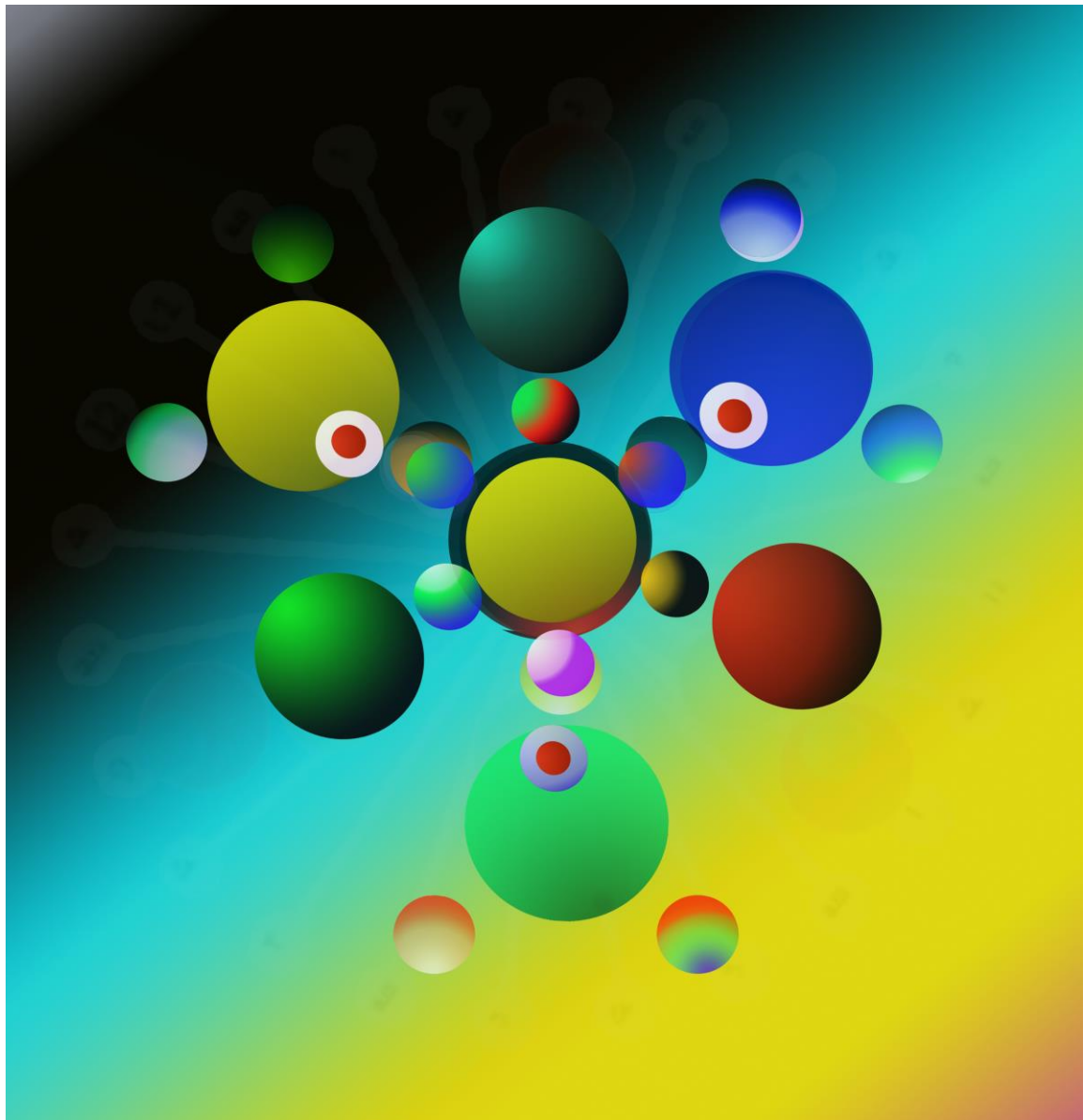


# **Number, space and logic**

**Volume I**

**Superstructures**



**Jim Hamilton Adams**

[www.jimhadams.org](http://www.jimhadams.org)

Copyright © 2018 Jim Hamilton Adams

This work may be translated or copied in whole or in part without the permission of the author, Jim Hamilton Adams, provided that he is given as the source of the text, which is asserted by him unless indicated otherwise. It can be used in information storage and retrieval systems, in electronic adaption, computer software, or by any method now known or later developed.

The cover artwork is by Ian Dell.

---

Jim Hamilton Adams is a researcher in the concepts of mathematics and their explicit representation. This work is a guide to some of these ideas, and an encouragement to the graduate to develop mathematical research productively and continually.

