CHAPTER II

The meaning of branched spaces

2.1. Introduction.

We repeat the axioms for a *vector space* given in [Ad15], Volume I, chapter II, and give the definition of its generalisation for a *module*, together with examples.

Topology uses *open sets*, an example of which is the real line interval]0, 1[, with the points 0 and 1 removed, and *closed sets*, an example of which is the closed real line interval [0, 1], with the points 0 and 1 included. Set theory is related to logic. We describe two examples of logic. Untwisted logic maps sets to oriented manifolds, for example a plane, whereas twisted logic maps sets to antioriented manifolds, for example a Möbius strip. Since an antioriented manifold has only one side, in twisted logic the global statement NOT B includes an image of the local statement B.

After describing the Euler-Poincaré characteristic, we extend this idea to branched spaces, where a line is a 2-branched space, and the removal of a point in an n-branched line splits it into n pieces. Descartes introduced an algebraic description of geometry by mapping lines in geometry in the plane to coordinate systems described by pairs of numbers. We can do the same for branched spaces, and consider mappings between subobjects in them.

Chapter II of volume II develops traditional geometries and extends them to branched spaces.

2.2. Vector spaces, block scalar products and modules.

A vector space is a collection of objects, **V**, called *vectors*, denoted in bold type, which can be added together and multiplied by numbers called *scalars*, given in ordinary letters. Scalars can be real numbers, but there can also be scalar multiplication by complex numbers, rational numbers or generally any field. The operations of vector addition and scalar multiplication satisfy the axioms

Axiom	*Meaning*
(1) Associativity of addition	**u** + (**v** + **w**) = (**u** + **v**) + **w**
(2) Commutativity of addition	**u** + **v** = **v** + **u**
(3) Identity element for addition	There exists an element **0** belonging to **V**, called the *zero vector*, so that **v** + $\mathbf{0}_\mathbf{v}$ = **v** for all **v** that belong to **V**.
(4) Inverse elements for addition	For every **v** that belongs to **V**, there exists an element –**v** that belongs to **V** called the additive inverse of **v**, with the property **v** + (–**v**) = $\mathbf{0}_\mathbf{v}$.
(5) Compatibility of scalar multiplication with field multiplication	a(b**v**) = (ab)**v**.
(6) Scalar multiplication identity element	There is a scalar 1 satisfying 1**v** = **v** for all **v**.
(7) Distributivity of scalar multiplication with respect to vector addition	a(**u** + **v**) = a**u** + a**v**.
(8) Distributivity of scalar multiplication with respect to field addition	(a + b)**v** = a**v** + b**v**.

Vectors may be represented by diagrams. The *base* of a vector a $\mathbf{v}$ is the vector $\mathbf{0_v}$, and its *tip* is the vector $\mathbf{v}$ itself.

(1)

We will use, colloquially, the word dimension in what follows, where the technical name is *rank*. We will introduce the definition of this difference now.

Two vectors $\mathbf{u}$, $\mathbf{v} \neq \mathbf{0_v}$ are *linearly independent* if there are no scalars a_u, a_v not both zero satisfying

$$a_u\mathbf{u} + a_v\mathbf{v} = 0. \qquad (2)$$

(3)

$\mathbf{u}$, $\mathbf{v}$ linearly dependent $\qquad$ $\mathbf{u}$, $\mathbf{v}$ linearly independent

This may be generalised to n dimensions, or as we should say, rank n. Vectors $\mathbf{u}_1, ..., \mathbf{u}_k, ..., \mathbf{u}_n \neq \mathbf{0}_{\mathbf{u}k}$ are linearly independent if there are no scalars $a_1, a_2, ... a_n$ not all zero satisfying

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + ... + a_n\mathbf{u}_n = 0. \qquad (4)$$

We say the vectors $\mathbf{u}_1, ..., \mathbf{u}_k, ..., \mathbf{u}_n$ form a *basis* of the vector space. It is usual to consider that this basis is linearly independent, but we may need to say so explicitly, because it is not within the definition. So a selected linearly independent representation of vector space, where all other vectors are represented in terms of sums of this representation, we know as a *basis* for a vector space. Actually, we need to prove that all vectors can be represented by a basis, but we will not be so fussy. If you are not aware of the proof, to do this is an ideal exercise in rigorous mathematical reasoning.

Then the *dimension*, n, of a vector space is the maximum number of its linearly independent vectors. By definition, its *nullity* is its *rank* minus its *dimension*.

Intuitively, a vector space with *base point* is a vector space in which all vectors are attached to the base point $\mathbf{b}$.

Vector spaces may, or may not, have base points, that is, a constant vector $\mathbf{b}$, so that all vectors $\mathbf{v}$ in the vector space are of the form $\mathbf{v} + \mathbf{b}$, and $\mathbf{b}$ is included in $\mathbf{v}$ for all $\mathbf{v}$, where we can extend the inclusion idea to say that $\mathbf{b}$ is a linearly dependent vector for all $\mathbf{v}$. Such a vector $\mathbf{b}$ exists. It could be a selected $\mathbf{0_v}$ for a $\mathbf{v}$, so that the zero vector $\mathbf{0_v}$ is unique for all $\mathbf{v}$, for example 0 in a coordinate system, which we will describe next. Note that two possible ideas for base points are to attach vectors either at their bases or their tips.

