CHAPTER II

The meaning of branched spaces

2.1. Introduction.

We repeat the axioms for a *vector space* given in [Ad15], Volume I, chapter II, define its generalisation for a *module*, together with examples, and describe *boxes*.

We describe colour sets with order relations: posets, lattices, filters and ultrafilters.

Topology uses *unfinished sets*. An example is the real line interval]0, 1[, with the points 0 and 1 removed. For *finished sets*, an example is the real line interval [0, 1], with the points 0 and 1 included. Set theory is related to logic. We describe probability logic and other logics. Untwisted logic maps sets to oriented manifolds, for example a plane, whereas twisted logic maps sets to antioriented manifolds like a Möbius strip. Since an antioriented manifold has only one side, in twisted logic the global statement NOT B includes an image of the local statement B.

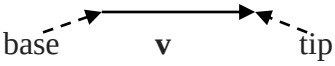
After describing the Euler-Poincaré characteristic, we extend this idea to branched spaces, where a line is a 2-branched space, and the removal of a point in an n-branched line splits it into n pieces. Descartes introduced an algebraic description of geometry by mapping lines in geometry in the plane to coordinate systems described by pairs of numbers. We can do the same for branched spaces, and consider mappings between subobjects in them.

2.2. Vector spaces, scalar products and modules.

A vector space is a collection of objects, **V**, called *vectors*, denoted in bold type, which can be added together and multiplied by numbers called *scalars*, given in ordinary letters. Scalars can be real numbers, but there can also be scalar multiplication by complex numbers, rational numbers or generally any field. The operations of vector addition and scalar multiplication satisfy the axioms

Axiom	*Meaning*
(1) Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
(2) Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(3) Identity element for addition	There exists an element **0** belonging to **V**, called the *zero vector*, so that $\mathbf{v} + \mathbf{0}_v = \mathbf{v}$ for all **v** that belong to **V**.
(4) Inverse elements for addition	For every **v** that belongs to **V**, there exists an element $-\mathbf{v}$ that belongs to **V** called the additive inverse of **v**, with the property $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}_v$.
(5) Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$.
(6) Scalar multiplication identity element	There is a scalar 1 satisfying $1\mathbf{v} = \mathbf{v}$ for all **v**.
(7) Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
(8) Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Vectors may be represented by diagrams. The *base* of a vector a **v** is the vector $\mathbf{0}_v$, and its *tip* is the vector **v** itself.



(1)

We will use, colloquially, the word dimension in what follows, where the technical name is *rank*. We introduce the definition of this difference now.

Two vectors $\mathbf{u}$, $\mathbf{v} \neq \mathbf{0}_v$ are *linearly independent* if there are no scalars a_u, a_v not both zero satisfying

$$a_u\mathbf{u} + a_v\mathbf{v} = 0. \tag{2}$$

(3)

u, **v** linearly dependent **u**, **v** linearly independent

This may be generalised to n dimensions, or as we should say, rank n. Vectors $\mathbf{u}_1$, ..., $\mathbf{u}_k$, ..., $\mathbf{u}_n \neq \mathbf{0}_{uk}$ are linearly independent if there are no scalars a_1, a_2, ... a_n not all zero satisfying

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + ... + a_n\mathbf{u}_n = 0. \tag{4}$$

We say the vectors $\mathbf{u}_1$, ..., $\mathbf{u}_k$, ..., $\mathbf{u}_n$ form a *basis* of the vector space. It is usual to consider that this basis is linearly independent, but we may need to say so explicitly, because it is not within the definition. So a selected linearly independent representation of vector space, where all other vectors are represented in terms of sums of this representation, we know as a *basis* for a vector space. Actually, we need to prove that all vectors can be represented by a basis, but we will not be so fussy. If you are not aware of the proof, to do this is an ideal exercise in rigorous mathematical reasoning.

Then the *dimension*, n, of a vector space is the maximum number of its linearly independent vectors. By definition, its *nullity* is its *rank* minus its *dimension*.

In a vector space with *base point* all vectors are attached to the base point **b**.

Vector spaces may, or may not, have base points, a constant vector **b**, so that all vectors **v** in the vector space are of the form **v** + **b**, and **b** is included in **v** for all **v**, where we can extend the inclusion to say that **b** is a linearly dependent vector for all **v**. Such a vector **b** exists. It could be a selected $\mathbf{0}_v$ for a **v**, so that the zero vector $\mathbf{0}_v$ is unique for all **v**, for example 0 in a coordinate system. Note that two possible ideas for base points are to attach vectors either at their bases or their tips.

Colloquially, a vector is n-dimensional if it is represented by n scalars (a_1, a_2, ... a_n). Although the choice is arbitrary, the representation two dimensionally on a page can be a *row vector* shown by the n scalars shown in the row above, or a *column vector*, represented by n scalars in a column, where if it is in a column it is represented by [a_1, a_2, ... a_n].

The *scalar product* of two vectors $\mathbf{v} = (c_1, c_2, ..., c_n)$ and $\mathbf{u} = (d_1, d_2, ..., d_n)$, or alternatively $\mathbf{u} = [d_1, d_2, ..., d_n]$, is the scalar value

$$c_1d_1 + c_2d_2 + ... + c_nd_n.$$

When $\mathbf{u} = \mathbf{v}$ this is the size of the vector **v**, given from Pythagoras's theorem by its distance.

An $m \times n$ *matrix* can be represented by a 2 dimensional *array* of m row vectors [v_1, v_2, ... v_m] in a column, or by n column vectors (v_1, v_2, ... v_n) in a row. Each entry, or element, of the

array is represented uniquely by the scalar s_{ij}, where the index i ranges from 1 to m and j ranges from 1 to n.

We can define a scalar product on an even dimensional matrix, either from the scalar product in pairs of vector rows, or the scalar product in pairs of vector columns, and sum the result, but this depends on the choice of pairs.

We can define a sum of the vectors, either rows or columns, forming a matrix. For vectors in two dimensions, this is the tip of the vector sum (in the case of a parallelogram, with common base point) of the two vectors. The vector sum is unique. The reader might like to show that this does not depend on whether we represent the vectors as rows or columns.

A better and more useful choice is to form a scalar for a matrix so the area of a parallelogram for a two dimensional matrix is defined by its vectors, generally as a hypervolume in n dimensions, a parallelepiped. This hypervolume is called the *determinant* of the matrix, described in [Ad15], chapter II, section 10.

A matrix is a *square matrix* when m = n.

The *matrix product* two matrices P and Q, where P is an m×h matrix and Q is a h×n matrix is the m×n matrix M, where each m_{ij} entry in M is the scalar product of an ith row vector **p** in P and a jth column vector **q** in Q.

So for example, for the 2×2 square matrices

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

$$\mathrm{i} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{7}$$

their matrix product satisfies

$$\alpha\mathrm{i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \phi \neq \mathrm{i}\alpha = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{8}$$

In a vector space, its scalars are in a field, where the scalars operate on the vectors by scalar multiplication. In a *module*, the scalars need only be in a ring, which does not always have a division operation, and thus allows a wider choice. So a module is a generalisation of a vector space. To satisfy examples of modules, we merely have to provide examples of rings.

Examples of multiplicatively commutative rings are the complex numbers, which have an 'intricate representation', defined by the matrices 1, and i given in equation (7). These form a field, since the nonzero complex numbers have a multiplicative inverse for (a + ib) given by $(a - ib)/(a^2 + b^2)$. Other intricate commutative rings are the actual numbers a + αb, where α is defined in equation (6), which do not always have an inverse since $\alpha^2 = 1$, the inverse should be $(a - \alpha b)/(a^2 - b^2)$, and the denominator could be zero, likewise phantom numbers defined by a + ϕb in equation (8) have inverses $(a - \phi b)/(a^2 - b^2)$ when these exist.

An extension for commutative rings is a polynomial ring. These polynomials could consist, say, of variables with values in a field, or of actual or phantom type. If the coefficients of the polynomial act as symbols which are real, so that they commute with any matrix, then since X^n for a matrix X commutes with these symbols, these polynomials form a ring too.

Polynomials are compatible with differential and integral calculus, so that these ideas can be extended to modules in this case.

2.3. Boxes and box scalar products.

An n dimensional *box* is an array in n dimensions. So a 3 dimensional box is represented by entries or elements s_{ijk}. A box is sometimes called a *tensor*, but this has connotations with differentiable manifolds, and we will introduce definitions not commonly used with tensor geometry, which locally has a distance described by a scalar product, and thus a quadratic form. One of our objectives is to generalise the distance idea so that it is described by a multipolynomial. But we go beyond multipolynomials to include superstructures, which are described in chapter III.

