part of the characteristic $\chi_E^{\times}$ given by the product

$$\chi_E^{\times} = \prod_{i=1}^{k}(b_i - x).$$

where g is the bijective map

$$\chi_E^{+} \leftrightarrow \chi_E^{\times}.$$

We call the global Euler characteristic that global value χ_E^{+} or $\chi_E^{\times}$ obtained additively by cutting and pasting objects described locally by χ_E^{+} or $\chi_E^{\times}$.

2.11. The Gauss-Bonnet theorem.

A space X is *simply-connected* if and only if it is path-connected, and whenever p: [0,1] → X and q: [0,1] → X are two paths given by continuous maps with the same start and endpoint, then p can be continuously deformed to get q while keeping the endpoints fixed. Thus for any two given points in X, there is one and only one type of path connecting them.

Shorn of sophisticated description, a Riemannian manifold is an n-dimensional space given by a local quadratic metric

$$\sum_{j,k} g_{jk} dx_j dx_k = g_{00}dx_0^2 + g_{01}dx_0dx_1 + g_{10}dx_1dx_0 + \ldots$$

The negative derivative of the unit vector field which is normal, that is, at right angles, to a surface is called the shape operator (or Weingarten map or second fundamental tensor). The shape operator is a curvature of a submanifold of a manifold which depends on its particular embedding, also called the extrinsic curvature, and the Gaussian curvature is given by the determinant of it, and does not depend on the embedding.

For a two-dimensional Riemannian manifold without boundary, the integral of the Gaussian curvature over the entire manifold with respect to area is 2π times the Euler characteristic of the manifold. This is the Gauss-Bonnet theorem. It is interesting that the total Gaussian curvature is differential-geometric in character, but the Euler characteristic is topological and does not depend on differential geometry. So if we distort the surface and change the curvature at any location, regardless of how we do it, the same total curvature is maintained.

2.12. Embeddings and projections.

Chapter IV defines a superexponential variety. Reduced to the case of a matrix variety, a typical example could be

$$aX^3Y + bX^2YZ^5 + cZ = 0,$$

where X, Y and Z are matrices and a, b, c are coefficients. In this section we will deal with quadratic forms where the matrices and coefficients are reduced to complex variables.

Chapter XII, section 2, of *Superexponential algebra* proved Sylvester's law of inertia, which states that the signature, or difference between positive and negative terms, derived from the quadratic form $\sum x_j b_{jk} x_k$, where b_{jk} is symmetric, is invariant under any similarity transformation to

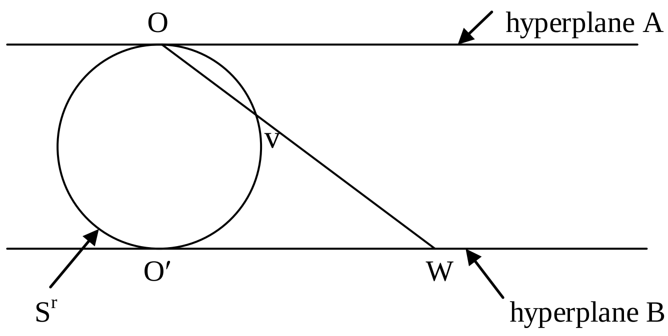
$$x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_n^2. \qquad (1)$$

We have already seen such forms in the norm squared of an intricate number, and as an everywhere positive squared norm for a quaternion, octonion, or otherwise n-novanion. The determinant can be defined from the norm squared, but from the n-hyperintricate number $Я_n$ multiplied by its conjugate $Я_n^*$ to form $\det Я_n$, then if the norm squared is quadratic and n > 1 there is cancellation of common factors between $Я_n^*$ and $\det Я_n$.

A classical case we wish to consider is stereographic projection from a sphere to a complex plane, which we will generalise to an r-sphere mapped to an r-hyperplane. A sphere has n = r in equation (1) above, and we wish to generalise this to stereographic projection when n > r, so this corresponds to hyperbolic geometry, and we will look at this.

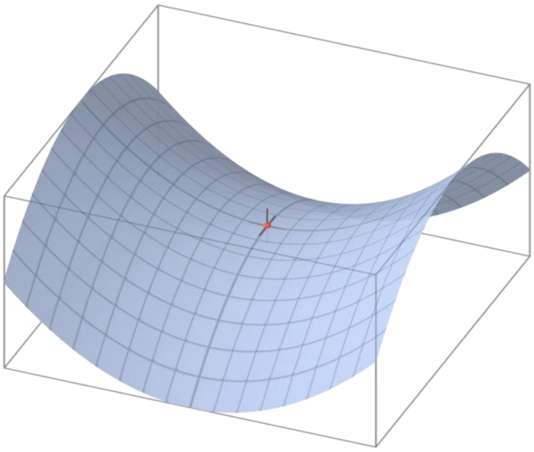
Let S^r be the r-sphere. The r-component of numbers with quadratic norms of the form (1), represented by some points on S^r, may be mapped to the tangent hyperplanes A and B below. Let the stereographic map from B, at W in the diagram, to S^r be p^{-1}.