Colloquially, a vector is n-dimensional if it is represented by n scalars $(a_1, a_2, ... a_n)$. Although the choice is arbitrary, the representation can be, if it is represented two dimensionally on a page, either a *row vector* shown by the n scalars shown in the row above, or a *column vector*, represented by n scalars in a column, where the fact that it is in a column is represented by $[a_1, a_2, ... a_n]$.

The *scalar product* of two vectors $\mathbf{v} = (c_1, c_2, ..., c_n)$ and $\mathbf{u} = (d_1, d_2, ..., d_n)$, or alternatively $\mathbf{u} = [d_1, d_2, ..., d_n]$, is the scalar value

$$c_1d_1 + c_2d_2 + ... + c_nd_n.$$

When $\mathbf{u} = \mathbf{v}$ this is the size of the vector $\mathbf{v}$, given from Pythagoras's theorem by its distance.

An $m \times n$ *matrix* can be represented by a 2 dimensional *array* of m row vectors $[v_1, v_2, \ldots v_m]$ in a column, or equivalently by n column vectors $(v_1, v_2, \ldots v_n)$ in a row. Each entry, or element, of the array must be represented uniquely by the scalar s_{ij}, where the index i ranges from 1 to m and j ranges from 1 to n.

By these means we can define a scalar product on an even dimensional matrix, either from the scalar product in pairs of vector rows, or the scalar product in pairs of vector columns, and sum the result. But then this depends, apparently on the choice of pairs.

We can, however define a sum of the vectors, either rows or columns, forming a matrix. In the case of vectors in two dimensions, this is the tip of the vector sum (in the case of a parallelogram, with common base point) of the two vectors. This vector sum is unique. The reader might like to show that this does not depend on whether we represent the vectors as rows or columns.

A better and useful choice to form a scalar for a matrix is to form the area of a parallelogram for a two dimensional matrix defined by its vectors, in the general case as the hypervolume of the generalisation to n dimensions of a parallelogram, a parallelepiped. This hypervolume is known as the *determinant* of the matrix, described further in [Ad15], chapter II, section 10.

A matrix is a *square matrix* when m = n.

The *matrix product* two matrices P and Q, where P is an m×h matrix and Q is a h×n matrix is the m×n matrix M, where each m_{ij} entry in M is the scalar product of an ith row vector **p** in P and a jth column vector **q** in Q.

Thus for example, for the 2×2 square matrices

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

$$i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{7}$$

their matrix product satisfies

$$\alpha i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \phi \neq i\alpha = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{8}$$

A n dimensional *block* is an array in n dimensions. So a 3 dimensional block is represented by entries or elements s_{ijk}. A block is sometimes called a *tensor*, but this has connotations with differentiable manifolds, and we will introduce definitions not commonly used with tensor geometry, which locally has a distance described by a scalar product.

For instance, for two blocks $B = b_{ijk}$ and $C = c_{ijk}$ we can introduce the reduced block product

$$b_1(b_{ij})c_1(c_{jk}) + b_2(b_{ij})c_2(c_{jk}) + \ldots + b_n(b_{ij})c_n(c_{jk}), \tag{9}$$

provided we can represent b_h in a consistent way so that

$$b_h(b_{ij}) = b_{hij}. \tag{10}$$

We know that $b_h = 1$ is such a representation, but is the reduced block product then unique, so that it gives the same representation with $b_i = 1$ in b_{hij}?

Proof. The reduced block product consists of terms in multiplications with, b_{hij}, b_{ihj} and b_{ijh}. The order given by ij matters in matrix multiplications, but the b_h term commutes, therefore all three terms above are equal, and since the reduced block product ranges over every value available to it, the reduced block product defined in this way is unique.

But then, we can define the *block scalar product* as the determinant of the reduced block product. Since both the reduced block product and the determinant are unique, this product is unique. Further, we can generalise, so that we descend in a chain of such operations from the product of two blocks of dimension n, to a scalar value. I will leave to the reader the task of defining the various types of ways that block scalar products can be defined from scalar products and determinants.

In a vector space, its scalars are in a field, where the scalars operate on the vectors by scalar multiplication. In a *module*, the scalars need only be in a ring, which does not always have a division operation, and thus allows a wider set of structures. So a module is a generalisation of a vector space. Thus to satisfy examples of modules, we hope we merely have to provide examples of rings.

Some examples of multiplicatively commutative rings are the complex numbers, which we have seen in [Ad15] have an 'intricate representation', defined by the matrices 1, and i given in equation (7). These form a field, since the nonzero complex numbers have a multiplicative inverse for (a + ib) given by $(a - ib)/(a^2 + b^2)$. Other intricate examples of commutative rings are the actual numbers a + αb, where α is defined in equation (6), which do not always have an inverse since if it exists, because $\alpha^2 = 1$, it is $(a - \alpha b)/(a^2 - b^2)$, and the denominator could be zero, and likewise phantom numbers defined by a + ϕb in equation (8), which for the same reason have inverses $(a - \phi b)/(a^2 - b^2)$ when these exist.