After a career in IT, Jim joined New Music Brighton as a composer and performer of his own works. He has been extensively involved in mathematical research.

The eBooks published in 2014 were *The climate and energy emergencies* and *Innovation in mathematics*. Volumes I, II and III of *Superexponential algebra* were completed in 2017, and *Number, space and logic* is a mathematical sequel to that work.

*Keywords:* analysis, nonconformal analysis, branched spaces, division algebras, novanions, vulcanions, zargonions, elliptic curves, modular forms, p-adic numbers, polynomial wheels, Riemann hypothesis, simple groups, superstructures.

---

## Foreword

*Number, space and logic* could be viewed as a mathematical manifesto and encyclopaedia. It defines and interrelates three ideas, all in some ways differing from current understandings. The first classifies all possible universes with space and time. The second idea states that all consistent problems are decidable, and gives methods for solving all such problems. The third extends our new ideas of infinity, including transcendental infinity, to computable models.

The Irish mathematician William Rowan Hamilton discovered the noncommutative numbers called the quaternions. This book extends the programme of Hamilton in novel ways, which reveals that the geometrical and physical intuition of this approach, long thought to have been superseded, is still alive.

George Birch Jerrard, who studied at Trinity College Dublin 1821–1827, developed methods on the solvability of the quintic polynomial equation begun by Tschirnhaus in 1683 and E.S. Bring in 1786, and extended them to show the coefficients of  $x^{n-1}$ ,  $x^{n-2}$  and  $x^{n-3}$  of a polynomial of degree  $n > 4$  can always be eliminated. In 1859 he wrote *An essay on the resolution of equations*. One version contains an epilogue by James Cockle (who became governor of New Zealand) stating that Jerrard's insistence that the quintic was solvable was incorrect. In support of Jerrard, it is well known by using the Cayley-Hamilton theorem and using companion matrices that matrix solutions of polynomial equations of arbitrary large degree can be directly written down. Galois theory, which for over 150 years has eclipsed these investigations, predicts that the solution of the sextic containing duplicate roots reduces to the standard Galois case of unsolvability, the theory of inseparable extensions, but it can be directly demonstrated that this sextic can be solved by radicals. Further, linear transformations of variables, which are always used to solve polynomial equations of degree  $n \leq 4$  cannot be accommodated in Galois theory. Galois theory reduces the study of solvability to multiplicative group theory based on permutations of its roots, but it can be shown that an extension of this theory, to ring automorphisms, transformations with  $+$  and  $\times$ , predicts that the swap of two roots will not in general leave a third root intact, so a correct solvability theory cannot be based on permutations. Nevertheless, a dependency theory, based on the special case of 'killing central terms' to obtain solutions, shows that no general polynomial of degree  $> 4$  is solvable by radicals by these means. But this does not imply that other methods, say involving comparison of polynomials of different types, cannot yield solutions by radicals. We will show both theoretically and by direct computation, that complex solutions of complex polynomial equations of arbitrary large degree exist.

The Riemann hypothesis, which predicts important results on the distribution of primes, is proved by two methods in this book. The earliest technique uses what we call Dw exponential algebras. These were developed by the author, and the idea was extended by the physicist David Bohm, in conversations with me via an intermediary, Ebrahim Baravi, in the early 1980's. David thought that this had applications to the Riemann hypothesis, but at the time I did not. On reopening this research over 30 years later, I rediscovered this idea. It is correct. The second proof involves extending the idea of infinity, which was mentioned to me as an approach by a researcher on this problem, whose name I do not know, and he provided no other details. The techniques we have developed here quite independently use the idea of infinities called capital Zeta functions, which incorporates the transcendental real number system within a computable model. We then apply this model to the Weil conjectures for finite fields in algebraic curves, fully proved after work by A. Grothendieck and co-workers by P. Deligne in a paper of 1974.

*Number, space and logic* combines a practical graduate level textbook with a research project in a commentary and development of the mathematics of the late 20<sup>th</sup> and the 21<sup>st</sup> centuries, just as *Superexponential algebra* [Ad15] does for the 20<sup>th</sup> century before that covered in this work. As a guide to this, we include the mathematics needed to prove the general Riemann hypothesis and from it the weak Goldbach conjecture, that any odd number greater than 5 is the sum of three primes. There is an exposition of the theory, to be described later in this foreword, of zargonions, polynomial wheels and branched spaces, and as detailed in volume III, a proof on the consistency of analysis by extending the work of Gentzen.