For two boxes $B = b_{ijk}$ and $C = c_{ijk}$ we can introduce the reduced box product

$$b_1(b_{ij})c_1(c_{jk}) + b_2(b_{ij})c_2(c_{jk}) + ... + b_n(b_{ij})c_n(c_{jk}), \tag{9}$$

provided we can represent b_h in a consistent way so that

$$b_h(b_{ij}) = b_{hij}. \tag{10}$$

We know that $b_h = 1$ is such a representation, but is the reduced box product then unique, so that it gives the same representation with $b_i = 1$ in b_{hij}?

Proof. The reduced box product consists of terms in multiplications with, b_{hij}, b_{ihj} and b_{ijh}. The order given by ij matters in matrix multiplications, but the b_h term commutes, therefore all three terms above are equal, and since the reduced block product ranges over every value available to it, the reduced box product defined in this way is unique. □

Determinants, describing the hypervolume of an n-dimensional parallelepiped, in turn can be described by n vectors usually with base point, either as row vectors or as column vectors.

The *box scalar product* is the determinant of the reduced box product. Since the reduced box product and the determinant are unique, this product is unique. We can descend in a chain of such operations from the product of two boxes of dimension n, to a scalar value. I will leave to the reader the task of defining the ways that box scalar products can be defined from scalar products and determinants.

The attachment of cells called CW complexes, which we can describe by such hypervolumes, is part of the theory of paths called homotopy theory. This and the relationship of extended types of box with entries which include the zargonions of chapter IV, called zargon boxes, related to Hodge structures, are discussed in volume II.

2.4. The probability logic of dependent and independent events.

Probability is the assignment of values to logic. It has values in two valued, or Boolean, logic, and in more general logics called colour logics. We assign intermediate values called probabilities between values of true and false, or between colours. We think of this as being a retract structure, applicable to the theory of paths called homotopy theory, or generally as an interpretation in colour logic for branched spaces.

The probability P(A) and P(B) of two independent events A and B respectively, satisfies the multiplicative relation

$$P(A \cap B) = P(A)P(B) \tag{1}$$

so, given by the equivalence demonstrated in *Superexponential algebra* [Ad15], chapter XIV, of set theory to arithmetic, there is a bijective mapping between sets and probabilities, say, represented by numbers in the interval [0, 1].

In the case we have considered, namely that impossible is 0 and certain is 1, the probability of the complement of A, C(A) is the additive relation

$$P(C(A)) = 1 - P(A). \quad (2)$$

In Boolean set theory given by the propositional calculus the de Morgan laws apply:

$$A \cap B = C\big(C(A) \cup C(B)\big) \quad (3)$$

and

$$A \cup B = C\big(C(A) \cap C(B)\big). \quad (4)$$

Using (1) and (2) in (3) or (4) gives, as has been proved in [Ad15], chapter XIII,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \quad (5)$$

Since (3) and (4) are equivalent to (1), (2) and (5), the distributive rules D1 and D2 discussed below are equivalent to (3) and (4), assuming (1) and (2). Note that

$$P(A \cup B) - P(A \cap B)$$

is the probability of A OR B but not both, which in the case $P(A \cap B) = 0$ is the disjoint sum of P(A) and P(B), but in general we can use (1) to define multiplication and (5) to define addition in these logics, so if we wished we could define superexponential operations as an extension of this idea. □

The probability for identical events is not given by (1) or (5). We allocate, which is consistent with (3) and (4)

$$A \cap A = A$$
$$A \cup A = A$$

So the associated probabilities are

$$P(A \cap A) = P(A) \quad (6)$$
$$P(A \cup A) = P(A). \quad (7)$$

In propositional calculus we have

$$(A \text{ \& NOT } B) \text{ \& } (B \text{ \& NOT } A) = (A \text{ OR } B) \text{ \& NOT}(A \text{ \& } B). \quad (8)$$

We can find, given evaluations joining A's together and B's together in the above expression, that the probabilities of the left hand side and right hand side are equal, provided we use dependent and independent probability expressions (1), (5), (6) and (7) together. □

Sets A, B and C are taken to obey the *distributive rules* for intersection and union

D1 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

D2 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Probability logic is consistent under distributive rules. For instance

$$P(A \cup (B \cap C)) = P(A) + P(B)P(C) - P(A)P(B)P(C)$$

$$\begin{aligned} P[(A \cup B) \cap (A \cup C)] &= [P(A) + P(B) - P(A)P(B)][P(A) + P(C) - P(A)P(C)] \\ &= P(A) + P(A)P(C) + P(B)P(C) + P(A)P(B) - P(A)P(B) \\ &\quad - P(A)P(C) - P(A)P(B)P(C) - P(A)P(B)P(C) + P(A)P(B)P(C). \end{aligned}$$

We develop these ideas to include partial dependencies in linear probability logic. Suppose an event A is compared with an event B, and there is a partial dependency between A and B. We will express, say, P(A) as rP(B) + (1 – r)P(C), where C is independent of B. We evaluate probabilities when there are partial dependencies between events ranging over dependencies.

These dependencies are *correlated* between A and B if, as P(A) increases so does P(B) in its dependent part, and *anticorrelated* when as P(A) increases, P(B) decreases. If A and B are anticorrelated and linearly dependent over their entire range, then P(A) = 1 – P(B). □

2.5. Posets and lattices.

Partially ordered sets, or *posets*, are described in category theory and superstructure theory in chapter III. A poset is a set with subsets ordered, or nested, by inclusion. These inclusions may be pictured using arrows. We look at one order relation associated with a set.

A general idea is that sets can be given an ordering, $\leq$. For elements {a, b} this satisfies

(A) if $a \leq b$ and $b \leq a$ then $a = b$

(B) if $a \leq b$ and $b \leq c$ then $a \leq c$.

If $a \leq b$ and $b \leq a$ entails $a = b$, then the order is called a *partial order*, and a *total order*, or linear order, if for all elements $a \leq b$, $b \leq a$ or both.

For a poset, $(P, \leq)$, we use the notation $b \geq a$ for $a \leq b$. We admit posets with neither $a \leq b$ nor $b \leq a$. If it is not the case that $a \leq b$, we write $a > b$, or equivalently $b < a$.

The arrow functions defined by inclusion $A \subseteq B \subseteq C$ and the reverse arrow functions defined by restriction $C \supseteq B \supseteq A$ form posets.

If we map $\leq$ to $\subseteq$ and $\geq$ to $\supseteq$, then we can map $<$ to the $\subset$ 'is strictly contained in' relation or $>$ to the $\supset$ 'strictly includes' relation. By rule (A), for sets A and B we deduce

if $A \subseteq B$ and $B \subseteq A$ then $A = B$

NOT($A \subseteq B$ & $B \subseteq A$ entails $A = B$) is false

[NOT($A \subseteq B$)] OR [NOT($B \subseteq A$)] entails $A \neq B$

$B \subset A$ or $A \subset B$ or both entail $A \neq B$,

so for posets with strict inclusion condition (A) never applies. □

To repeat this, the poset

$C \subset B \subset A$

does not possess an equivalence relation given by =. We use this when we discuss open sets, where we gave the example interval of real numbers, r, between 0 and 1 with their boundary points 0 and 1 removed, $0 < r < 1$, redefined as unfinished sets, and look at their relation to Boolean lattices, which must everywhere have an = equivalence.

A *maximal element*, max, in a subset S of a poset satisfies for all $s \in S$, that if $max \leq s$ then $max = s$ (thus there is no $max < s$). *Minimal elements* are defined substituting $\geq$ for $\leq$. Unique maximal elements need not exist. A *fence* consists of minimal and maximal elements only.

In a *well-ordered set*, every non-empty subset of the set has a minimal element.

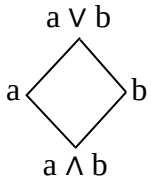
Definition 2.5.1. A poset $(P, \leq)$ is a *lattice* if and only if (written as *iff*) every two elements a, b in P have a unique *meet* given by $a \wedge b = \max\{m \in P, m \leq a \text{ and } m \leq b\}$ and a unique *join* given by $a \vee b = \min\{j \in P, a \leq j \text{ and } b \leq j\}$.

Examples: Meet for a poset under inclusion maps meet, $\wedge$, to *set intersection*, $\cap$, and join, $\vee$, to *set union*, $\cup$.

Using addition and its inverse, subtraction, for two real numbers a and b, $a \leq b$ iff $b - a$ is positive or zero. Meet on a real line is the *greatest lower bound* for a and b, and join is the *least upper bound*. Lower bounds are less than or equal to the corresponding upper bounds.

Using multiplication and its inverse, division, for two positive natural numbers a and b, $a \leq b$ iff b/a is a natural number. Meet on a set of natural numbers generated by multiplication is the *greatest common divisor* for a and b, and join is the *least common multiple*. Divisors are less than or equal to the corresponding multiples.

