CHAPTER I

The meaning of the finite and the infinite

1.1. Introduction.

Transfinite number theory is considered for the transnatural numbers $\mathbb{M}_t$, where the theory is an extension of finite arithmetic. We repeat a proof given in [Ad15] on the inconsistency of the uncountable continuum hypothesis with respect to the countability of the rationals. Then we extend results on the ladder algebra of ordinal arithmetic given in [Ad15] to $\mathbb{M}_t$.

1.2. The Peano axioms for the transnatural numbers, $\mathbb{M}_t$.

The axioms for a set specify in the axiom of infinity that there exists a non-trivial set called the natural numbers. We will work with the set of natural numbers, $\mathbb{N}_{\neq 0} = \mathbb{N} = \{1, 2, 3, ... \}$, $\mathbb{N}_{\cup 0} = \mathbb{N} \cup \{0\}$ and the integers $\mathbb{Z}$ given by $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

Equality, =, satisfies the property that it is an *equivalence relation*, namely if m, n, p are members in a set then = is

reflexive: $m = m$ (1)

symmetric: if $m = n$ then $n = m$ (2)

transitive: if $m = n$ and $n = p$ then $m = p$. (3)

If it is not the case that $m = n$, then we write $m \neq n$.

If elements of a set S satisfy the property of being in an equivalence relation given by =, its application to elements which belong to S forms a *partition* of S:

if $m \in S$, then the intersection of all elements $n \in S$ which equal m, with the set of all $n' \in S$ which do not equal m is the empty set:

for every $m \in S$, $\{n; n = m \in S\} \cap \{n'; n' \neq m \in S\} = \varnothing$,

where also

for every $m \in S$, $\{n; n = m \in S\} \cup \{n'; n' \neq m \in S\} = S$.

The natural numbers satisfy the Peano axioms describing a recursive procedure to generate them.

$1 \in \mathbb{N}$ (4)

for every $n \in \mathbb{N}$, there exists an $S(n)$ interpreted as $(n + 1) \in \mathbb{N}$ (5)

there is no number $0 \in \mathbb{N}$ with $S(0) = 1$ (6)

for two numbers $m, n \in \mathbb{N}$, $S(n) = S(m)$ implies $n = m$. (7)

(induction) a subset of $\mathbb{N}$ containing 1 and $S(n)$ whenever $n \in \mathbb{N}$, is $\mathbb{N}$. (8)

We can express in an axiom system for sets for any property P that is not self-referentially true, that for the set $\{x: P; x \in X\}$, there exists a set $\{y: \text{NOT } P; y \in Y\}$. Since by this means we can introduce uncountable sets, we can now establish uncountable induction.

Definition 1.2.1.The transfinite natural numbers $\mathbb{M}_t$, where t belongs to an index set which also satisfies the properties below, satisfy the following axioms

$1 \in \mathbb{M}_t$ (9)

for every $m \in \mathbb{M}_t$ there exists an $S(m)$ interpreted as $(m + 1) \in \mathbb{M}_t$ (10)

there is no number $0 \in \mathbb{M}_t$ with $S(0) = 1$ (11)

for two numbers m, n $\in \mathbb{M}_t$, S(m) = S(n) implies m = n (12)
$\mathbb{M}_t \subset \mathbb{M}_{S(t)}$ (13)
$\mathbb{M}_t$ is not bijective to $\mathbb{M}_{S(t)}$ (14)
there is no proper subset $\mathbb{M}'$ with the above properties satisfying $\mathbb{M}_t \subset \mathbb{M}' \subset \mathbb{M}_{S(t)}$ (15)
(induction) a subset of $\mathbb{M}_t$ containing 1 and S(m) whenever m $\in \mathbb{M}_t$ is $\mathbb{M}_t$. (16)

Notation 1.2.2. $\mathbb{M}_t$ with the number 0 appended to it is denoted by $\mathbb{M}_{t\cup 0}$.

We define < by the property, if m, n $\in \mathbb{M}_t$, then m < n if and only if there exists a p $\in \mathbb{M}_t$ so that

m + p = n.

Definition 1.2.3. The *transrational numbers* $\mathbb{Q}_{Mt}$, t ≠ 1, are the set of numbers ±m/n where m $\in \mathbb{M}_{t\cup 0}$ and n $\in \mathbb{M}_t$.

Definition 1.2.4. A *transprime* number is a transnatural number which is only divisible without remainder by 1 and itself.

Definition 1.2.5. A *transinteger* is a transnatural number multiplied by 1, 0 or -1.

Definition 1.2.6. A *transcomposite* number is the product of either a transnatural number of transintegers or may be used in a more restricted sense as a product of a transnatural number of transnatural numbers.

1.3. The Euclidean algorithm.

Strictly speaking, before proving the division with remainder theorem, we need to prove that *any transnatural number* n *greater than* 1 *is either transprime or a product of transprimes*.