A further example, or extension, of these for commutative rings, is a polynomial ring. Indeed, these polynomials could consist, say, of variables with values in a field, or of actual or phantom type. If the coefficients of the polynomial act as symbols which are real, so that they commute with any matrix, then since X^n for a matrix X commutes with these symbols, these polynomials form a ring too.

Polynomials are compatible with differential and integral calculus, so that these ideas can be extended to modules in this case.

2.3. Open and closed sets in twisted and untwisted logic.

Sets S and T define a logic in the propositional calculus, in which the set membership relation for a set

$$s \in S$$

is mapped onto statements with true or false values. The propositional calculus defines NOT, & and OR by truth tables on true and false values for statements, so these mappings include

$$CS \rightarrow \text{NOT } x \in S \quad (1)$$

$$S \cup T \rightarrow x \in S \text{ OR } x \in T \quad (2)$$

$$S \cap T \rightarrow x \in S \,\&\, x \in T. \quad (3)$$

We have given examples of open and closed sets in the introduction. Formally, we can define their properties by axioms. If a set T is open, then its *set complement*, CT, given by elements which are not in T, is closed. Our purpose will be to give rules for open and closed sets that are sensible, irrespective of their conformity or otherwise with standard definitions.

We consider an operation on a set T called the *closure* of T, defined by the mapping

$$T \rightarrow \overline{T}, \quad (4)$$

and say $\overline{T}$ is a *closed* set. We define an *open* set as $C\overline{T}$.

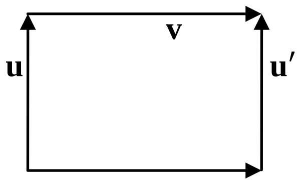
We define the empty set as open. The complement of this set in its set universe is then closed.

Consider sets S and T. The two sets are *separate* if their intersection is the empty set

$$S \cap T = \emptyset. \qquad (5)$$

Sets may not be entirely open or closed. If S is an open set and $\overline{T}$ is a closed set, and they are not separate, nor is one set properly included in the other, then $S \cup \overline{T}$ is neither completely open nor completely closed. We say the resulting set is half-empty and half-closed. I leave to the reader the case of extending this idea to n sets.

A set may be situated in a vector space, defined in section 2. Consider two vectors **u** and **v** without the same base point. We can consider scalars which vary so that the scalar s_x applied to the vector u changes in value x from 0 to 1, possibly continuously if we can define this. Then s_0u is at the base of u and s_1**u** at the tip. We call this path a *retract* along **u**. We can then define a retract of **u** along **v** as a mapping of the base and tip of u so that it begins from the base of **v** and ends with its tip. This retract has defined a rectangle shown below.



(6)

This idea can be extended to give retracts of retracts, and thus define an n-dimensional rectangle. A rectangle is *flat* if, for constant values of the retracts along **u**, every retract along **v** increases monotonically or is constant.

A *local* object L is a set defined within an n-dimensional rectangle, by inclusion within it.

The *boundary* of a rectangle is the closed part of the rectangle without the open subset within it. The *interior* of the rectangle is the union of all open sets included in the closure of the rectangle.

There exist a number of operations which can be done on these rectangles. For instance we can glue the vector **u** with its rectangle retract **u′** so that the base of **u** corresponds with the base of **u′**, and the tip of **u** corresponds with the tip of **u′**. Then we say that the rectangle forms an untwisted manifold, a cylinder. An untwisted manifold is usually called an oriented manifold. Alternatively we can glue **u** and **u′** so the base of **u** corresponds with the tip of **u′**, and the tip of **u** corresponds with the base of **u′**. Then we say the rectangle forms a Möbius strip, which is a twisted manifold, which we will also call an antioriented manifold.

A local object is in one part, so that there are no two subsets within L so that they are separate and the separate sets contain sets which are not in L. The Jordan curve theorem, which we do not prove here, states that in a flat untwisted manifold the boundary of a rectangle, which is the closed part of the rectangle without the open subset within it, divides the manifold into two parts. For a cylinder with two boundary circles, this means the cylinder has two sides. However, the case for a Möbius strip means that it has one side.

We can now define a mapping on a local object to form the closure of its interior, which will be on one side of its surface for an untwisted surface. The complement of this is open, and since the union of the open and closed sets is the whole surface, this local object will divide the surface into two components.

Then for untwisted logic if we make a local statement which maps in propositional calculus to a local set, then for a statement B, the statement NOT B is not included in it. These two statements are separate.

But for twisted logic, because a twisted manifold has globally only one side, if we make a local statement B, an image of this statement locally on the other side is entirely included within NOT B.

Thus the observation long made by some sections of the population that politicians not only use twisted logic, but were the first to discover it, may have some genuine foundations based on a mathematical theorem.