We extend the stereographic projection map p to the mapping including the punctured point O in S^r to the codomain given by hyperplane A, tangent at O. This hyperplane belongs to an *indeterminate zero algebra* [Ad14], chapter II, defined by 0/0 and which differs from the usual extension to a 'point at infinity'. Alternatively O is at c0 for a *determinate zero algebra* and cЧ is ultrainfinity.



Near O, an infinitesimal $h \notin p^{-1}(A)$, and $h \notin p^{-1}(B)$ provided all elements of $B < 1/h$, but if B contains ordinal infinities, $h \in p^{-1}(B)$. ☐

[Wikipedia] *Hyperbolic space* is a homogeneous space that has a constant negative curvature. It is hyperbolic geometry in more than 2 dimensions, and is distinguished from Euclidean spaces with zero curvature that defines the Euclidean geometry, and elliptic geometry with a constant positive curvature.



When embedded to a Euclidean space (of a higher dimension), every point of a hyperbolic space is a saddle point. Another distinctive property is the amount of space covered by the n-ball in hyperbolic n-space: it increases exponentially with respect to the radius of the ball for large radii, rather than as a polynomial.

Hyperbolic n-space, denoted $\mathbb{H}^n$, is a maximally symmetric, simply connected, n-dimensional Riemannian manifold with a constant negative curvature. Hyperbolic space is a space exhibiting hyperbolic geometry. It is the negative-curvature analogue of the n-sphere. Hyperbolic 2-space, $\mathbb{H}^2$, is also called the hyperbolic plane. Although hyperbolic space $\mathbb{H}^n$ is diffeomorphic to $\mathbb{U}^n$, its negative-curvature metric gives it very different geometric properties.

We will now discuss models of hyperbolic space.

Hyperbolic space, developed independently by Nikolai Lobachevsky and János Bolyai, is a geometrical space analogous to Euclidean space, but such that Euclid's parallel postulate is no longer assumed to hold. Instead, the parallel postulate is replaced by the following alternative in two dimensions:

- Given any line L and point P not on L, there are at least two distinct lines passing through P which do not intersect L.

It is then a theorem that there are infinitely many such lines through P. This axiom still does not uniquely characterise the hyperbolic plane up to isometry. There is an extra constant, the curvature $K < 0$, which must be specified. However, it does uniquely characterise it up to bijections which only change the notion of distance by an overall constant, called *homothety*. By choosing an appropriate length scale, we can thus assume, without loss of generality, that $K = -1$.

Models of hyperbolic spaces that can be embedded in a flat, for example Euclidean, space may be constructed. In particular, the existence of model spaces implies that the parallel postulate is logically independent of the other axioms of Euclidean geometry.

There are several important models of hyperbolic space: the *Klein model*, the *hyperboloid model*, the *Poincaré ball model* and the *Poincaré half space model*. These all model the same geometry in the sense that any two of them can be related by a transformation that preserves all the geometrical properties of the space, including isometry (though not with respect to the metric of a Euclidean embedding). An isometry is a transformation which maps elements to the same or another metric space such that the distance between the image elements in the new metric space is equal to the distance between the elements in the original metric space.

(i) The hyperboloid model.

The hyperboloid model realises hyperbolic space as a hyperboloid in $\mathbb{U}^{n+1} = \{(x_0, ..., x_n): x_i \in \mathbb{U}, i = 0, 1, ..., n\}$. The hyperboloid is the locus $\mathbb{H}^n$ of points whose coordinates satisfy

$$x_0^2 - x_1^2 - ... - x_n^2 = 1, x_0 > 0.$$

In this model a *line* (or geodesic) is the curve formed by the intersection of $\mathbb{H}^n$ with a plane through the origin in $\mathbb{U}^{n+1}$.

The hyperboloid model is closely related to the geometry of Minkowski space. The quadratic form

$$Q(x) = x_0^2 - x_1^2 - ... - x_n^2,$$

which defines the hyperboloid, polarises to give the bilinear form

$$\begin{aligned} B(x, y) &= [Q(x + y) - Q(x) - Q(y)]/2 \\ &= x_0y_0 - x_1y_1 - ... - x_ny_n, \end{aligned}$$

The space $\mathbb{U}^{n+1}$, equipped with the bilinear form B, is an (n + 1)-dimensional Minkowski space $\mathbb{U}^{n,1}$.