Proof. By the *method of induction*, if we wish to prove a statement for a positive transnatural number n, we can assume it has been proved for any number less than n. If n = 2, the theorem is proved. If n is transcomposite, it can be represented as ab, where both a and b are greater than 1 and less than n, but then we know by the induction method that a and b are either transprimes or the product of transprimes. ☐

The *fundamental theorem of arithmetic* states that *this factorisation is unique up to order of factors*.

Proof. If n is transprime, the theorem is proved. Suppose n is transcomposite and there are two factorisations

n = pqr ... = p′q′r′ ...

Let the transprimes be ordered in increasing size. By an induction hypothesis, no transprime p, q or r, ... can be the same as any of p′, q′, r′, ... otherwise we could divide n by it and get two representations of a smaller number, where we would continue the proof with this smaller number. Since n is transcomposite it consists of at least two transprimes, so $n \geq p^2$ and $n \geq p'^2$ which implies n > pp′ with strict inequality, since p and p′ are unequal. Now form n – pp′. This has pp′ as a factor, so dividing pqr ... by p, p′ must be a factor of qr ..., a contradiction. ☐

Suppose we represent the integer parts of the transrational numbers by notches along a line, for instance

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|____|____|____|____|____|____|____|____|____|____|____|__

-1    0    1    2    3    4    5    6    7    8    9
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then any transrational number may be represented uniquely by a transinteger plus a transrational number q equal to or greater than 0 and less than 1. This is an example of the theorem given by Euclid, in book 7 of the Elements.

The algorithm may be restated as:
Every positive transnatural number n *can be written uniquely in terms of a positive transnatural number* w *less than* n *multiplied by another transnatural number* k > 0 *with a unique remainder* $0 \leq u < w$:

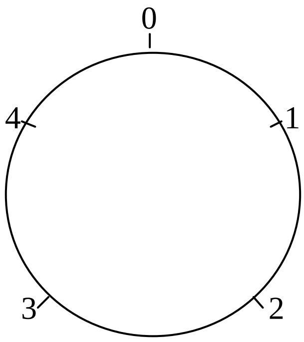
$$n = kw + u. \qquad (1)$$

Proof. If w divides n, then u = 0 and we are done. Otherwise assume (1). If n comes between kw and (k + 1)w, then (1) holds with 0 < u < w. ☐

Using the theorem, we can develop an arithmetic for fixed w in which we only consider u above. This transfinite arithmetic is known as congruence, or clock arithmetic. The equation above is then written, in a notation due to Gauss

$$u = n \pmod{w},$$

which can be depicted in the example diagram for w = 5:



n (mod 5)

1.4. Fermat's little theorem.

Theorem 1.4.1. Fermat's little theorem extended to the transnatural numbers states that for a transprime p, verifiable directly for p = 2

$$x^p – x = bp. \qquad (1)$$

for some unique b dependent on x.

Proof. We prove this by induction. For x = 0

$$0^p – 0 = 0p.$$

Assume (1) holds. Then for $x \to x + 1$, by the binomial theorem and the transnatural primality of p, so p does not divide any denominator

$$(x + 1)^p – (x + 1) = x^p – x + px^{p-1} + [p(p – 1)/2]x^{p-2} + ... + 1^p – 1 = bp + cp$$

for some unique c. ☐ (2)

1.5. Euler's totient theorem.

Let u be any positive transinteger, and let the totient $\varphi(u)$ denote the number of positive transintegers, 1 included, which are coprime to u and not greater than u. Since primality makes sense for transnatural numbers, it also makes sense for relative primality.

By definition $\varphi(1) = 1$. Also if p is a transprime number

$$\varphi(p) = p - 1.$$

Next suppose u transcomposite, and let p, q, r, s, ... be the different transprimes dividing u.

Consider the series of transintegers, 1, 2, 3, ... u. Of these the following are multiples of p:

$$p, 2p, 3p, \ldots (t/p).p,$$

(u/p in all).

Write these down with the sign +. Similarly, write down all the multiples of q, r, s, ... each with the sign +.

In the same series there are u/(pq) multiples of pq. Write these down with the sign –: and do the same with all the multiples of pr, ps, qr, ... (taking all the products of p, q, r, s, ... two at a time).

Next write down all the multiples of the triple products pqr, pqs, ... each with the sign +, and so on, until at last we come to the multiples of pqrs... with sign $(-1)^{k-1}$, k being the number of different primes.

Now take any number θ which is not greater than t and not coprime to it. It will involve in its composition a certain number (λ say) of the different primes p, q, r, ... How many times will it occur among the multiples already written down?

By the binomial theorem, the number of combinations of λ things taken v at a time is

$$\lambda!/[u!(\lambda - v)!].$$

Evidently, taking its appearances in the order of the sets of multiples, θ will occur for λ times, the binomial coefficient, for v = 1 with the sign +, then for $\lambda(\lambda - 1)/2$ times for v = 2 with the sign –, then $\lambda(\lambda - 1)(\lambda - 3)/3!$ times for v = 3 with the sign +, and so on.