Meet for a lattice ordered by $\leq$ is its greatest *minimal element*, of which there is only one, and join is its least *maximal element*, where again there is only one, shown in the example *Hasse diagram* below, which we may take additively



Proposition 2.5.2. A lattice satisfies the following rules

L1, *commutativity*

($L1_A$) $a \vee b = b \vee a$

($L1_B$) $a \wedge b = b \wedge a$

L2, *associativity*

($L2_A$) $a \vee (b \vee c) = (a \vee b) \vee c$

($L2_B$) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

L3, *idempotence*

($L3_A$) $a \vee a = a$

($L3_B$) $a \wedge a = a$

L4, *absorbtion*

($L4_A$) $a \vee (a \wedge b) = a$

($L4_B$) $a \wedge (a \vee b) = a$. □

Corollary 2.5.3. By the idempotence rules, a lattice whose non-meet and non-join elements are restricted to just one element consists of one element. □

Remark 2.5.4. Rules L1 to L4 hold under a bijective map $\vee \leftrightarrow \wedge$. Under this mapping $\leq$ is replaced by $\geq$ and vice versa.

Definition 2.5.5. A *distributive lattice* has a meet $\wedge$ matching $\cap$, with a join $\vee$ matching $\cup$ satisfying the distributive rules D1 and D2 of section 4.

A *ring* is a field except there is not necessarily a multiplicative inverse. The definitions we use here for meet and join do not correspond to usual ideas of minimum and maximum. We can define meet for a ring by $a \wedge b = ab$ for $a \neq b$, or a when $a = b$. If join is defined by $a \vee b = a + b - ab$ for $a \neq b$ and a otherwise then the ring becomes a probability logic with P the identity, $P(a) = a$, so the ring is distributive. If join were defined by $a \vee b = a + b$, then L4 is not satisfied and because $a + (bc) \neq (a + b)(a + c)$ the distributive rule fails. □

Theorem 2.5.6. D1 holds iff D2 holds.

Proof. By the remark we need only prove that D1 entails D2. Assume D1 for the lattice. The left hand side of D2 is

$a \vee (b \wedge c) = (a \vee (a \wedge c)) \vee (b \wedge c)$ by $L4_A$

$= a \vee ((a \wedge c) \vee (b \wedge c))$ by $L2_A$

$= a \vee ((c \wedge a) \vee (c \wedge b))$ by $L1_B$

$= a \vee (c \wedge (a \vee b))$ by D1

$= a \vee ((a \vee b) \wedge c)$ by $L1_B$
$= (a \wedge (a \vee b)) \vee ((a \vee b) \wedge c)$ by $L4_B$
$= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)$ by $L1_B$
$= (a \vee b) \wedge (a \vee c)$. □

We give a criterion for distributive lattices later. Here is a taster.

Theorem 2.5.7. A lattice with at most two elements which are not maximal or minimal for the lattice is distributive.

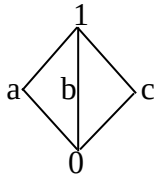
Proof. There are two variables a and b. Consider rule D1. There are three typical cases.

$a = a \vee (a \wedge a)$ by L3
iff the distributive
$(a \vee a) \wedge (a \vee a) = a$ by L3 holds.
$a = a \vee (a \wedge b)$ by $L4_A$
iff the distributive
$(a \vee a) \wedge (a \vee b) = a \wedge (a \vee b)$ by $L3_A$
$= a$ by $L4_B$ holds.
$a \vee b = a \vee (b \wedge b)$ by $L3_B$
iff the distributive
$(a \vee b) \wedge (a \vee b) = a \vee b$ by $L3_B$ holds.

The case D2 follows from this reasoning on applying theorem 2.5.6. □

A *sublattice* is a subset that is closed under meet and join.

Nondistributive lattices exist. The *diamond lattice* M_3 given by the Hasse diagram



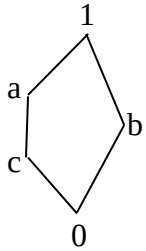
satisfies

$a \wedge (b \vee c) = a \wedge 1 = a \neq 0$
$0 = 0 \vee 0 = (a \wedge b) \vee (a \wedge c)$,

and so is nondistributive. □

Taking meet as the greatest common divisor and join as the least common multiple, the lattice represented by pqr = 1 with p, q and r coprime, and 0 in the Hasse diagram represented by 1 is a nondistributive M_3 lattice.

The *pentagon lattice* N_5 has Hasse diagram



and is nondistributive because

$a \wedge (b \vee c) = a \wedge 1 = a \neq c$
$c = 0 \vee c = (a \wedge b) \vee (a \wedge c)$. □

The lattice with the same meet and join as above, $1 = p^2q$, $a = p^2$, $b = p$ and $c = q$, with p and q coprime, where 0 in the Hasse diagram is 1, is of N_5 type and thus nondistributive.

We will see in chapter V that normal subgroups of a group can form a nondistributive lattice. The analogy with the above example is direct.

The analogy of normal subgroups with the multiplicative features of positive whole numbers leads us to introduce reciprocal groups. If a group G has order, or number of elements, |G| then its reciprocal group has order 1/|G|. In this way we can introduce fractional groups.

A weaker rule than distributivity is modularity. Every distributive lattice is modular but not every modular lattice is distributive.

Definition 2.5.8. A lattice is called *modular* if it satisfies the modular law

M if $a \leq b$ then $a \vee (b \wedge c) = a \wedge (b \vee c)$.

Theorem 2.5.9. A distributive lattice is modular.

Proof. By (ii)

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

and if $a \leq b$ then $a \vee b = b$. □

Remark 2.5.10. Every lattice satisfies

if $a \leq b$ then $a \vee (b \wedge c) \leq b \wedge (a \vee c)$,

since if $c \leq a$ this reduces to $a \leq a$, and if $c \geq a$ then the expression becomes $(b \wedge c) \leq (b \wedge c)$.

So to verify the modular law we need to check

if $a \leq b$ then $b \wedge (a \vee c) \leq a \vee (b \wedge c)$. □

Theorem 2.5.11. (Dedekind) A lattice is modular iff it has no pentagon lattice as a sublattice.

Proof. If there do not exist distinct a, b and c within a lattice, it has at most distinct a and b, so is modular. From the preceding discussion a pentagon lattice is not modular, thus we need to prove the converse, that if a lattice is not modular

$a \leq b$ and $a \vee (b \wedge c) < b \wedge (a \vee c)$

then we can always find a pentagon lattice N_5 within it.

Let $a' = a \vee (b \wedge c)$ and $b' = b \wedge (a \vee c)$. Then

$c \wedge b' = c \wedge [b \wedge (a \vee c)]$

so by associativity of $\wedge$, $L2_B$, and commutativity of $\vee$ and $\wedge$, $L1_A$ and $L1_B$,

$= [c \wedge (c \vee a)] \wedge b$

and by absorbtion, $L4_B$,

$= c \wedge b$.

Now

$c \vee a' = c \vee [a \vee (b \wedge c)]$

so by associativity of $\vee$ and commutativity

$= [c \vee (c \vee b)] \vee a$

and by absorbtion

$= c \vee a$.

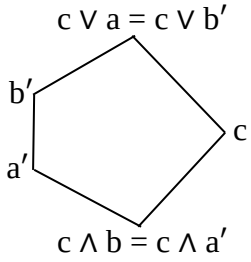
Hence

$c \wedge a' = c \wedge b'$

and

$$c \vee b' = c \vee a' = c \vee a,$$

and this is the pentagon diagram N_5 below.



Lemma 2.5.12. For a nondistributive lattice

$$a \wedge (b \vee c) > (a \wedge b) \vee (a \wedge c).$$

Proof. For any lattice $(a \wedge b) \leq (c \vee b)$. Thus

$$(a \wedge b) \vee (a \wedge c) \leq (c \vee b) \wedge (a \wedge c) = (a \wedge c) \vee (b \vee c). \qquad (1)$$

For a nondistributive lattice

$$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c).$$

Hence if $a \wedge (b \vee c) < (a \wedge b) \vee (a \wedge c)$, then by (1)

$$a \wedge (b \vee c) < (a \wedge c) \vee (b \vee c),$$

which is impossible. □

Theorem 2.5.13. (Birkhoff) A lattice is nondistributive iff its sublattices are isomorphic to M_3 or N_5.

Proof. A proof is given in ref. [BS12]. We have proved the theorem for the pentagon lattice N_5 which is the only nonmodular case, and we know that the lattice M_3 is not distributive. Thus we need to prove that any modular nondistributive lattice is isomorphic to M_3.