2.4. The Euler-Poincaré characteristic.

As mentioned in [AH35] and [BLW86] the number of vertex points – edges + areas

$$\chi = P - E + A$$

as an invariant of a simplical decomposition of a polyhedron – that is, the space is divided up into polygons and simplexes of higher dimension, was first put in an equivalent form by Descartes. χ itself, the Euler-Poincaré characteristic of a manifold, was discovered by Euler [Eu1752]. It is the number of vertices, minus the number of edges, plus the number of faces, etc., as an alternating sum, and describes an invariant of the space – provided the topological shape remains the same – as a sphere or torus, etc. An example is a surface of a cube, which has 8 vertices, 12 edges and 6 faces, so its Euler-Poincaré characteristic is $\chi = 8 - 12 + 6 = 2$, and this is a topological invariant which describes a 2-sphere, in which the vertices, edges and faces can be embedded, and gives the same Euler-Poincaré characteristic for a surface of a tetrahedron (a triangular pyramid) with 4 vertices, 6 edges and 4 faces: $\chi = 4 - 6 + 4 = 2$.

It would be impossible to situate Riemann except in the middle of a long tradition, yet the paper [Ri1851] which defines the *genus*, $g = 1 - \chi/2$ for a surface (this is the number of handles), is often taken as the starting point of our subject. The idea of connectivity given there was then extended to higher dimensions by Betti [Be1871].

The paper of Poincaré on Analysis Situs, and the five supplements to it has been translated into English by John Stillwell [Po10]. The work that developed in topology up to the mid 1930's was vast, particularly in Germany. For a bibliography of this period the reader could consult [ST80]. Of note is Herman Weyl's work on Riemannian surfaces [We47], and Emmy Noether, who further developed the idea of homology groups [No83].

There are 3 main ways to describe the Euler-Poincaré characteristic.

(i) $\chi = \Sigma_i(-1)^i a_i$,
where a_i is the number of i-dimensional faces.

(ii) From discussing the genus, $\chi = \Sigma_i(-1)^i p_i$,
where p_i are the Betti numbers of the space, defining the i-dimensional connectivity, for example as handles. The p_i are as defined by Poincaré, not Betti.

(iii) χ = number of pits – number of passes + number of peaks of a surface as studied by Cayley [Ca1859]. This can be generalised to an n-dimensional manifold by considering a height function of the manifold immersed in $\mathbb{U}^{2n}$.

To generalise and specialise at the same time, the Euler characteristic of a n-surface becomes expandable in two sorts of ways as a hyperintricate polynomial in degree n with variables x, additively as the Euler characteristic

$$\chi^{+} + (-x)^{n} = \sum_{i=1}^{n} a_i(-x)^{i} \quad (1)$$

where $a_n = 1$, and multiplicatively as the characteristic $\chi^{\times}$ given by the product

$$\chi^{\times} + (-x)^{n} = \prod_{i=1}^{n}(b_i - x), \quad (2)$$

in which these two representations are equivalent

$$\chi^{+} = \chi^{\times}. \quad (3)$$

The Euler characteristic is obtained from equations (1) and (2) by putting x = 1.

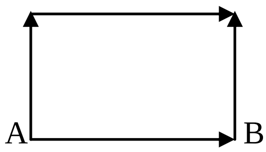
To give an example, the square is related to the polynomial $\{2 - x\}^2$, and the cube has a mapping to $\{2 - x\}^3$, in particular the coefficients in the binomial expansion of $\{2 - x\}^2$ give respectively the number of vertex points, edges and areas of the square, and the coefficients in the binomial expansion of $\{2 - x\}^3$ give the number of vertex points, edges, areas and volumes of the cube. This fits in with a description of the cylinder mapped to $\{1 - x\}\{2 - x\}$, and to the torus, mapped to $\{1 - x\}^2$.

We then introduce *branched spaces* via the polynomial $\{m - x\}^n$, where to begin with m is a whole number for n-dimensional such spaces. For example, the 3-branched cube maps on to the polynomial $\{3 - x\}^3$, and has 27 vertex points, 27 edges, 9 areas and 1 volume.

Non-oriented manifolds were introduced by Möbius [Mö1887], and first systematically classified by von Dyck [Dy1882], [Dy1885], [Dy1890]. The value of χ for these is obtained by putting the coefficients b_i negative in formula (2). An important idea is the representation of branched Möbius strips. The standard Möbius strip is given by

$$(-1 - x)(2 - x)$$

in which we are denoting the twist by the presence of a minus sign: $(-1 - x)$. The standard cylinder is represented by



in which vertices on both edges of A and B are identified and A and B are identified. Thus its Euler characteristic of $(1 - x)(2 - x)$ tells us that the cylinder has one area, 3 edges and two vertices, as is obtained under this identification. For the Möbius strip, the vector at B is in the opposite direction, so A and B under vector identification cancel. Because of the twist, there is now only one edge. The rule is, under gluing add the edge vectors and subtract one of them. Then $(-1 - x)(2 - x)$ gives one area, one edge and two points. The generalisation to branched spaces is clear, so that all positive and negative values are admissible. A Klein bottle is now represented by $\chi = -(1 + x)(1 - x)$, and $\chi = -(2 + x)(2 - x)$ by a pair of oppositely oriented edges derived from the diagram above.

We can consider rational, algebraic, transcendental and complex numbers of hypervolumes, volumes, areas, edges and points. More abstractly, we can consider matrices and more general objects. Where the numbers are ladder numbers and the coefficients are ordinal infinites, we describe the branched space as an *explosion*, and the case where the coefficients are infinitesimals, as an *implosion*. The spaces we have so far been considering are not the most general. Firstly we have considered so far only one variable, x. This may be expanded to a variety in a number of variables. The generalisation we consider in chapter IV is a superexponential variety.