If then we take the algebraic sum of all the sets, we have θ occurring with a coefficient

$$\lambda - \lambda(\lambda - 1)/2! + \lambda(\lambda - 1)(\lambda - 3)/3! - \ldots = 1 - (1 - 1)^{\lambda} = 1.$$

Thus the algebraic sum in question is the sum of all positive transintegers not greater than u and not coprime to it. Now the *number* of these integers is equal to the excess of the number of positive terms in the whole sum, as originally written, above the number of negative terms:

$$u\{(1/p + 1/q + 1/r + \ldots) - (1/(pq) + 1/(pr) + 1/(qr) + \ldots) + (1/(pqr) + 1/(pqs) + \ldots) - \ldots + (-1)^{k-1}.1/(pqrs \ldots)\}.$$

Subtracting this from u, we have finally

$$\varphi(u) = u(1 - 1/p)(1 - 1/q)(1 - 1/r) \ldots \square \qquad (1)$$

Corollary 1.5.1. If u is odd, $\varphi(u)$ is even. ☐

1.6. Trans p-adic numbers.

The p-adic number system for any transprime number p extends the ordinary arithmetic of the rational numbers in a different way from the extension of the rational number system to the real and complex number systems. The extension is achieved by an alternative interpretation of the concept of *closeness* or *absolute value*. In particular, trans p-adic numbers have the property that they are considered closer the more their differences are divisible by higher powers of p. These numbers exist also for transprimes. A trans p-adic number b is represented for $j \geq k \in \mathbb{Z}_t$ and $0 \leq a_i < p \in \mathbb{M}_t$ by

$$b = \sum_{i=j}^{k} \frac{a_i}{p^i}. \quad (1)$$

Congruence arithmetic can then be encoded if

$$(j - k) = 0 \pmod{q} \quad (2)$$

usually for some transprime q, but now an ordering is not naturally defined.

1.7. The inconsistency of the UCH property for $\mathbb{N}^{\mathbb{N}}$.

We repeat a small part of [Ad14] on countable and uncountable sets and ordinal infinities, and extend it, but first in this section we will find a rule for the conditions under which the principle of induction is valid for the properties of sets.

Doly García remarks that if a property holds for a set indexed by 1, n and n + 1, one cannot argue that it holds for the entire set, an instance being given by the set $A_1 = \{1\}$, with the property that $2 \notin A_1$, and in general for $A_n = \{1, 2, ... n\}$, where $n + 1 \notin A_n$, so that each complement exists for finite n, but not the entire set $\mathbb{N}$. This is part of the reason for us introducing ladder numbers, but we need to view this where nonstandard analysis is not used.

To respond to this criticism, if we look at the definition of the empty set just given, we notice that mZFC deals with predicates in a nonstandard way, in that the same sentence may range over true and false is admissible in the definition of a set, and false defines the void set in mZFC, and not otherwise. Then if we look at the complement of A_1, this is $\mathbb{N}\backslash\{1\}$, and the complement of A_n is $\mathbb{N}\backslash\{1, 2, ... , n\}$, so that applying the principle of induction, over the set $A_{\mathbb{N}}$ we are dealing with $\mathbb{N}\backslash\mathbb{N}$, which is the empty or void set, and thus the same property ranges over true and false in accordance with its principle of induction for sets. However
Proposition 1.7.1. *If the property for a set does not range over the void set, it does not become false.*

In mZFC, if a predicate holds for all finite $n \in \mathbb{N}$, either the predicate does not hold for all n, which could be the case for standard set theory, where the set of all n not including all finite n is empty, which satisfies a false predicate, or the predicate holds for all n, which is the case for nonstandard set theory, since now the set of all n not including all finite n has elements.

When mapping properties, for instance a bijection $\{\mathbb{V}, \emptyset\} \leftrightarrow \{\mathbb{V}, \emptyset\}$ restricted to $\mathbb{V} \rightarrow \mathbb{N}$, for standard set theory possible mappings are $\mathbb{V} \leftrightarrow \mathbb{V}$, $\mathbb{V} \leftrightarrow \emptyset$, $\emptyset \leftrightarrow \mathbb{V}$ and $\emptyset \leftrightarrow \emptyset$. Since $\mathbb{V}$ maps to true and $\emptyset$ to false, we may and will define that $\leftrightarrow$ satisfies the truth table for IF and only IF, written as $\Leftrightarrow$, with $A \leftrightarrow B \leftrightarrow C$ satisfying $(A \Leftrightarrow B)$ & $(B \Leftrightarrow C)$.