In order to present our work in a coherent way, we have tended to proceed top down, from the abstract to the specific, but so that this generality can be understood we first give an account of the meanings of these abstractions. First we describe our approach to the finite and the infinite, then how our generalisations of space fit into the theory of trees and amalgams. Finally we show how superexponential structures, which we call *superstructures*, can be used to extend these ideas in an overarching theory of mathematics.

In the chapter on *the meaning of the finite and the infinite*, after giving the rules for set theory we give an overview of standard and significant results in finite and discrete number theory, in the context of nonstandard extensions. Firstly we discuss extended sets of positive whole numbers, which we call transnatural numbers. We give a description of capital  $\Xi$  functions, which can be used to give a model of the real numbers. This idea is employed in Volume III to give a proof of the general Riemann hypothesis from a local field point of view. Transfinite number theory is considered for the transnatural numbers  $\mathbb{M}_t$ , where the theory is an extension of finite arithmetic. The essential feature of this is that, unlike our results in nonstandard set theory, these numbers are not two-way, or bijective, to the natural numbers, instead they have a transfinite number of members, but otherwise they share the properties of natural numbers. For instance they contain transfinite prime numbers and can be used to define transfinite rational numbers. We also deduce the inconsistency of the uncountable continuum hypothesis, which states there is no set of size strictly between the natural numbers and the real numbers, if we describe real numbers derived uniquely only from countably infinite Cauchy sequences, where elements become arbitrarily close to one another as the sequence progresses, implying they are countable. We give an account of ladder algebra which incorporates a reformulation and extension of nonstandard analysis. Then we extend results on the ladder algebra of infinite ordinal number arithmetic to  $\mathbb{M}_t$ .

The chapter on *the meaning of branched spaces* connects ideas about vector spaces to sets and logic. It then introduces us to our extension of the domain of the Euler-Poincaré characteristic used in topology (for a graph this is the number of vertices, minus the number of edges plus the number of faces), related to étale cohomology theory introduced by Grothendieck to prove the Weil conjectures. In its familiar form our description appears more restricted than usual, although other formulations are equivalent to it. In its extended form it describes types of topology which have no analogue in standard topology. The model we introduce is one of branched spaces. We can put it this way: when a point is removed from the equivalent of an n-branched line, it divides into n pieces. An *explosion* is a set that contains an infinite number of branches. Indeed these are the topological analogues of other behaviour we will meet in exponential and superexponential number systems.

Concerning *the meaning of superstructures*, which is the topic of the last chapter of the eBook *Superexponential algebra* [Ad15], this idea is an extension of the repetitive process which generates multiplication from addition, and exponentiation from multiplication. We generate an  $(n + 1)$ th suoperator, short for superexponential operation, by repeating an nth suoperator.

In that book we discussed nonstandard exponential and suoperator algebras, which we describe here by saying that they are unbranched compared with standard such algebras. A real number we say has a *sunorm* defining its size under exponentiation or a suoperator which is similarly real. So a suoperator on numbers has two features that are interesting to investigate: sunorms and branching.

Having built up the meaning of these abstractions, from their axiomatic formulation we then provide a detailed exposition of these ideas in reverse order: of superstructures, trees and amalgams, and the finite and the infinite, and look at some of their consequences.

Thinking these ideas could be developed to give an overarching description of mathematics, it has only been in the writing of this work that I realised to my astonishment that a detailed description was available not only in theory, but it was being built in practice.

Two essential leading ideas which are further components in this supermathematics are those of the nonassociative zargonions and the deconstruction of the current theory of polynomial equations and its replacement, now completed, by polynomial wheel methods, so called because I ask whether their implications are as important as the invention of the wheel, giving access to computable structures for the whole of mathematics. The difference between our approach and the standard one is that we say that there is no accommodation of nonabelian group theory in modelling the solution sets of abelian varieties, although there is always a mapping from noncommutative structures to commutative ones. Our methods reveal a link between the solvability of abelian varieties, elliptic curves and modular forms. The structural problems we solve can be generalised to all mathematical problems. Mathematics hitherto has stated that there exist undecidable problems. But the statement that there are consistent undecidable problems is false; we have proved that all consistent problems are decidable. When these computable structures combine with zargonion superstructures, mathematics describes number, space and logic in full and coherent generality.

## **Volume I in more detail**

Resulting from our investigations, we are inclined to introduce a new paradigm in mathematics, as is very necessary replacing for mathematical objects with addition and multiplication the theory of groups by ring theory, and in general replacing the theory of rings by superstructures. We describe the basic meaning of terms used in category theory, which looks at mathematics from the point of view of transformations, called morphisms, and develop its technical results. We define superstructures as nonassociative, or parenthesis dependent, generalised categories. Group theory is limited to a special case of these superstructures, but for groups there is only one operation, associative multiplication.