We introduce as examples a conceptual model in the 1-dimensional case of what is meant by branched lines and points, describing this by what is known as generalized 'Dedekind cuts', and in the 2-dimensional case provide a model of a branched square.

We discuss branched deformation retracts, branched orientation, n-branched surgery with h-handles and h-crosscaps, and that $\partial\partial = 0$ can fail for k-*explosions*.

2.5. The familiar square, cylinder, torus and cube.

The number of points, number of edges and area of a square, each with sign given by the Euler characteristic χ, are related by

$$\chi = P - E + A$$

for P the number of vertex points, E the number of edges and A the number of areas, and these are given in sequence by the coefficients of

$$\{2 - x\}^2 = 4 - 4x + x^2,$$

so P = 4, -E = -4 and A = 1.

For a cylinder, formed when two opposite edges and two opposite points of a square are identified, the values of P, -E and A are given in sequence by the coefficients of

$$\{1 - x\}\{2 - x\} = 2 - 3x + x^2.$$

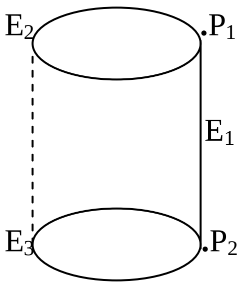


Figure. The familiar torus with one vertex point, two edges and one area is obtained from the familiar cylinder shown, by gluing the top and bottom edges E_2 and E_3 together, so P_1 and P_2 coincide.

For a torus, where I identify equally oriented edges E_2 and E_3 and points P_1 and P_2 above, P, -E and A are given in sequence by the coefficients of

$$\{1 - x\}^2 = 1 - 2x + x^2.$$

For a cube, the volume, and the values of P, -E and A are given by the coefficients of

$$\{2 - x\}^3 = 8 - 12x + 6x^2 - x^3.$$

For the cube with two opposite faces identified, and the two sets of 4 points of those square faces identified, these are given by the coefficients of

$$\{1 - x\}\{2 - x\}^2 = 4 - 8x + 5x^2 - x^3,$$

with two sets of two opposite faces identified as

$$\{1 - x\}^2\{2 - x\} = 2 - 5x + 4x^2 - x^3,$$

etc., and for a 4-dimensional hypercube, by the coefficients of

$$\{2 - x\}^4.$$

2.6. Branched spaces.

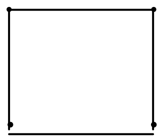
I now generalise this idea to *branched* spaces. A 3-branched space, for example a 3-branched square, has P, -E and A given by the coefficients of $\{3 - x\}^2$. It therefore has:

9 points 6 sides and 1 area.

The reader will with difficulty develop a visual model for this topology, but the idea is as consistent as $\{3 - x\}^2$, and I provide a model example at the end of this chapter. Recall that imaginary numbers were first thort of as not describing the 'real' world.

The question then arises, how do I compute the number of points etc., of a general branched simplex?

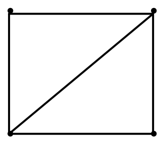
Consider a familiar square and a 3-branched square



?

4 points, 4 edges, 1 area — 9 points, 6 edges, 1 area

I can triangulate the familiar square by forming a diagonal



I now have 4 points, 5 edges and 2 areas.

So I have for the untriangulated familiar square topology

$\chi = 4 - 4 + 1 = 1$,

for the cylinder

$\chi = 2 - 3 + 1 = 0$

and for the torus

$\chi = 1 - 2 + 1 = 0$.

These values of χ are invariant under a change of triangulation that maintains the topological shape. Can I assume the same for branched simplexification?

If I do, then the branched Euler characteristics are

$\{3 - x\}^2$: $\chi = 9 - 6 + 1 = 4$,

$\{1 - x\}\{2 - x\}$: $\chi = 3 - 4 + 1 = 0$,

$\{2 - x\}\{3 - x\}$: $\chi = 6 - 5 + 1 = 2$,

and $\{3 - x\}^3$: $\chi = 27 - 27 + 9 - 1 = 8$, etc.

Suppose for $\{3 - x\}^3$ I add one edge, but keep the number of points constant. Then I must create an extra area to keep χ the same. I can always add points and increment the number of edges correspondingly. Inductively, for any dimension I can add a hyper-area and add a hyper-edge whilst retaining χ invariant.

I note that for $\{3 + x\}^2$, (with a *plus* sign) if I add a hyper-area I must subtract a hyper-edge to retain χ invariance, likewise for $\{k + x\}^n$, k a complex number.

For complex hyper-volumes, take the example of adding a *semi-point*, say half a point, then the addition of the corresponding compensating semi-edge must be adjusted to leave χ invariant.

2.7. Models for branched lines and areas.

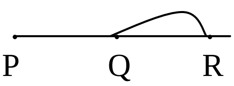
I now provide model examples of branched spaces, firstly in one dimension.

To begin with, consider {1 – x}, which represents a circle. If the circle consists of real numbers, then a 'Dedekind cut' – a removal of one point – leaves the resulting 'manifold' in one piece.