A ⇔ B	A	B
T	T	T
F	T	F
F	F	T
T	F	F

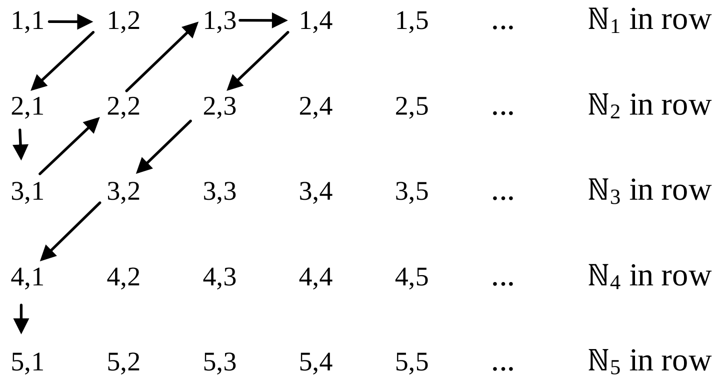
This allows us to state that the principle of induction applies to bijective properties in standard set theory. □

For the argument which follows there is no element where the bijection between the set of rationals, $\mathbb{Q}$, and what we define as $\mathbb{N}^{\mathbb{N}}$ does not hold; nowhere does the induction we have carried out range over the predicate for a void set, since its function of functions $\mathbb{N} \to \mathbb{N} \to$ ... mapped in sequence to $\mathbb{N} \to \mathbb{N} \times \mathbb{N} \to$ ... has a fixed and not exhausted function as its domain. This assertion is proved in chapter XIV section 6 of [Ad15], which is developed from [5Co66], where cardinals defined by bijections are shown to be lowest ordinals, ordinals are countable if there is a bijection to $\mathbb{N}$, and it is proved that because a union of countably many countable sets is countable, exponential and superexponential operations on countable ordinals are countable.

For finite or countably infinite sets S, T or U we define $\equiv$ by the existence of at least one bijection within the natural numbers $\mathbb{N}$. This is an equivalence relation, in that $S \equiv S$, if $S \equiv T$ then $T \equiv S$, and if $S \equiv T$ and $T \equiv U$, then $S \equiv U$, so this forms a partition between those sets belonging to the equivalence class, and those outside it. Then for sets S_n, T_n, $n \in \mathbb{N}$, if for each n $S_n \equiv T_n$, then for the set of all S_n, $\{S_{nn}\} \equiv \{T_{nn}\}$, where this means if $S_n \equiv T_n$, then the bijection is maintained for $S_{n+1} \equiv T_{n+1}$, and the second subscript indicates a distinguished copy for $S_n \neq \varnothing$, defined inductively: $S_{n1} = \{S_n\}$, $S_{n2} = \{\{S_n\}\}$, $S_{n\,p+1} = \{S_{np}\}$.□

In this work we will adopt the argument of Cantor that *the set* $\mathbb{Q}$ *of rationals is countable*. Define the Cartesian product of all natural number pairs $\mathbb{N} \times \mathbb{N}$ as $\mathbb{N}^2$. Consider the rational numbers not in lowest terms given by the set $\mathbb{Q} \equiv \{\{1/n\}, \{2/n\}, \{3/n\}, . . .\} \equiv$ the unordered distinguished copies$\{\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3, . . .\} \equiv \mathbb{N} \times \mathbb{N}$ (a set of ordered pairs) which by the Cantor argument given next is $\equiv \mathbb{N}$. The mapping from $\mathbb{N}$ to $\mathbb{Q}$ is given in the following diagram.

$(m, n) \to m/n \in \mathbb{Q}_{\neq 0}$.



What is meant by the symbols . . . in the sets just given? This indicates that if the pth position is occupied, then a similar item exists at position p + 1, although we can remove . . . from the language and use the properties of $\mathbb{N}$ itself given in chapter III, section 3. Then by induction defined through the properties of $\mathbb{N}$, we have $\mathbb{N} \equiv \mathbb{N}^p$ for p a natural number, so that the set

$\{\mathbb{N}_1, \mathbb{N}_2{}^2, . . . , \mathbb{N}_p{}^p, . . .\}$

$\equiv \{\mathbb{N}_1, \mathbb{N}_2 \times \mathbb{N}_2, . . , \mathbb{N}_p \times \mathbb{N}_p \times . . \text{(p terms)}, . . .\} \equiv \mathbb{N},$ (1)

contains by definition $\mathbb{N}^{\mathbb{N}} \equiv \{\mathbb{N} \times \mathbb{N} \times . . .\}$. This is in violation of the assumptions of the uncountable continuum hypothesis (UCH) in set theory, that $\{0, 1\}^{\mathbb{N}}$ is uncountable. ☐

Repeated proof. By the definition of a union of sets of chapter III section 2 and induction

$\bigcup_{\mathbb{N}} n \equiv \mathbb{N},$

the union of all $n \in \mathbb{N}$ is $\mathbb{N}$.