One can associate a *distance* on the hyperboloid model by defining the distance between two points x and y on $\mathbb{H}$ to be

$$d(x, y) = \operatorname{arcosh} B(x, y).$$

This function satisfies the axioms of a metric space. It is preserved by the action of the Lorentz group on $\mathbb{U}^{n,1}$. Hence the Lorentz group acts as a transformation group preserving isometry on $\mathbb{H}^n$. ☐

(ii) The Klein model.

An alternative model of hyperbolic geometry is on a certain domain in projective space. The Minkowski quadratic form Q defines a subset $V^n \subset \mathbb{U}\mathbf{P}^n$ given as the locus of points for which $Q(x) > 0$ in the homogeneous coordinates x. The domain V^n is the *Klein model* of hyperbolic space.

The lines of this model are the open line segments of the ambient projective space which lie in V^n. The distance between two points x and y in V^n is defined by

$$d(x, y) = \operatorname{arcosh}\left(\frac{B(x,y)}{\sqrt{Q(x)Q(y)}}\right).$$

This is well-defined on projective space, since the ratio under the inverse hyperbolic cosine is homogeneous of degree 0.

This model is related to the hyperboloid model as follows. Each point $x \in V^n$ corresponds to a line L_x through the origin in $\mathbb{U}^{n+1}$, by the definition of projective space. This line intersects the hyperboloid $\mathbb{H}^n$ in a unique point. Conversely, through any point on $\mathbb{H}^n$, there passes a unique line through the origin (which is a point in the projective space). This correspondence defines a bijection between V^n and $\mathbb{H}^n$. It is an isometry, since evaluating d(x,y) along Q(x) = Q(y) = 1 reproduces the definition of the distance given for the hyperboloid model. ☐

(iii) The Poincaré ball model.

A closely related pair of models of hyperbolic geometry are the Poincaré ball and Poincaré half-space models.

The ball model comes from a stereographic projection of the hyperboloid in $\mathbb{U}^{n+1}$ onto the hyperplane $\{x_0 = 0\}$. In detail, let S be the point in $\mathbb{U}^{n,1}$ with coordinates (−1, 0, 0, ..., 0): the *south pole* for the stereographic projection. For each point P on the hyperboloid $\mathbb{H}^n$, let P^* be the unique point of intersection of the line SP with the plane $\{x_0 = 0\}$.

This establishes a bijective mapping of $\mathbb{H}^n$ into the unit ball

$$B^n = \{(x_1, ..., x_n): x_1^2 + ... + x_n^2 < 1\}$$

in the plane $\{x_0 = 0\}$.

The geodesics in this model are semicircles that are perpendicular to the boundary sphere of B^n. Isometries of the ball are generated by spherical inversion in hyperspheres perpendicular to the boundary. ☐

(iv) The Poincaré half-space model.

The half-space model results from applying inversion in a circle with centre a boundary point of the Poincaré ball model B^n above and a radius of twice the radius.

This sends circles to circles and lines, and is moreover a conformal transformation. Consequently, the geodesics of the half-space model are lines and circles perpendicular to the boundary hyperplane. ☐

Considering hyperbolic manifolds, every complete, connected, simply connected manifold of constant negative curvature -1 is isometric to the real hyperbolic space $\mathbb{H}^n$. As a result, the universal cover of any closed manifold M of constant negative curvature -1, which is to say, a hyperbolic manifold, is $\mathbb{H}^n$. Thus, every such M can be written as $\mathbb{H}^n/\Gamma$ where Γ is a finite discrete group of isometries on $\mathbb{H}^n$. That is, Γ is a lattice in $SO^+(n, 1)$.

For Riemann surfaces, two-dimensional hyperbolic surfaces can also be understood according to the language of Riemann surfaces. According to the uniformisation theorem, every Riemann surface is elliptic, parabolic or hyperbolic. Most hyperbolic surfaces have a non-trivial fundamental group $\pi_1 = \Gamma$, where the groups that arise this way are known as Fuchsian groups. The quotient space $\mathbb{H}^2/\Gamma$ of the upper half-plane modulo the fundamental group is known as the Fuchsian model of the hyperbolic surface. The Poincaré half plane is also hyperbolic, but is simply connected and noncompact. It is the universal cover of the other hyperbolic surfaces.

The analogous construction for three-dimensional hyperbolic surfaces is the Kleinian model.