If I consider {2 – x}, representing a real line with two end points, each end point of which is connected in only one way with the rest of the interval – in other words the line is a *closed* interval, then removal of an interior point leaves the resulting manifold in two pieces.

Now look at {3 – x}. I consider three end points, each end point of which is connected in only one way with the rest of the interval, so by analogy with the previous case I will call this interval again closed. Then a Dedekind cut – the removal of one interior point – leaves the resulting manifold in *three* pieces. Thus a branched line represented by {n – x} with n a variable, under removal of a unique interior point, divides the line into n pieces. Normally, if the point were not unique, there would be more than n ends. To bypass this, an alternative is that the branched line is considered *affine* or *relativ*e, so that always the removal of the first selected point (so that the axiom of choice is restricted to a first selection) divides the line into n pieces and there are n ends. These are extended meanings of line or 'edge'.

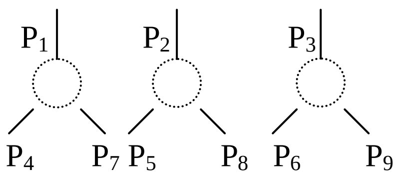
The sequence of points in this interval can be reconnected in its interior, for example:



where I have shown three points, P, Q and R. All such points can be reconnected in this way.

If there are no interior reconnections, so that all the points are connected in an expanding tree, I call the resulting analogue of a real number interval an *explosion*.

The next model example, of a 3-branched square, was first developed by Doly García. All sets of interior edges except for one are reconnected, or the space is affine. I represent 3 sets of '3 vertex points and one edge' as follows:



I then connect vertex point P_1 with an 'edge' to simultaneously P_2 and P_3, then P_4 with an edge to P_5 and P_6, and P_7 to P_8 and P_9, making 9 points, 6 edges and 1 area. The closed end points are here connected as a 2-branch.

Further we note that removing an edge from the 3-branched square reduces the dimension by one – the area dimension disappears. Reversibly, in the process of adding an edge, the number of areas is increased by one, thus retaining the Euler characteristic.

2.8. Deformation retracts and orientation.

Our definition of the Euler characteristic, χ, of a familiar m-dimensional hypercube, given as the sum of the coefficients of $(2 - x)^m$, corresponds with its assignation as a deformation retract with ends two (m – 1)-dimensional copies of a hypercube.

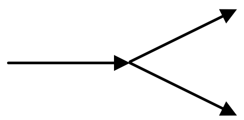
It is possible to amalgamate these two copies by gluing at both ends an opposite retract, with glued retracts corresponding to two orientation types – with the same or reversed orientation.

The question then arises whether this definition of a deformation retract is extendable to branched spaces. It is. We consider this as a global phenomenon, where a localisation of it is visible at the ends of the retract. Since our philosophy is that the retract is built out of objects which are not necessarily real numbers, the question is evident as to how the localisation is manifest in the interior of the retract. In terms of connectivity a branched retract is bijective to the inverse operation of what we previously called a generalised Dedekind cut. We regard this retract not only relatively in terms of connectivity, but also as a state.

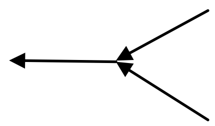
Consider $(n - x)^m$ for a branched space. We will call n the *branched root* and m the *branched degree*. We will show the branched root for an n-edge differs in general from the number of orientations of the n-edge.

To give an example, consider the García diagram for a 3-branched square given above. Its 1-dimensional subobjects are the 3-edges of which we have displayed 3.

The orientation of each 3-edge may be represented by



which can be subject to a threefold rotation or a reflection. The opposite orientation, which is a dual map, may be represented by



again with a threefold rotational symmetry, or combined with a reflection about the horizontal axis.

If we consider the reflections as equivalent orientations then the total number orientations for a 3-edge is 6. The three 3-edges in the García diagram are free to have each of the six possibilities.

If we select a set of these, then the orientation of a new connection between P_1, P_2 and P_3 etc. to flow as a continuation in the same direction, is fixed. Thus P_1 is connected to three possibilities: P_2, P_5 and P_8, then to a further 3 possibilities: P_3, P_6 and P_9, making 9 possible connections with P_1.

The number of connections with P_2 is then reduced, since, say, P_1 has been already selected with P_2, making two possibilities with P_5 and P_8, and again 2 possibilities with, say, P_6 and P_9.

Finally, P_3 has only one set of connections available. Thus the number of orientations for a 3-branched square is $3x6 + 3^2 + 2^2 + 1^2 = 32$.

For a line segment, the number of orientations corresponds with its number of end-points. We have seen this is not the case for an n-edge. Thus what in former considerations was isomorphic has become distinct.

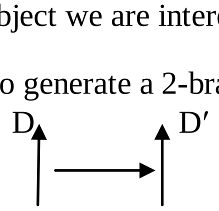
2.9. Branched handles, crosscaps and surgery of branched spaces.