Define

$$d = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$$
$$e = (a \vee b) \wedge (a \vee c) \wedge (b \vee c).$$

If we put

$$a' = (a \wedge e) \vee d$$
$$b' = (b \wedge e) \vee d$$
$$c' = (c \wedge e) \vee d$$

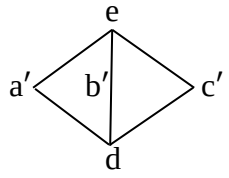
then

$$d \leq a', b', c' \leq e.$$

We can narrow this down to replace $\leq$ by $<$. We represent the application of an interchange under the modular law M by an underscore.

$$a \wedge e = a \wedge (b \vee c) \text{ by } L4_B.$$
$$a \wedge d = \underline{a} \wedge (\underline{(a \wedge b) \vee (a \wedge c)} \vee (b \wedge c))$$
$$= ((a \wedge b) \vee (a \wedge c)) \vee (a \wedge (b \wedge c)) \text{ by M.}$$

Hence $d < e$.



We now wish to show that the diagram above is a copy of M_5, so that

$$a' \wedge b' = a' \wedge b' = b' \wedge c' = d \quad (2)$$

$$a' \vee b' = a' \vee b' = b' \vee c' = e. \quad (3)$$

These hold under permutation of a, b and c and thus under permutation of a′, b′ and c′. We will verify a generic case of (2). The case (3) arises under interchange of meet and join, when the variables d and e switch round.

$$\begin{aligned} a' \wedge b' &= ((a \wedge e) \vee \underline{d}) \wedge (\underline{(b \wedge e) \vee d}) \\ &= ((a \wedge e) \wedge ((b \wedge \underline{e}) \vee \underline{d})) \vee d \quad \text{by M} \\ &= ((a \wedge e) \wedge ((b \vee d) \wedge e)) \vee d \quad \text{by M} \\ &= ((a \wedge e) \wedge e \wedge (b \vee d)) \vee d \\ &= ((a \wedge e) \wedge (b \vee d)) \vee d \quad \text{by } L3_B \\ &= (a \wedge (\underline{b \vee c}) \wedge (b \vee (a \wedge c))) \vee d \quad \text{by M} \\ &= (\underline{a} \wedge (b \vee (\underline{a \wedge c})) \vee d \quad \text{since } a \wedge c \leq b \vee c \\ &= (a \wedge c) \vee (b \wedge a)) \vee d \quad \text{by M} \\ &= d. \ \square \end{aligned}$$

If we represent the minimal element of a lattice by the base of an arrow and its maximal element by the tip, then we may introduce an arrow between a lattice and a copy of the lattice, known as a *lattice retract*. The arrow we have introduced may itself be a lattice. We may wish to populate corresponding intermediate nodes of the lattice pair with copies of introduced arrows.

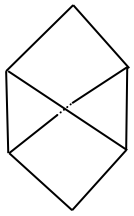
Then the resulting diagram is itself a lattice, with a priority of ordering (lattice, introduced arrow) or (introduced arrow, lattice). These orderings are bijective. □

2.6. Multilattices and the Dedekind-MacNeille construction.

We discuss here sets with multiobjects. We look at one order relation associated with a set. The case of multiple order relations is dealt with elsewhere.

Definition 2.6.1. A *multilattice* is a lattice except multiple minimal and maximal elements are admitted.

A Hasse diagram for a multilattice which is not a lattice is



We have shown two lines that do not intersect in the middle. Meet and join are colours or multiobjects in a multilattice.

A development occurs when multiplicative objects a, b in a multilattice are noncommutative. From chapters III, IV, V and the Heegner numbers of chapter VIII, these objects exist and are nontrivial. The Heegner numbers allow the existence of complex primes and so products of them. Noncommutative groups containing elements $(J_k)^2 = -1$ can have all coefficients Heegner numbers or their products. For such objects the meet $a \wedge b$ is distinct from the meet $b \wedge a$, similarly the join. So this multilattice is a lattice with up to twofold meet and join for all objects.

Formally, this multilattice is a lattice on commutative objects satisfying a noncommutative logic.

A *lattice with abelian logic* is an amalgam of a multilattice where all meet colours are identified with themselves and likewise all join colours.

The Dedekind-MacNeille construction takes a multilattice and forms the smallest lattice that contains the multilattice. There may be several such constructions.

We show a multilattice on the left and construct its Dedekind-MacNeille lattice on the right.

The diagram on the left has four nodes and the diagram on the right five. Wikipedia takes an example of four diagrams like the one on the left, where one node from each of the diagrams again forms a multilattice like the one on the left, and forms the Dedekind-MacNeille lattice from it.

In category theory, reversing the direction of arrows is known as a *dual* operation.

2.7. Finished and unfinished sets.

Sets S and T define a logic in the propositional calculus, in which the set membership relation for a set

$$s \in S$$

is mapped onto statements with true or false values. The propositional calculus defines NOT, & and OR by truth tables on true and false values for statements, so these mappings include

$$CS \to \text{NOT } x \in S \tag{1}$$

$$S \cup T \to \ x \in S \text{ OR } x \in T \tag{2}$$

$$S \cap T \to \ x \in S \ \& \ x \in T. \tag{3}$$

We have given examples of finished and unfinished sets in the introduction. Formally, we can define their properties by axioms. If a set T is finished, then its *set complement*, CT, given by elements which are not in T, is unfinished. Our purpose is to give rules for unfinished and finished sets that are sensible, irrespective of their conformity or otherwise with standard definitions of respectively open and closed sets.

We consider an operation on a set T called the *finishing* of T, defined by the mapping

$$T \to \overline{T}, \tag{4}$$

and say $\overline{T}$ is a *finished* set. We define an *unfinished* set as $C\overline{T}$.

We define the empty, or void, set as unfinished and not finished. This definition is different from an open set, which is defined as open and closed. The complement of this set in a set universe is finished.

In chapter 1, section 12, we gave a model where for the set $\mathbb{R}$ of real numbers we identify $\boldsymbol{\Omega}_{\mathbb{R}}$ with a finished set and $\boldsymbol{\Omega}_{\mathbb{R}} - 1$ with an unfinished set. This means for a finished interval [a, b] its unfinished interval could be interpreted as $[a(1 - \boldsymbol{\Omega}_{\mathbb{R}})^{-1}, b(1 - \boldsymbol{\Omega}_{\mathbb{R}})^{-1}]$. We have interpreted infinities in terms of colour logics, where one type of colour is not reducible to another. In this model a finished set [a, b] is interpreted as in an odd number of colours everywhere on its

boundary, and the unfinished set]a, b[is a set in an even number of colours everywhere on its boundary. The void boundary has no colours. We will introduce colour logic in volume II.

Consider sets S and T. The two sets are *separate* if their intersection is the empty set

$$S \cap T = \emptyset. \tag{5}$$

Sets may not be entirely finished or unfinished. If S is an unfinished set and $\overline{T}$ is a finished set, and they are not separate, nor is one set properly included in the other, then $S \cup \overline{T}$ is neither completely finished nor completely unfinished. We say the resulting set is half-finished and half-unfinished. I leave to the reader the case of extending this idea to n sets.

Since we define

$$\mathbf{\Omega}_{\mathbb{R}t} = \sum_{\text{all } \mathbb{R}t} 1 \tag{4}$$

an infinite set of size $\mathbf{\Omega}_{\mathbb{R}t} + 1$ can be reduced to 1 by subtracting 1 $\mathbf{\Omega}_{\mathbb{R}t}$ times. An unfinished set $[1, \mathbf{\Omega}_{\mathbb{R}t} + 1]$ intersected with sets of this type obtained from subtracting 1 from the least upper bound $\mathbf{\Omega}_{\mathbb{R}t}$ times then reduces to [1], which is a finished set. This motivates

Definition 2.7.1. Let T be a set, finished or unfinished. For $i \in \mathbb{R}_t$ consider sets $\{S_i\}$ indexed by i. A topology o satisfies

(a) $S_i \in o$
(b) $\cup_i\{S_i\}$ is unfinished
(c) For i finite, $\cap_i\{S_i\}$ is unfinished

Abstractly, a *filter* is a subset of a partially ordered set.

Definition 2.7.2. A *filter*, F, of a poset $(P, \leq)$ satisfies

(A) F is not the void, or empty set.
(B) for every $x, y \in F$, there is a $z \in F$ with $x \leq z$ and $z \leq y$.
(C) for every $x \in F$ and $y \in P$ then $x \leq y$ implies that $y \in F$.

We will say the dual notion of a filter is an *ideal*.

A filter of a poset with maximal elements is an *ultrafilter*.

The next definition was introduced by the German mathematician Felix Hausdorff. Trapped in Nazi Germany, he committed suicide in 1942 after being visited by the Gestapo.