This paradigm shift goes far beyond the theory of groups, restricted to one operation, which has dominated the mathematical perspective for well over 150 years. It replaces the theory of symmetries under transformations by a theory based on multiple operations acting on many states. As we show, the latter theory provides a key to solvability that appears unattainable by classical group methods or their extensions.

There is a sequence of algebraic structures which begins with commutative real and complex numbers, and then the quaternions which are multiplicatively noncommutative, so  $AB \neq BA$ . Quaternions were introduced by the Irish mathematician William Rowan Hamilton in 1843. They can be represented by arrays of numbers with  $+$  and a special type of  $\times$ , called matrices,

although matrices can be singular so there is no general multiplicative inverse, but nonzero quaternions always have an inverse. The quaternions are conformal structures, meaning that they keep angles under rotations. Following on after the quaternions a generalisation is the octonions, which are usually nonassociative,  $A(BC) \neq (AB)C$ . Since they are nonassociative, they cannot be represented by matrices, but they are conformal structures. We say these structures are division algebras. The proof by J. F. Adams *On the nonexistence of elements of Hopf invariant one* [AdJF60], stated that there are no division algebras beyond the octonions. Indeed, the Cayley-Dickson construction taken up to the 16-dimensional sedenions, which applied to the quaternions gives the octonions, correctly shows that the sedenions are not division algebras. However, we have reviewed the technical case-by-case for 16 dimensions, which is external to [AdJF60], and not understanding this, we have looked into it using our own simplified methods derived from the Fano diagram describing the octonions, which we call fan methods. This shows there is a 14-dimensional division algebra, an alien mathematical object which for no other reason we call a 14-vulcannion. There are no 16-vulcannions. For a Vulcan number  $v$ , a  $(v + 1)$ -vulcannion generates a new Vulcan number  $6 + v$ . The only positive distinct Vulcan generators begin at  $v = 1$ .

Beyond the division algebras are the novanions. Only in the case when the real, or scalar, part of the novanion is zero can the product of two novanions be zero, otherwise they are division algebras like the octonions. We say that novanions form a novanion algebra. Like octonions, novanions are conformal, or angle preserving, structures. We say classical novanions are generated by division algebra objects up to the octonions, otherwise they are nonclassical. The survey given in chapter IV collects together results on division algebras and novanions. Additional to the results documented in *Superexponential algebra*, we show that 10- and 26-novanions have both fermion (odd number of twists) and boson (even twists) representations, and there is an enveloping infinite dimensional classical novanion containing all finite classical novanions within it, which is contained in an enveloping nonclassical novanion.

Since we are in rebellion, we have also abandoned here the practice that the coining of new mathematical terms should be based on the names of long-dead mathematicians with no direct links to an idea, or that they should be built from words and thinking current in ancient Rome or classical Greece. Continuing with the science-fiction naming style, a zargonion is generated using vulcannion and novanion components. Since a vulcannion algebra is a type of novanion algebra, so that a zargonion algebra includes all others, we use the generic name zargon algebra.

One of our interests is to investigate the classification of simple groups. Chapter V provides a description of the basic features of group theory, including a proof of the three isomorphism theorems, and introduces the Schur multiplier, used in this classification. We provide a table of the sporadic simple groups, which contrary to current understandings are part of an infinite system and introduce the orbit-stabiliser, Jordan-Hölder and Sylow's theorems.

We introduce an analysis of group theory derived from the theory, developed in [Ad15], *New physics for engineers* [Ad18a], *Elementary universal physics* [Ad18b] and *Investigations into universal physics* [Ad18c], of the nonassociative algebras we call zargonions. An interesting construction arises from a speculation of Daniel Hajas that 10-novanions are connected with the heterotic string in physics, so that 26-novanions, containing octonion type algebras would be present as a string. We know that octonions have a (mod 12) Jacobi identity mapping to algebras, in particular octonions are related to some exceptional Lie algebras, like  $E_8$ , with a mapping to groups, which is part of a more general result for zargonions. We state this because override conditions on novanions of dimension equal to or greater than 26 lead to simple groups via Jacobi identity mappings to algebras (mod 12) and by standard arguments, since overrides

create one structure. The case is similar for vulcannions. Picking up the work of Borcherds linking the monster simple group to orbifolds in the 26 dimensions of the heterotic string, we investigate and show that there are simple groups beyond the monster, in direct conflict with current understandings.