In order to understand how we can extend the idea of gluing handles and Möbius strips to holes in 2-branched spaces (manifolds), to encompass n-branched spaces, we need a formulation that is first of all compatible with our previous considerations relating χ from $(2 – x)^3$, topologically a ball, to χ for $(1 – x)^3$, a handlebody – which can be pictured as a torus in classical 3-space swept out and reconnected along a fourth dimension, and likewise a disk, $(2 – x)^2$, to a handle, $(1 – x)^2$.

In our model, an n-branched object will be called *closed* when its boundary (of say vertices) is present, and *open* when it is absent.

There are two basic modes of construction we can perform. The first is, having been provided with a ready-made n-object with boundary, to identify parts of this boundary, possibly via other objects. The second is to perform surgery to remove a number of n-object copies and then glue other derived n-objects. To do this we need a concept of the interior of an n-object, and in order to introduce this, it will be useful to describe the abutment of n-objects to create an extended n-object. We discuss to begin with the first of these ideas, then for a 3-branched object we are interested in surgery involving u = 1 and 2.

To generate a 2-branched torus from a 2-branched square

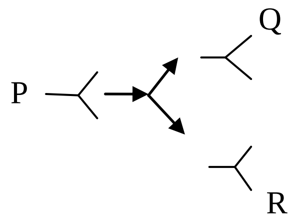


identify C and C′, D and D′ along the entirety of the retracts CD and C′D′, and then identify at the C, C′, D and D′ boundaries the oriented 2-edges CC′ and DD′.

To generate a Möbius strip, identify C and D′, D and C′ along the entirety of the retracts CD and C′D′.

For the formation of the 3-branched square we have inserted and connected three more 3-edges from those at the vertices of P, Q and R shown below, where the arrows are the retract.

There are six 3-edges.



Now identify the three 3-edges P, Q and R, corresponding to the initial retract, which are to be amalgamated at the retraction of their vertices and to a common 3-edge. If we allocate these vertices in the order they are connected by the remaining three 3-edges, and then amalgamate these remaining three 3-edges, this is the 3-branched torus.

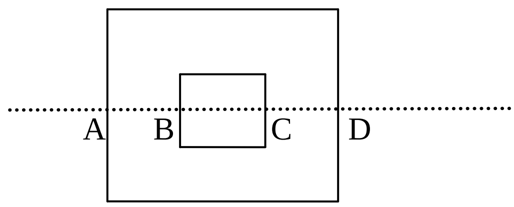
If the 3-edges, P, Q and R, are amalgamated at their vertices in an order that is different than the initial retract, this is a 3-branched Möbius strip. The amalgamation can be cyclic, in which case there is one boundary, or a swap, in which case there are two boundaries, one corresponding to the swap and one corresponding to the identity.

In general the amalgamation is given by the group of permutations on n objects, called the symmetric group, and the number of boundaries is equal to the number of cycles, including individual retract identities.

The 3-branched retract we have been considering has an (m – 1)-dimensional 3-object on the left and $(3^m – 1)$ 3-objects on the right. To form an abutment of these $(3^m – 1)$ 3-objects on the right, for each of these amalgamate a 3-object on the left associated with $(3^m – 1)$ 3-objects on the right. Then for k such iterated abutments, there will exist $(3^m – 1)^{k + 1}$ 3-objects each of dimension (m – 1) on the right.

It is possible to form h – 1 further copies of this abutted object and amalgamate the retracted part of the boundaries of the h versions. *Object* A will leave the left hand of these retracts unamalgamated.

In order to deal with surgery, we first need to explore its simplest instances. For the square with a hole



the hole can be considered as the removal of a subobject of the same type as the containing square. For the 1-dimensional subobject given by the horizontal line we can also consider this as three retracts (synonymous in this case with two abutments) given by

A —— B C—— D

the (point) retract AB, the surgery subobject BC, and the retract CD. The subobject classifier here is defined as Boolean. Extensions beyond the Boolean are given in chapter XII. Then as probabilities AB and CD map to τ, or *certain*, and BC to υ as *impossible*.

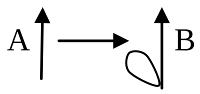
There are possible three types of horizontal line in the above diagram, as above the hole, where the standard retract holds, intersecting with the hole, as given, and below the hole. Correspondingly there are 3 vertical lines, under the designation of the squares as Cartesian products.

For a 3-square, consider 2 further abutments. Let surgery be performed, represented along a *horizontal* 3-edge, as an allocation of a τ or υ classifier, and a τ classifier above and below. To allow this existence of the unimpeded retract above and below, form two further *vertical* abutments of the already abutted object. Then the new object has an interior hole which is classified by υ as four 3-branched squares, two for each of the horizontal and vertical assignments.

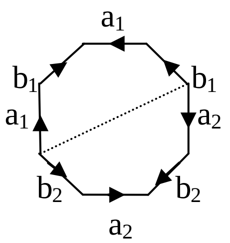
That there are *two* such assignments, horizontal and vertical, follows from the *two* pairs of three 3-edges for each 3-branched square.

The classification of derived objects can be developed further. We have mentioned only object A. Interior holes as already described can be glued to objects of type A. For a normal handle, it would not seem reasonable to glue h copies of a cylinder to a single hole. The conceptual model of n-branched spaces liberates us from that constraint.

A construction which generates from the 2-branched square a 2-branched torus with an extra handle is shown below.