Definition 2.7.3. A *topological space* is a set X of points $x \in X$, with a function $x \rightarrow H(x)$ for each point giving a neighbourhood $H(x) \in X$. The neighbourhood axioms satisfy

(i) If $N \in H(x)$ then $x \in N$.
(ii) H(x) is included in X, then H(x) is a neighbourhood of x.
(iii) If H_1 and H_2 are neighbourhoods of x then their intersection is a neighbourhood of x.
(iv) Any neighbourhood $H_1(x)$ includes a neighbourhood $H_2(x)$ such that H_1 is a neighbourhood of every $y \in H_2(x)$.

We can obtain an equivalent formulation of a topological space in terms of the abstract definitions of finished and unfinished sets, for which we have given a colour model. Then in a topological space a *neighbourhood* of a point x is a possibly finished subset T that includes an unfinished set S.

Definition 2.7.4. A *neighbourhood system*, or neighbourhood filter, is a filter F(x) of the set X if

(1) the point x is an element of each U in F(x)
(2) each U in F(x) contains some V in F(x) such that for every y in V, U is in F(y).

A set is a topological space iff it has a neighbourhood filter system.

Theorem 2.7.5. If F is a neighbourhood filter system and if we define
$o = \{A \in P(X)$: for all $x \in A$, $A \in F(x)\} \cup \{\emptyset\}$
then o is a topology.

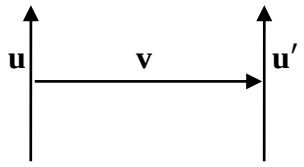
Proof. (a) For every $x \in X$, $F(x) \neq 0$, so by (A), let $A \in F(x)$ for some $A \subseteq X$. By (B) $x \in F(x)$. Since $x \in X$ is arbitrary, we get $X \in F(x)$ for all $x \in X$. This means $X \in o$.

(b) Suppose $\{U(i): i \in I\}$ is a family of sets of o. To show V: $\cup_{i \in I} U(i) \in o$, we pick any $x \in V$ and show that $V \in F(x)$ by the definition of o. Again using (B) and $U(i) \subseteq V$ we get $V \in F(x)$.

(c) Let $A, B \in o$. We want to show that $A \cap B \in o$. Then for every $a \in A$ and $b \in B$ we can find an a = b, c, say, otherwise (1) is not satisfied, so for the collection of all c, $A \cap B \in o$.

2.8. Twisted and untwisted logic.

A set may be situated in a vector space, defined in section 2. Consider two vectors **u** and **v** without the same base point. We can consider scalars which vary so that the scalar s_x applied to the vector u changes in value x from 0 to 1, possibly continuously if we define this. Then s_0u is at the base of u and s_1**u** at the tip. We call this path a *retract* along **u**. We can define a retract of **u** along **v** as a mapping of the base and tip of u so that it begins from the base of **v** and ends with its tip. This retract has defined the diagram shown below.



(6)

This idea can be extended to give retracts of retracts, and thus define an n-dimensional rectangle given by the bases and tips of the arrows. A rectangle is *flat* if, for constant values of the retracts along **u**, every retract along **v** increases monotonically or is constant.

A *local* object L is a set defined within an n-dimensional rectangle, by inclusion within it.

The *boundary* of a rectangle is the finished part of it without the unfinished subset within it. Its *interior* is the union of all unfinished sets included in the finishing of the rectangle.

There exist a number of operations which can be done on these rectangles. For instance we can glue the vector **u** with its rectangle retract **u′** so that the base of **u** corresponds with the base of **u′**, and the tip of **u** corresponds with the tip of **u′**. Then we say that the rectangle forms an untwisted manifold, a cylinder. An untwisted manifold is usually called an oriented manifold. Alternatively we can glue **u** and **u′** so the base of **u** corresponds with the tip of **u′**, and the tip of **u** corresponds with the base of **u′**. Then we say the rectangle forms a Möbius strip, which is a twisted manifold, which we call an antioriented manifold.

A local object is in one part, so that there are no two subsets within L so that they are separate and the separate sets contain sets which are not in L. The Jordan curve theorem, which we do not prove here, states that in a flat untwisted manifold the boundary of a rectangle, which is the finished part of the rectangle without the unfinished subset within it, divides the manifold into two parts. For a cylinder with two boundary circles, this means the cylinder has two sides. However, the case for a Möbius strip means that it has one side.

We can now define a mapping on a local object to form the finishing of its interior, which will be on one side of its surface for an untwisted surface. Its complement is unfinished, and since the union of the finished and unfinished sets is the whole surface, this local object will divide the surface into two components.

Then for untwisted logic if we make a local statement which maps in propositional calculus to a local set, then for a statement B, the statement NOT B is not included in it. These two statements are separate.

For twisted logic, a twisted manifold has globally only one side. If we make a local statement B, an image of this statement locally on the other side is entirely included within NOT B.

Thus the observation long made by some sections of the population that politicians not only use twisted logic, but were the first to discover it, may have some genuine foundations based on a mathematical theorem.

2.9. The Euler-Poincaré characteristic.

As mentioned in [AH35] and [BLW86] the number of vertex points – edges + areas

$$\chi = P - E + A$$

as an invariant of a simplical decomposition of a polyhedron – that is, the space is divided up into polygons and simplexes of higher dimension, was first put in an equivalent form by Descartes. χ itself, the Euler-Poincaré characteristic of a manifold, was discovered by Euler [Eu1752]. It is the number of vertices, minus the number of edges, plus the number of faces, etc., as an alternating sum, and describes an invariant of the space – provided the topological shape remains the same – as a sphere or torus, etc. An example is a surface of a cube, which has 8 vertices, 12 edges and 6 faces, so its Euler-Poincaré characteristic is $\chi = 8 - 12 + 6 = 2$, and this is a topological invariant which describes a 2-sphere, in which the vertices, edges and faces can be embedded, and gives the same Euler-Poincaré characteristic for a surface of a tetrahedron (a triangular pyramid) with 4 vertices, 6 edges and 4 faces: $\chi = 4 - 6 + 4 = 2$.

It would be impossible to situate Riemann except in the middle of a long tradition, yet the paper [Ri1851] which defines the *genus*, $g = 1 - \chi/2$ for a surface (this is the number of handles), is often taken as the starting point of our subject. The idea of connectivity given there was then extended to higher dimensions by Betti [Be1871].

The paper of Poincaré on Analysis Situs, and the five supplements to it has been translated into English by John Stillwell [Po10]. The work that developed in topology up to the mid 1930's was vast, particularly in Germany. For a bibliography of this period the reader could consult [ST80]. Of note is Herman Weyl's work on Riemannian surfaces [We47], and Emmy Noether, who further developed the idea of homology groups [No83].

There are 3 main ways to describe the Euler-Poincaré characteristic.

(i) $\chi = \Sigma_i(-1)^i a_i$,
where a_i is the number of i-dimensional faces.

(ii) From discussing the genus, $\chi = \Sigma_i(-1)^i p_i$,
where p_i are the Betti numbers of the space, defining the i-dimensional connectivity, for example as handles. The p_i are as defined by Poincaré, not Betti.

(iii) χ = number of pits – number of passes + number of peaks of a surface as studied by Cayley [Ca1859]. This can be generalised to an n-dimensional manifold by considering a height function of the manifold immersed in $\mathbb{U}^{2n}$.

To generalise and specialise at the same time, the Euler characteristic of a n-surface becomes expandable in two sorts of ways, as a hyperintricate polynomial in degree n with variables x, additively as the Euler characteristic

$$\chi^+ + (-x)^n = \sum_{i=1}^{n} a_i(-x)^i \qquad (1)$$

where $a_n = 1$, or multiplicatively given by the characteristic $\chi^\times$ with a product

$$\chi^\times + (-x)^n = \prod_{i=1}^{n}(b_i - x). \qquad (2)$$

These two representations are equivalent

$$\chi^+ = \chi^\times. \qquad (3)$$

The Euler characteristic is obtained from equations (1) and (2) by putting x = 1.

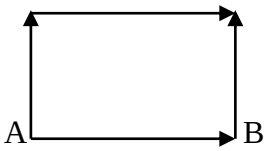
For example, the square is related to the polynomial $\{2 - x\}^2$, and the cube has a mapping to $\{2 - x\}^3$. In particular the coefficients in the binomial expansion of $\{2 - x\}^2$ give respectively the number of vertex points, edges and areas of the square, and the coefficients in the binomial expansion of $\{2 - x\}^3$ give the number of vertex points, edges, areas and volumes of the cube. This fits in with a description of the cylinder mapped to $\{1 - x\}\{2 - x\}$, and to the torus, mapped to $\{1 - x\}^2$.