When we go over to countably infinite sets, there is a property which does not hold for finite sets, that for distinct copies $\mathbb{N}_n$

$\bigcup \mathbb{N}_n \equiv \mathbb{N}.$

This is extended and is inherent in the Cantor argument for the countability of $\mathbb{Q}$, that there exists a constructible bijective mapping

$\mathbb{N} \equiv \bigcup_{\mathbb{N}} \mathbb{N}_n \equiv (\bigcup_{\mathbb{N}} n) \times \mathbb{N} \equiv \mathbb{N} \times \mathbb{N}.$ (2)

If the size of the nth diagonal, up or down, in the previous diagram is n, and the sum of the diagonals is given by the arithmetic series

$1 + 2 + 3 + ... + n = n(n + 1)/2,$

then for each $m \in \mathbb{N}$ there is an $n(n + 1)/2$ and $t \in \mathbb{N}$ with the last diagonal of size n, $0 \leq t < n < n(n + 1)/2$, and by the Euclidean algorithm of chapter III section 9, a bijection

$m \leftrightarrow n(n + 1)/2 + t.$

This is a bijection

$\mathbb{N} \times \mathbb{N} \leftrightarrow \mathbb{N},$

or as we have written

$\mathbb{N} \times \mathbb{N} \equiv \mathbb{N}.$

Let $\mathbb{N}^1$ be $\mathbb{N}$, and $\mathbb{N}^{p-1} \times \mathbb{N}$ be $\mathbb{N}^p$. By the definition of the natural numbers in section 2 the principle of induction states that because $\mathbb{N}^1$ is a subset of $\mathbb{N}$ containing 1 and $(n + 1)$ whenever $n \in \mathbb{N}$, and if $\mathbb{N}^{p-1} \leftrightarrow \mathbb{N}$ holds for 1 and $(p - 1) + 1$, then $\mathbb{N}^p \leftrightarrow \mathbb{N}$, so we have shown by the definition of $\mathbb{N}^p$,

$\mathbb{N} \equiv \mathbb{N} \times \mathbb{N} \equiv \mathbb{N}^{p-1} \times \mathbb{N} \equiv \mathbb{N}^p.$

There is a bijection

$\mathbb{N}_n \leftrightarrow \mathbb{N}$

obtained by stripping out all containing parentheses. Thus

$\mathbb{N}_n \equiv \mathbb{N},$ (3)

and we have proved the equivalence (2), where we can add the comment for any n,

$n \times \mathbb{N} \equiv \mathbb{N}.$

From the equivalence (3) we get

$\mathbb{N}_p{}^p \equiv \mathbb{N}.$ (4)

Thus we have shown directly that equation (1) holds, where

$\mathbb{N}^p \rightarrow \mathbb{N}$ is injective implies $\mathbb{N}^{p+1} \rightarrow \mathbb{N}$ is injective,

so, because there is no void set in the mappings, by induction we obtain its consequence

$\mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$ is injective,

and since for the constant $\{1\}^{\mathbb{N}}$

$\mathbb{N} \leftrightarrow \{1\}^{\mathbb{N}} \times \mathbb{N} \rightarrow \mathbb{N}^{\mathbb{N}}$ is injective

we derive the result

$\mathbb{N}^{\mathbb{N}} \equiv \mathbb{N}.$ ☐ (5)

Another way of stating this feature is that the uncountable continuum hypothesis is not an axiom that is independent of the countability of $\mathbb{Q}$ and this rule of induction, and the countability of $\mathbb{Q}$ naturally takes precedence over the continuum hypothesis.

Thus in a version of the second order logic developed here, all superexponential operations resulting in the construction of sets build countable sets from countable sets. This is a new conclusion, and must replace the uncountable continuum hypothesis. ☐

The previous proof raises the issue of the status of the Cantor diagonal argument on the uncountability of the real numbers. Before dealing with this, we address its counterpart for finite sets, where the Cantor argument does not work. We extend this idea to infinite but countable sets.

The Cantor diagonal argument is deconstructed as follows. First note the example of a finite set consisting of elements ordered as (a, b), where a and b $\in$ {0, 1}, can be described as two finite sets E, F where E = {(0, 0), (0, 1)} and F = {(1, 0), (1, 1)}. In this example we define *diagonal* E to be found from

$(\underline{0}, 0)$
$(0, \underline{1})$,

that is, (0, 1) as underlined, so NOT *diagonal* E = NOT (0, 1) = (1, 0). So E is finite, and the fact that NOT *diagonal* E $\notin$ E (but $\in$ F) does not show {E, F} is not a finite set. We have shown for a finite set, the diagonalisable elements form a subset of the finite set under an ordering.
In the finite case the Cantor theorem holds that the power set (the set of subsets) of any set is larger than the set, but the NOT *diagonal* relation for finite sets does not give the correct semantic interpretation for infinity. The correct definition of a set S_m being finite is that it is empty or has all elements $(s_1, s_2, ... s_m)$ with fixed m $\in \mathbb{N}$, and an infinite set is not finite. ☐

Definition 1.7.2. A *Eudoxus number*, the set of which we denote by $\mathbb{U}$, is a bounded number representable at most by a sum of a countably infinite set of rational numbers.