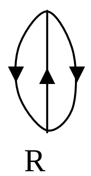


We have obtained the 2-branched square, by inserting and connecting two more 2-edges from those at the vertices of A and B, where the horizontal arrow is the retract, and produced a torus from this, except for the loop shown at the bottom vertex of B, by identifying the four 2-edges in pairs with the A retracted to B pair matched. The loop may be detached at the base, but is reconnected under the identification of the base vertex of B with the base vertex of A. This now forms a hole in the torus, which can be glued to the hole of a copy of that torus with a hole. This torus with one handle, that is, a sphere with two handles, can be represented by the diagram below



where the a_1's, b_1's, a_2's and b_2's are identified by gluing in matched directions and the identified hole is given by the dashed line.

Analogously, for a 3-branched torus with one 3-branched handle, we need the equivalent of a loop. This is shown below as a self-attached 3-edge,



so the 3-branched loop is all reconnected at R.

To the 3-branched torus, now attach a 3-branched loop at R. The loop can be detached at R, but is reconnected under the identification of vertices for the 3-branched torus, so it forms a 3-branched hole. To form a 3-branched h-handle, identify by gluing onto the 3-branched hole h copies of this 3-branched torus with hole.

Thus a 3-branched torus with hole can be identified by gluing with a 3-branched Möbius strip. This is the crosscap construction for 2-branched spaces to produce non-orientable manifolds.

The constructions we have mentioned can be extended in a natural manner to n-branched spaces. If we were to follow Steenrod, the n-branched m-sphere identifies the boundary of an n-branched m-hypercube to a point [St51].

2.10. Explosion boundaries.

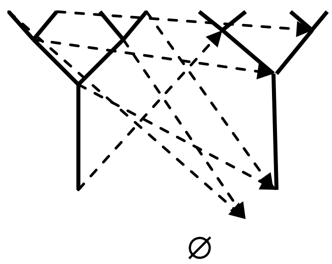
The analogue of a real line is an *explosion*, for which we include next a discussion in terms of explosion analysis. Note that, say, a 3-explosion has an infinity of ends, which themselves may be uncountable and be reassembled to form a manifold in the usual sense. Thus we open ourselves to the possibility of a triple boundary $\partial\partial\partial = 0$, more generally of a k-branched explosion with boundary a k – 1 branched explosion, so $\partial^k = 0$. Here is an example of $\partial^k = 0$ and $\partial^{k-1} \neq 0$, k > 2.

Let И be a k-explosion, k > 2. Consider a real interval (2-explosion), R, within it. Let there be a metric on this real line, and let the total length of the interval be t. For each point p_i of R, select a further 2-explosion not belonging to R except at p_i, with length $u(p_i)$ from all p_i. Let the end point of this line be at q_i. Then the boundary of R is the end-points of R together with all q_i.

For each p_i, $p_j \in R$ with distance interval t_{ij}, consider an *induced metric* on q_i, q_j with length also t_{ij}. Then the boundary of R includes the q_i, and the q_i have induced the structure of a real line, which itself will have two boundary points, the boundary of which is zero. Thus if this is the only line selected $\partial\partial\partial = 0$, but $\partial\partial \neq 0$.

2.11. Trees and amalgams.

The retract structure we have developed may be described in the finite case by finite trees [Se00]. Reconnecting nodes to other nodes may then be represented by a mapping of trees, which is itself a graph, where the diagram shows such a mapping from tree T to T, some arrows being to the empty set.



This mapping may itself be a multifunction, that is, described by a tree mapping. We can also discuss trees with a finite, or an infinite countable or uncountable number of nodes.

2.12. The general polynomial.

If Π indicates multiplication from i = 1 to m, the branched spaces given by the coefficients of an mth degree polynomial are represented by

$$\prod_{i=1}^{m}\{n_i - x\},$$

This assignation is consistent with the idea of a topos in which its morphisms lie in a category, in particular when the union of an element and a negative element is the initial object. Further examples are Grothendieck groups [Ro84].

We now extend the idea of the branched representation where we had x, n and m at most as complex numbers, to x, n and m *matrices*.

We detail in chapter III the hyperintricate representation of matrices, in which the complex numbers occur as subobjects of intricate numbers – representable by 2 x 2 real matrices. In this formalism, the branched representation now becomes expandable as a hyperintricate polynomial.

Thus having described the branched Euler characteristic in terms of a polynomial, questions of polynomial representations arise, even hyperintricately. In particular, we now derive a polynomial isomorphism, g, between the additive part of the characteristic

$$\chi_E{}^{+} = \sum_{i=1}^{k} a_i x^i$$

with $a_k = 1$, and the multiplicative part of the characteristic $\chi_E{}^{\times}$ given by the product

$$\chi_E{}^{\times} = \prod_{i=1}^{k}(b_i - x).$$

where g is the bijective map

$$\chi_E{}^{+} \leftrightarrow \chi_E{}^{\times}.$$

We call the global Euler characteristic that global value $\chi_E{}^{+}$ or $\chi_E{}^{\times}$ obtained additively by cutting and pasting objects described locally by $\chi_E{}^{+}$ or $\chi_E{}^{\times}$.