We then introduce *branched spaces* via the polynomial $\{m - x\}^n$. To begin with consider m is a whole number for n-dimensional branched spaces. So the 3-branched cube maps on to the polynomial $\{3 - x\}^3$, and has 27 vertex points, 27 edges, 9 areas and 1 volume.

Non-oriented manifolds were introduced by Möbius [Mö1887], and first systematically classified by von Dyck [Dy1882], [Dy1885], [Dy1890]. The value of χ for these is obtained by putting the coefficients b_i negative in formula (2). An important idea is the representation of branched Möbius strips. The standard Möbius strip is given by

$$(-1 - x)(2 - x).$$

We are denoting the twist by the presence of a minus sign: $(-1 - x)$. The standard cylinder is represented by



in which vertices on both edges of A and B are identified and A and B are identified. Thus its Euler characteristic of $(1 - x)(2 - x)$ tells us that the cylinder has one area, 3 edges and two vertices obtained under this identification. For the Möbius strip, the vector at B is in the opposite direction, so A and B under vector identification cancel. Because of the twist, there is now only one edge. The rule is, under gluing add the edge vectors and subtract one of them. Then $(-1 - x)(2 - x)$ gives one area, one edge and two points. The generalisation to branched spaces is that all positive and negative values are admissible. A Klein bottle is now represented by $\chi = -(1 + x)(1 - x)$, and $\chi = -(2 + x)(2 - x)$ by a pair of oppositely oriented edges derived from the diagram above.

We can consider rational, algebraic, transcendental and complex numbers of hypervolumes volumes, areas, edges and points. More abstractly, we can consider matrices and more general objects. Where the numbers are ladder numbers and the coefficients are ordinal infinites, we describe the branched space as an *explosion*, and the case where the coefficients are infinitesimals, as an *implosion*. The spaces we have so far been considering are not the most general. Firstly we have considered so far only one variable, x. This may be expanded to a variety in a number of variables. We consider a superexponential variety in chapter VII.

We introduce as examples a conceptual model in the 1-dimensional case of what is meant by branched lines and points, describing this by what is known as generalised 'Dedekind cuts', and in the 2-dimensional case provide a model of a branched square.

2.10. The familiar square, cylinder, torus and cube.

The number of points, number of edges and area of a square, each with sign given by the Euler characteristic χ, are related by

$$\chi = P - E + A$$

for P the number of vertex points, E the number of edges and A the number of areas, and these are given in sequence by the coefficients of

$$\{2 - x\}^2 = 4 - 4x + x^2,$$

so P = 4, -E = -4 and A = 1.

For a cylinder, formed when two opposite edges and two opposite points of a square are identified, the values of P, -E and A are given in sequence by the coefficients of

$$\{1 - x\}\{2 - x\} = 2 - 3x + x^2.$$

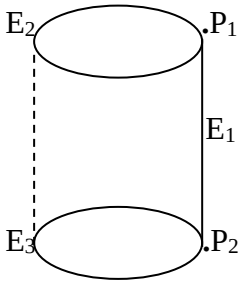


Figure. The familiar torus with one vertex point, two edges and one area is obtained from the familiar cylinder shown, by gluing the top and bottom edges E_2 and E_3 together, so P_1 and P_2 coincide.

For a torus, where I identify equally oriented edges E_2 and E_3 and points P_1 and P_2 above, P, -E and A are given in sequence by the coefficients of

$$\{1 - x\}^2 = 1 - 2x + x^2.$$

For a cube, the volume, and the values of P, -E and A are given by the coefficients of

$$\{2 - x\}^3 = 8 - 12x + 6x^2 - x^3.$$

For the cube with two opposite faces identified, and the two sets of 4 points of those square faces identified, these are given by the coefficients of

$$\{1 - x\}\{2 - x\}^2 = 4 - 8x + 5x^2 - x^3,$$

with two sets of two opposite faces identified as

$$\{1 - x\}^2\{2 - x\} = 2 - 5x + 4x^2 - x^3,$$

etc., and for a 4-dimensional hypercube, by the coefficients of

$$\{2 - x\}^4.$$

2.11. Branched spaces.

I now generalise this idea to *branched* spaces. A 3-branched space, for example a 3-branched square, has P, -E and A given by the coefficients of $\{3 - x\}^2$. It therefore has:

9 points 6 sides and 1 area.

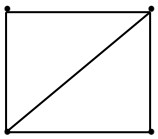
You may with difficulty develop a visual model for this topology, but the idea is as consistent as $\{3 - x\}^2$, and I provide a model example at the end of this chapter. Recall that imaginary numbers were first thought of as not describing the 'real' world.

The question then arises, how do I compute the number of points etc., of a branched simplex? Consider a familiar square and a 3-branched square

?

4 points, 4 edges, 1 area 9 points, 6 edges, 1 area

I can triangulate the familiar square by forming a diagonal



I now have 4 points, 5 edges and 2 areas.

So I have for the untriangulated familiar square topology
$\chi = 4 - 4 + 1 = 1,$
for the cylinder
$\chi = 2 - 3 + 1 = 0$
and for the torus
$\chi = 1 - 2 + 1 = 0.$

These values of χ are invariant under a change of triangulation that maintains the topological shape. Can I assume the same for branched simplexification?

If I do, then the branched Euler characteristics are
$\{3 - x\}^2$: $\chi = 9 - 6 + 1 = 4,$
$\{1 - x\}\{2 - x\}$: $\chi = 3 - 4 + 1 = 0,$
$\{2 - x\}\{3 - x\}$: $\chi = 6 - 5 + 1 = 2,$
and $\{3 - x\}^3$: $\chi = 27 - 27 + 9 - 1 = 8,$ etc.

Suppose for $\{3 - x\}^3$ I add one edge, but keep the number of points constant. Then I must create an extra area to keep χ the same. I can always add points and increment the number of edges correspondingly. Inductively, for any dimension I can add a hyper-area and add a hyper-edge whilst retaining χ invariant.

I note that for $\{3 + x\}^2$, (with a *plus* sign) if I add a hyper-area I must subtract a hyper-edge to retain χ invariance, likewise for $\{k + x\}^n$, k a complex number.

For complex hypervolumes, take the example of adding a *semi-point*, say half a point, then the addition of the corresponding compensating semi-edge must be adjusted to leave χ invariant.

2.12. Models for branched lines and areas.

I now provide model examples of branched spaces, firstly in one dimension.

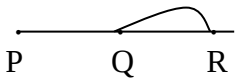
To begin with, consider $\{1 - x\}$, which represents a circle. If the circle consists of real numbers, then a 'Dedekind cut' – a removal of one point – leaves the resulting 'manifold' in one piece.

If I consider $\{2 - x\}$, representing a real line with two end points, each end point of which is connected in only one way with the rest of the interval – in other words the line is a *closed* interval, then removal of an interior point leaves the resulting manifold in two pieces.

Now look at $\{3 - x\}$. I consider three end points, where each end point is connected in only one way with the rest of the interval, so by analogy with the previous case I will call this interval again closed. Then a Dedekind cut – the removal of one interior point – leaves the

resulting manifold in *three* pieces. Thus a branched line represented by {n – x} with n a variable, under removal of a unique interior point, divides the line into n pieces. Normally, if the point were not unique, there would be more than n ends. To bypass this, an alternative is that the branched line is considered *affine* or *relativ*e, so that always the removal of the first selected point (so that the axiom of choice is restricted to a first selection) divides the line into n pieces and there are n ends. These are extended meanings of line or 'edge'.

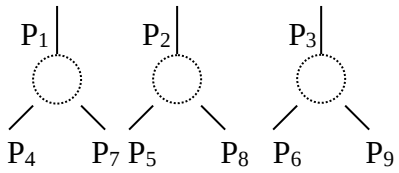
The sequence of points in this interval can be reconnected in its interior, for example:



where I have shown three points, P, Q and R. All such points can be reconnected in this way.

If there are no interior reconnections, so that all the points are connected in an expanding tree, I call the resulting analogue of a real number interval an *explosion*.

The next model example, of a 3-branched square, was first developed by Doly García. All sets of interior edges except for one are reconnected, or the space is affine. I represent 3 sets of '3 vertex points and one edge' as follows:



I then connect vertex point P_1 with an 'edge' to simultaneously P_2 and P_3, then P_4 with an edge to P_5 and P_6, and P_7 to P_8 and P_9, making 9 points, 6 edges and 1 area. The closed end points are here connected as a 2-branch.

Further we note that removing an edge from the 3-branched square reduces the dimension by one – the area dimension disappears. Reversibly, in the process of adding an edge, the number of areas is increased by one, thus retaining the Euler characteristic. □

2.13. Models for multiobjects.

Multiobjects exist in different equivalence classes given by equalites. If colours are ordered by inclusion, as for ordinal infinities, they are not mutually bijective.