For the diagonal argument for infinite sets, consider the list of Eudoxus numbers denoted in binary and indexed by $2n_i$

$2\mathbb{N}$	The set of all diagonalisable Eudoxus numbers with respect to an ordering
$2n_i$	u_i

Firstly, we will consider a specific ordering of $\mathbb{U}$. By a similar argument to the finite case *diagonal* $\{u_i\}$ = {the Eudoxus number with the ith digit taken from u_i}, so NOT *diagonal* $\{u_i\} \notin \{u_i\}$ is consistent, since $\mathbb{N} \equiv 2\mathbb{N} \equiv (2\mathbb{N} + 1) \equiv 2\mathbb{N} \cup (2\mathbb{N} + 1)$, and all Eudoxus numbers are indexed by $n_i \in \mathbb{N} \equiv \mathbb{N}_{\text{diagonal}} \cup \{n_i\}_{\text{nondiagonal}}$. In this ordering $\{n_i\}_{\text{nondiagonal}}$ is one element appearing after the diagonal terms, or can be inserted in first to show the bijection with $\mathbb{N}$. Thus the utility of the diagonal argument disappears under a specific ordering.

We will now consider a generic ordering covering all possible orderings of $\mathbb{U}$. Then the larger set of $\{n_i\}_{\text{nondiagonal}}$ now corresponds to the set of possible $\mathbb{U}$, so each instance of this diagonal can appear, say, as the first element of $\mathbb{U}$ in a different ordering. Thus the utility of the diagonal argument now disappears under a generic ordering, and we are back to proving $\mathbb{N}^{\mathbb{N}} \equiv \mathbb{N}$, which we have already done, and so the Eudoxus numbers are countable. ☐

This theorem is in conflict with the assertion that $\{0, 1\}^{\mathbb{N}}$ is uncountable, which in turn uses the unacceptable semantic definition that the NOT *diagonal* relation for countably infinite sets acts as a criterion for uncountability. However as mentioned by P. J. Cohen in [1Co63], [1Co64] 'one can construct models in which the set of constructible reals is countable'. □

1.8. Ordinal (ladder) arithmetic in $\mathbb{M}_t$.

Definition 1.8.1. We adopt the ***standard protocol*** for ladder algebra:

$$\mathbf{\Omega}_{\mathbb{M}t} = \sum_{\text{all } \mathbb{M}t} 1. \qquad (1)$$

For any n $\in \mathbb{M}_t$, n < $\mathbf{\Omega}_{\mathbb{M}t}$, so $\mathbf{\Omega}_{\mathbb{M}t} \notin \mathbb{M}_t$. We will treat $\mathbf{\Omega}_{\mathbb{M}t}$ as being irreducible. This means we do not split $\mathbf{\Omega}_{\mathbb{M}t}$ into noncontiguous components, or truncate or extend it. We adopt for $\mathbf{\Omega}_{\mathbb{M}t}$ the negation shown below of a property attributed by Archimedes to Eudoxus of Cnidus for finite natural numbers: the ordinal infinity $\mathbf{\Omega}_{\mathbb{M}t}$ is inaccessible with respect to n and obeys the rule

for every m $\in \mathbb{M}_t$ *and for* $\mathbf{\Omega}_{\mathbb{M}t}$ *there does not exist an* n $\in \mathbb{M}_t$: $\mathbf{\Omega}_{\mathbb{M}t}$ < mn. □ (2)

Definition 1.8.2. We adopt the ***strict transfer principle*** for ladder algebra:
the axioms for a field or zero algebra hold with respectively a$\mathbf{\Omega}_{\mathbb{M}t}$, b$\mathbf{\Omega}_{\mathbb{M}t}$ *and* c$\mathbf{\Omega}_{\mathbb{M}t}$ *replacing some or all of* a, b *and* c *in these axioms.* □

An example is 1$\mathbf{\Omega}_{\mathbb{M}t}$ = $\mathbf{\Omega}_{\mathbb{M}t}$.

Definition 1.8.3. *Ladder transnatural numbers* $\mathbf{L}_{\mathbb{M}t\cup0}$ are defined by

$$\mathbf{L}_{\mathbb{M}t\cup0} = \cup_m[\mathbb{M}_{t\cup0}(\mathbf{\Omega}_{\mathbb{M}t})^m],\ m \in \mathbb{M}_{t\cup0}.$$

The algorithmic proof of a proposition by induction: choose a start value, assume for n and then prove for n + 1, now extends to n $\in \mathbf{L}_{\mathbb{M}t\cup0}$ under the strict transfer principle. However, the Peano axiom of induction states that the natural numbers are unique, so we must augment it for $\mathbb{N}_{\cup0}$ by saying it contains no elements $(\mathbf{\Omega}_{\mathbb{M}t})^m$. In ladder algebra the complement of all $\mathbb{M}_t$ is not empty, so an induction predicate can always be true, even in mZFC.