A model for a multiobject with n values can be given by evaluating a whole number (mod n). The multiobject is a set ℕ (mod n). The function (mod n) can be represented by $e^{2\pi im/n}$. This form can be extended to describe nonassociative objects, for example the zargon suboxes of chapter VII, section 7.

2.14. Deformation retracts and orientation.

Our definition of the Euler characteristic, χ, of a familiar m-dimensional hypercube, given as the sum of the coefficients of $(2 - x)^m$, is of a deformation retract with ends two (m – 1)-dimensional copies of a hypercube.

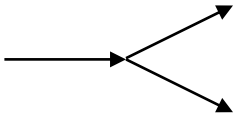
It is possible to amalgamate these two copies by gluing at both ends an opposite retract, with glued retracts corresponding to two orientation types – with the same or reversed orientation.

We can ask whether this definition of a deformation retract is extendable to branched spaces. It is. We consider this as a global phenomenon, where a localisation of it is visible at the ends of the retract. Since our philosophy is that the retract is built out of objects which are not necessarily real numbers, the question is evident as to how the localisation is manifest in the interior of the retract. In terms of connectivity a branched retract is bijective to the inverse operation of what we previously called a generalised Dedekind cut. We regard this retract not only relatively in terms of connectivity, but also as a state.

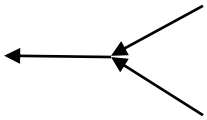
Consider $(n - x)^m$ for a branched space. We will call n the *branched root* and m the *branched degree*. We will show the branched root for an n-edge differs in general from the number of orientations of the n-edge.

To give an example, consider the García diagram for a 3-branched square given above. Its 1-dimensional subobjects are the 3-edges of which we have displayed 3.

The orientation of each 3-edge may be represented by



which can be subject to a threefold rotation or a reflection. The opposite orientation, which is a dual map, may be represented by



again with a threefold rotational symmetry, or combined with a reflection about the horizontal axis.

If we consider the reflections as equivalent orientations then the total number orientations for a 3-edge is 6. The three 3-edges in the García diagram are free to have each of the six possibilities.

If we select a set of these, then the orientation of a new connection between P_1, P_2 and P_3 etc. to flow as a continuation in the same direction, is fixed. Thus P_1 is connected to three possibilities: P_2, P_5 and P_8, then to a further 3 possibilities: P_3, P_6 and P_9, making 9 possible connections with P_1.

The number of connections with P_2 is then reduced, since, say, P_1 has been already selected with P_2, making two possibilities with P_5 and P_8, and 2 possibilities with, say, P_6 and P_9.

Finally, P_3 has only one set of connections available. Thus the number of orientations for a 3-branched square is $3x6 + 3^2 + 2^2 + 1^2 = 32$.

For a line segment, the number of orientations corresponds with its number of end-points. We have seen this is not the case for an n-edge. What used to be isomorphic has become distinct.

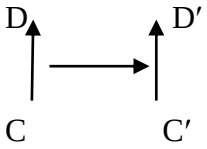
2.15. Branched handles, crosscaps and surgery of branched spaces.

To understand how we can extend the idea of gluing handles and Möbius strips to holes in 2-branched spaces (manifolds) to encompass n-branched spaces, we need a formulation that is compatible with our ideas relating the Euler characteristic χ from $(2 - x)^3$, topologically a ball, to χ for $(1 - x)^3$, a handlebody – which can be pictured as a torus in classical 3-space swept out and reconnected along a fourth dimension, and a disk, $(2 - x)^2$, to a handle, $(1 - x)^2$.

In our model, an n-branched object will be called *closed* when its boundary (of say vertices) is present, and *open* when it is absent.

There are two basic modes of construction we can perform. The first is, having been provided with a ready-made n-object with boundary, to identify parts of this boundary, possibly via other objects. The second is to perform surgery to remove a number of n-object copies and then glue other derived n-objects. To do this we need a concept of the interior of an n-object, and in order to introduce this, it will be useful to describe the abutment of n-objects to create an extended n-object. We discuss to begin with the first of these ideas, then for a 3-branched object we are interested in surgery involving u = 1 and 2.

To generate a 2-branched torus from a 2-branched square

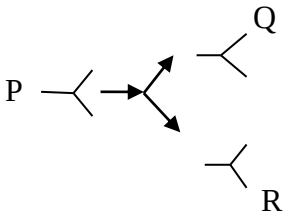


identify C and C′, D and D′ along the entirety of the retracts CD and C′D′, and then identify at the C, C′, D and D′ boundaries the oriented 2-edges CC′ and DD′.

To generate a Möbius strip, identify C and D′, D and C′ along the entirety of the retracts CD and C′D′.

For the formation of the 3-branched square we have inserted and connected three more 3-edges from those at the vertices of P, Q and R shown below, where the arrows are the retract.

There are six 3-edges.



Now identify the three 3-edges P, Q and R, corresponding to the initial retract, which are to be amalgamated at the retraction of their vertices and to a common 3-edge. If we allocate these vertices in the order they are connected by the remaining three 3-edges, and then amalgamate these remaining three 3-edges, this is the 3-branched torus.

If the 3-edges, P, Q and R, are amalgamated at their vertices in an order that is different than the initial retract, this is a 3-branched Möbius strip. The amalgamation can be cyclic, in which case there is one boundary, or a swap, in which case there are two boundaries, one corresponding to the swap and one corresponding to the identity.

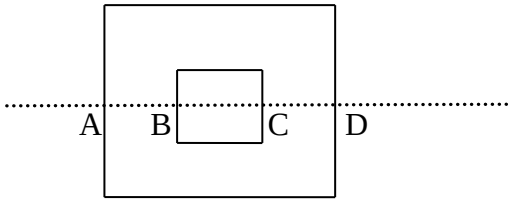
In general the amalgamation is given by the group of permutations on n objects, called the symmetric group, and the number of boundaries is equal to the number of cycles, including individual retract identities.

The 3-branched retract we have been considering has an (m – 1)-dimensional 3-object on the left and $(3^m – 1)$ 3-objects on the right. To form an abutment of these $(3^m – 1)$ 3-objects on the right, for each of these amalgamate a 3-object on the left associated with its $(3^m – 1)$ 3-objects on the right. Then for k such iterated abutments, there will exist $(3^m – 1)^{k + 1}$ 3-objects each of dimension (m – 1) on the right.

It is possible to form h – 1 further copies of this abutted object and amalgamate the retracted part of the boundaries of the h versions. *Object* A will leave the left hand of these retracts unamalgamated.

In order to deal with surgery, which involves cutting out parts of a space and gluing in other spaces along the boundaries of the cut pieces, we first need to explore its simplest instances.

For the square with a hole



the hole can be considered as the removal of a subobject of the same type as the containing square. For the 1-dimensional subobject given by the horizontal line we can also consider this as three retracts (synonymous in this case with two abutments) given by

A —— B C —— D

the (point) retract AB, the surgery subobject BC, and the retract CD. The subobject classifier here is defined as Boolean. Extensions beyond the Boolean are given in volume |II, chapter IV. In the Boolean case, as probabilities AB and CD map to τ, or *certain*, and BC to υ, as *impossible*.

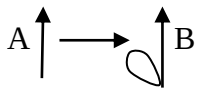
There are possible three types of horizontal line in the above diagram, as above the hole, where the standard retract holds, intersecting with the hole, as given, and below the hole. Correspondingly there are 3 vertical lines, with the squares as Cartesian products.

For a 3-square, consider 2 further abutments. Let surgery be performed, represented along a *horizontal* 3-edge, as an allocation of a τ or υ classifier, and a τ classifier above and below. To allow the unimpeded retract above and below to exist, form two further *vertical* abutments of the already abutted object. Then the new object has an interior hole which is classified by υ as four 3-branched squares, two for each of the horizontal and vertical assignments.

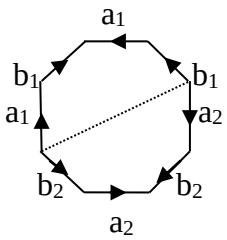
That there are *two* such assignments, horizontal and vertical, follows from the *two* pairs of three 3-edges for each 3-branched square.

The classification of derived objects can be developed further. We have mentioned only object A. Interior holes as already described can be glued to objects of type A. For a normal handle, it would not seem reasonable to glue h copies of a cylinder to a single hole. The conceptual model of n-branched spaces liberates us from that constraint.

A construction which generates from the 2-branched square a 2-branched torus with an extra handle is shown below.