What is the status of finite proofs in this situation? We have seen by the axiom of the strict transfer principle above, that proofs by countable induction over $\mathbb{N}$ reduce to finite proofs. Thus there exist some finite proofs over countably infinite sets.

We argue in terms of states and not processes. For infinite processes, obtained values may oscillate infinitely. There are at least two approaches that we can take to enable consistency.

The first looks at logical deduction. Valid reasoning based on the evaluation of finitely determined states by processes which terminate finitely are retained. Otherwise valid infinite reasoning is restricted, so that we may be able to obtain consistent results previously unavailable. We can find *preferred evaluations* of these infinitely determined states or those with infinite processes so that the induction procedure is restricted for these types. We adopt the following method employing arguments under the strict transfer principle. For reasoning using all $\mathbb{M}_t$, start with the first element of $\mathbb{M}_t$, which is 1, and employ *linear induction* by allocating either $\mathbb{M}_t$, or an ordered block of n elements which belong to $\mathbb{M}_t$, so that the same proof is valid in each block. *Nonlinear induction* can be defined for other partitions.

The second approach extends Boolean logic. General multivalued logics are discussed in volume II, chapter V. Thus, for example, for the infinite type of systems we have been considering, we could have multivalued logic operating on three states, 'true', 'false' and 'oscillates'. A particular type of multivalued logic discussed in chapter XIII of [Ad15] is *probability logic*. A Boolean type logic with two states τ and υ can be extended so that its values are linear combinations of τ and υ, in particular retaining the boundary condition that both τ and υ exist within the logic. A linear probability logic contains the states cτ + (1 – c)υ where c is, say, a real variable. It is a theorem that oscillating values may be allocated as the value ½τ + ½υ in a linear probability logic.

The model used to interpret this logic is not here the statistical correlation approach given in chapter XIII of [Ad15], although its evaluation may be found as a limit process acting on its correlation with oscillating values.

In the 'paradox of the liar' the value of " 'A is valid' and 'A is invalid'" is false, and is not the same as " 'A is valid' and 'A is invalid' is false" (which is true) on keeping track of the level of nesting of the quotes and their implicit parentheses.

Since for us, logic also includes parentheses in its syntax, validity or invalidity of a formula may depend on the presence or absence of such parentheses, and this includes in recursive statements, which must have them.

X is invalid means X is invalid
(X is invalid) is valid means X is invalid,

where similar and further nestings are possible. For example there is a possible chain

(((X is invalid) is invalid) is invalid).

Thus there are an infinite number of states and their infinite countable evaluation

(((X is invalid) is invalid) is invalid) ...

is equivalent to the evaluation

$$Y = \prod_{\text{all natural numbers } \mathbb{N}} -1.$$

For example, using the strict transfer principle to evaluate

$$Y = \prod_{\text{all } \mathbb{N}} -1, \qquad (3)$$

partition ℕ into blocks of pairs starting from 1, then for each pair evaluate as a member of

$$W = (-1)_{\text{odd } \mathbb{N}} \times (-1)_{\text{even } \mathbb{N}},$$

and the product is always 1. Thus under the linear induction principle,

Y = 1,

and consequently the infinite countable evaluation equivalent to finding Y

(((X is invalid) is invalid) is invalid) ...

evaluates to X which is valid. Its linear probability logic evaluation is ½τ + ½υ. ☐

There exist ideals of these blocks, in which proofs of the above type live, for the example above 1.W and -1.W, and say cosets of the ideal 1.W

0 + W, and -2 + W,

where only the 0 coset is the preferred evaluation. It follows that the preferred evaluation of $\mathbf{\Omega}_{\mathbb{M}_t}$ obtained additively is even and not odd, and its linear probability evaluates to

½(even = τ)τ + ½(odd = τ)υ. ☐

We have seen that an uncountable set ℝ is not generated recursively by any means from ℕ, even when it has a countably infinite set embedded within it. We say $\mathbf{\Omega}_{\mathbb{R}}$ with uncountable ℝ is strictly less than ultrainfinity nЧ, n ∊ ℕ, whereas there exist a hierarchy of ultrainfinities $n\text{Ч}^m$, m ∊ ℕ, including superexponential types of this sort of expression, all of which are greater than $\mathbf{\Omega}_{\mathbb{R}}$. This arises because $(\mathbf{\Omega}_{\mathbb{R}})^{-1} = \boldsymbol{\varepsilon}_{\mathbb{R}}$ is not a multizero; no a – a = $\boldsymbol{\varepsilon}_{\mathbb{R}}$. ☐

We can further introduce a sequence

$$\mathbf{\Omega}_{\mathbb{N}} < \mathbf{\Omega}_{\mathbb{R}} < \mathbf{\Omega}_{\mathbb{R}'} < \ldots$$

in which there are no bijections of the type ℝ → ℝ′. It can now occur that, say, an algorithm halts in ℝ′ but not in ℝ, so ℝ′ is inaccessible with respect to ℝ. ☐

The allocation of two values, True and False, to logical evaluation may be extended to an n-fold set of values. There is a relation and a difference between these values and recursive procedures.