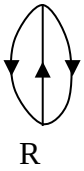


We have obtained the 2-branched square, by inserting and connecting two more 2-edges from those at the vertices of A and B, where the horizontal arrow is the retract, and produced a torus from this, except for the loop shown at the bottom vertex of B, by identifying the four 2-edges in pairs with the A retracted to B pair matched. The loop may be detached at the base, but is reconnected under the identification of the base vertex of B with the base vertex of A. This now forms a hole in the torus, which can be glued to the hole of a copy of that torus with a hole. This torus with one handle, that is, a sphere with two handles, can be represented by the diagram below



where the a_1's, b_1's, a_2's and b_2's are identified by gluing in matched directions and the identified hole is given by the dashed line.

Analogously, for a 3-branched torus with one 3-branched handle, we need the equivalent of a loop. Shown below is a self-attached 3-edge, so the 3-branched loop is all reconnected at R.



To the 3-branched torus, now attach a 3-branched loop at R. The loop can be detached at R, but is reconnected under the identification of vertices for the 3-branched torus, so it forms a 3-branched hole. To form a 3-branched h-handle, identify by gluing onto the 3-branched hole h copies of this 3-branched torus with hole.

Thus a 3-branched torus with hole can be identified by gluing with a 3-branched Möbius strip. This c*rosscap* construction for 2-branched spaces produces non-orientable manifolds.

The constructions we have mentioned can be extended in a natural manner to n-branched spaces. To follow Steenrod [St51], the n-branched m-sphere identifies the boundary of an n-branched m-hypercube to a point. □

2.16. Explosion boundaries.

The analogue of a real line is an *explosion*, which we discuss next in terms of explosion analysis. Note that, say, a 3-explosion has an infinity of ends. These may be uncountable and be reassembled to form a manifold in the usual sense. So we open ourselves to the possibility of a triple boundary $\partial\partial\partial = 0$, more generally of a k-branched explosion with boundary a k – 1 branched explosion, so $\partial^k = 0$. Here is an example of $\partial^k = 0$ and $\partial^{k-1} \neq 0$, $k > 2$.

Let И be a k-explosion, k > 2. Consider a real interval (2-explosion), R, within it. Let there be a metric on this real line, and let the total length of the interval be t. For each point p_i of R, select a further 2-explosion not in R except at p_i, with length $u(p_i)$ from all p_i. Let the end point of this line be at q_i. The boundary of R is the end-points of R together with all q_i.

For each p_i, $p_j \in$R with distance interval t_{ij}, consider an *induced metric* on q_i, q_j with length also t_{ij}. Then the boundary of R includes the q_i, and the q_i have induced the structure of a real line, which itself will have two boundary points, the boundary of which is zero. Thus if this is the only line selected $\partial\partial\partial = 0$, but $\partial\partial \neq 0$. □

2.17. The general polynomial.

If Π indicates multiplication from i = 1 to m, the branched spaces given by the coefficients of an mth degree polynomial are represented by

$$\prod_{i=1}^{m}\{n_i - x\},$$

This is consistent with the idea of a topos in which its morphisms lie in a category, in particular when the union of an element and a negative element is the initial object. Further examples are Grothendieck groups [Ro84].

We now extend the idea of the branched representation where we had x, n and m at most as complex numbers, to x, n and m *matrices*, or more generally *boxes*.

We detail in chapter III the hyperintricate representation of matrices, in which the complex numbers occur as subobjects of intricate numbers – representable by 2 x 2 real matrices. In this formalism, the branched representation now becomes expandable as a hyperintricate polynomial.

Thus having described the branched Euler characteristic in terms of a polynomial, questions of polynomial representations arise, even hyperintricately. In particular, we now derive a polynomial isomorphism, g, between the additive part of the characteristic

$$\chi_E{}^{+} = \sum_{i=1}^{k} a_i x^i$$

with $a_k = 1$, and the multiplicative part of the characteristic $\chi_E{}^{\times}$ given by the product

$$\chi_E{}^{\times} = \prod_{i=1}^{k}(b_i - x).$$

where g is the bijective map

$$\chi_E{}^{+} \leftrightarrow \chi_E{}^{\times}.$$

We call the global Euler characteristic that global value $\chi_E{}^{+}$ or $\chi_E{}^{\times}$ obtained additively by cutting and pasting objects described locally by $\chi_E{}^{+}$ or $\chi_E{}^{\times}$.

It is natural in this circumstance to generalize $\chi_E{}^{+}$ or $\chi_E{}^{\times}$ so we can consider sunomials not limited to + and ×. □

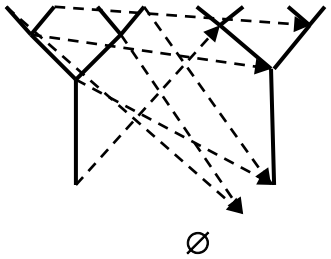
The foundations of the topologies we wish to introduce can then be based on local metrics described by box sunomials, and branched topologies using the same means.

As we will see and is developed in chapter IV, associative structures invariant under box scalar products are not the most general structures we can investigate, and there is a whole area of mathematics which is both interesting and computable using nonassociative structures in its many operations acting on general objects.

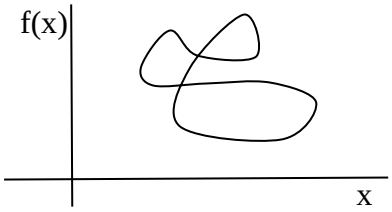
Briefly, category theory is included in the theory of associative branched spaces, where the application of the Euler characteristic to the branched space reduces this to an equivalence. Nonassociative categories, which are called superstructures, are included in a similar way in nonassociative branched spaces. A special and useful case of these nonassociative structures is the sunomials presented in the next chapter.

2.18. Trees, amalgams and xiqus.

The retract structure we have developed may be described in the finite case by finite trees [Se00]. We discuss trees with a finite, or an infinite countable or uncountable number of nodes. If we identify nodes within a tree so that different nodes become the same node and possibly do this for a number of sets of nodes in the tree, so that the identified nodes all end up different, then the result is called an *amalgam*. Reconnecting nodes to other nodes may be represented by an instance of mapping of trees, itself a graph, where the diagram shows such a mapping from tree T to T, some arrows being to the empty set.



The mapping may be a multifunction, also described by a tree. Multigraphs, their applications to category theory, and applications such as Petri nets are discussed in volume II. It is interesting that in what we have so far considered, objects are branched, and our mappings match these objects, so that the mappings are multifunctions. A bounded multifunction is shown below.



We can see it is possible to partition it into a number of separate functions, where each function value f(x) corresponds with multiple values of x, or a number of separate inverse functions $f^{-1}f(x) = x$, where each x corresponds to multiple values of f(x). Iteration of this alternating process generates a sequence of values of f(x) and x which may terminate, or may not.

In graph theory multifunctions can be considered between objects, but the objects themselves are not multiobjects.

This example may be generalised so that the mapping, or link, is not a multifunction but is described by an amalgam. So we can have links between amalgams, where the domain of the link is identified with the amalgam and the codomain of the amalgam is identified with it as well.

Finally, rather than consider one amalgam in the domain, one amalgam in the codomain and one amalgam link, we can think of many colours of amalgams the domain, many amalgams

of this type in many codomains, where this branching is not described by identifying start and end links by one amalagam, but by amalgams in many colours. This rich structure with many interlinks and many colours we call a *xiqu*, after the Chinese name for an opera. We believe it describes life.

Lattices and multilattices are types of amalgam. Lattices may have a distributive structure for +, for + and × as an instance of a probability logic, but for × alone under meet as the highest common factor and join as the least common multiple the lattice may not be distributive. To map bijectively to sets where the distributive rule holds, we are restricted at most to a ring structure of a dependent probability logic.

For suoperators beginning with exponentiation, ↑,

$$a \uparrow b \neq b \uparrow a \quad \text{and} \quad (a \uparrow b) \uparrow c \neq a \uparrow (b \uparrow c)$$

so the lattice description fails. If we wish to describe these by diagrams, we must formulate them in a new sense.

A question arising from the discussion of the Dedkind-MacNeille construction of section 6 is whether there is a typical example or model that has all the features of the general case that always describes this construction. We will address this question in chapter III. In order to prepare for it, we go into more detail of the properties of multifunctions.

If the maximum number of values of a multifunction is m, then only m functions are needed to describe it. The multifunction is the union of these functions. If the maximum number of values of its inverse multifunction is n, then only n inverse functions are needed to represent it. For a number of *bounces* of a multifunction f from x to f(x) and back, so that the bounces either return to themselves or the set of bounces is infinite, then the *bouncing set* of x, B(x), is a partition of x, and the map from B(x) to f(B(x)) is a bijective function. The set B(x) is a multiobject. □