In recursion, if we apply the NOT operator to say the property of being countable for a set, then we have seen we can substitute the ordinal infinity $\mathbf{\Omega}_{\mathbb{N}}$ describing ladder numbers in the countably infinite set $\mathbb{N}$ by the uncountable ordinal infinity $\mathbf{\Omega}_{\mathbb{R}}$ of $\mathbb{R}$ described by the standard protocol

$$\mathbf{\Omega}_{\mathbb{R}} = \sum_{\text{all } \mathbb{R}} 1. \quad (4)$$

If we now apply recursively this procedure to $\mathbf{\Omega}_{\mathbb{R}}$, we get a higher order uncountability of a set $\mathbb{R}'$ with ordinal infinity $\mathbf{\Omega}_{\mathbb{R}'}$. There is then a countable or uncountable sequence

$$\mathbf{\Omega}_{\mathbb{N}} \to \mathbf{\Omega}_{\mathbb{R}} \to \mathbf{\Omega}_{\mathbb{R}'} \to \ldots$$

corresponding to sets with successively higher cardinality.

$$\mathbb{N} \to \mathbb{R} \to \mathbb{R}' \to \ldots$$

For the allocation of NOT to the value True or False, we merely switch their values. Nevertheless, we can consider this mapping as a recursion (mod 2). For an n-valued logic this may be considered as recursive (mod n).

This comparison we have been making has up to now been restricted to algebras which are abelian and associative. It is our thesis that these ideas can be extended to nonabelian and nonassociative n-fold logics. ☐

The axioms for exponentiation are given as a case of those for superexponentiation in chapter XVII of [Ad15]. We extend the strict transfer principle to these axioms too. It follows that we may employ $(\mathbf{\Omega}_{\mathbb{M}t})^2$ and $(\mathbf{\Omega}_{\mathbb{M}t})^u$ for u an exponent $\in \mathbb{F}$ or $\mathbb{Y}$. For u a positive number, $(\mathbf{\Omega}_{\mathbb{M}t})^u$ becomes an example of a hyperinfinity.

We will denote $(\mathbf{\Omega}_{\mathbb{M}t})^{-1}$ by $\boldsymbol{\varepsilon}_{\mathbb{M}t}$. It follows from the property of $\mathbf{\Omega}_{\mathbb{M}t}$ that the ordinal infinitesimal $\boldsymbol{\varepsilon}_{\mathbb{M}t}$ satisfies

for every $m \in \mathbb{M}_t$ *and for* $\boldsymbol{\varepsilon}_{\mathbb{M}t}$ *there does not exist an* $n \in \mathbb{M}_t$: $n\boldsymbol{\varepsilon}_{\mathbb{M}t} > m$. (5)

Similarly $(\mathbf{\Omega}_{\mathbb{M}t})^{-u} = (\boldsymbol{\varepsilon}_{\mathbb{M}t})^u$ is a hyperinfinitesimal. ☐

If we define $\mathbf{\Omega}_{\mathbb{Z}t} = \sum_{\text{all } \mathbb{Z}t} 1$, then

$$\mathbf{\Omega}_{\mathbb{Z}t} = 1 + 2\mathbf{\Omega}_{\mathbb{M}t}. \ ☐ \quad (6)$$

The set $\mathbb{Q}$ of zero or positive rational numbers where $q = m/n \in \mathbb{Q}$, $m \in \{\mathbb{M}_t \cup 0\}$, $n \in \mathbb{M}_t$, ignoring the equivalence relation where m/n is in lowest terms, may be mapped bijectively to the set of pairs {m, n}. Then under the axiom system we are adopting, derived from the strict transfer principle acting on all pairs bounded by m and n, the number of elements of $\mathbb{Q}$ in not necessarily lowest terms is $\mathbf{\Omega}_{\mathbb{M}t} (\mathbf{\Omega}_{\mathbb{M}t} + 1)$. ☐

Using the strict transfer principle we obtain the Euler formula

$$e^{i\theta\mathbf{\Omega}} = \cos(\theta\mathbf{\Omega}) + i\sin(\theta\mathbf{\Omega}). \ ☐$$

We may obtain integrals using this principle:

$$\int_0^{\Omega} x^2\, dx = \frac{1}{3}\mathbf{\Omega}^3. \ ☐$$