We are interested in polynomial rings, and where these satisfy an equation, usually when they are in at least two variables, these are called varieties. The theory of polynomials extends of the work in [Ad15] which has mathematically deconstructed Galois solvability theory. That work left open a proof, rather than an assertion, of the fact that polynomial equations are solvable by radicals in conditions where the method does not involve ‘killing central terms’ of a polynomial equation, violating Galois solvability theory restrictions.

In chapter VI we discuss polynomial wheel methods, giving techniques to solve polynomial equations of any degree in radicals, and how they arose from an intense study of polynomial comparison theory. For comparison methods a polynomial with appended bogus roots is compared with a polynomial variety without killing central terms. The variables in this variety may themselves be polynomials. In the study of the quintic, comparison methods give a relationship with elliptic curves, of the form  $y^2 = x^3 + ax + b$ . Polynomial wheel methods reduce solutions of known complexity to simultaneous equations, which as a consequence solves the P/NP problem in encryption theory. Thus we have discovered a method holding to a principle put forward by another author: in mathematics what is not forbidden is allowed.

In chapter VII we discuss varieties and the theory of elliptic curves, backtracking from the relation established in the previous chapter with the quintic equation, extend these ideas to higher degree, and then move on to matrix and zargonion varieties. We are interested in extension and contraction maps, transforming varieties by the addition or removal of variables.

Chapter VIII discusses modular forms and the modularity theorem, which is used in volume II, chapter IV to give a proof of Fermat’s last theorem. The modularity theorem is essentially about Galois representation theory, or in less elevated language, finite arithmetic using the binomial theorem. We discuss some significant theorems in finite arithmetic, known as the quadratic reciprocity theorem and its extension from prime numbers to composite numbers, known as quadratic totient reciprocity. We are also interested in generalisations beyond a quadratic degree. We extend these discussions to the hyperpolyticate numbers introduced in [Ad15] chapter I as an exercise, as a generalisation of intricate numbers. We interrelate the j-abelian idea with eigenvalues, which are numbers obtained from matrices, and hyperactual numbers or their hyperpolyticate extensions. Our discussion on two dimensional lattices is a precursor to that of the next chapter IX on zargonion lattices in n dimensions. We discuss abelian groups, Heegner numbers, topology and divisors, elliptic curves and finally theta functions in the context of lattices.

In chapter X on conformal and nonconformal analysis we discuss Fourier series, convolutions and kernels and their various extensions. We introduce differential intricate manifolds in the intricate Cauchy-Riemann equations, discussing these in the angle-preserving, or conformal, situation and also transformations in the nonconformal case, and extensions of integration theory in the Cauchy integral formula. We include also zargonion manifolds in this.

Finally, chapter XI extends the discussion of the previous chapter to suoperators.

## **On inviting the reader to mathematics**

What is mathematics? Possible worlds! These worlds have strict definitions. They have states which describe what is in these possible worlds, and specified transformations of these states, which describe how these possible worlds may change.

It is my philosophy – a leap from some thinking about physics – that what is possible and consistent in mathematics matches what exists. Since mathematics is used to describe physics, consistent mathematics can describe the real world. If a symbol represents 1 as 0, in some systems this is inconsistent. Consistency and the stability of solutions of equations represented by symbols are related. In this way of thinking, states that evolve stably are what now exists in the physical universe, and other states have physically self-annihilated in an inconsistency.

It is then an interesting programme to document what mathematical worlds are possible. Some of these worlds appear to be hallucinations – they have no implementation as far as we can see in the physical world. Others may have implementations which may only be used by future generations on the discovery of new physics, or otherwise have seemingly obvious practicality but may be generalised in ways which have no apparent immediate physical significance.

The objective of this work is to introduce you the reader to these possible worlds and to ask you to become creatively involved in their construction and the analysis of their behaviour, so that giving new definitions and working out their consequences comes naturally to you. This programme is both abstract and potentially useful, to manipulate the physical world in ways which may be at the frontiers of human knowledge and experience, or may be mundane applications of direct and practical use.

Mathematics proceeds outside the social group by the analysis of abstract systems whose meaning is derived from features of the world, and also within this group, as culture, history, revolution, extension and rewriting of its basis. In order to connect with its modern modes of reasoning, we need also to be aware of the human features which have led to the development of mathematics.

The objective of creative mathematics is the production of ideas. These need caring support, and part of this objective is not just to replicate results, but to reconstruct mathematics from new principles, to have the resilience to deconstruct what has gone before if it is needed, and to extend results by an analysis of the core ideas of the subject as it currently stands. Thus it becomes feasible to develop generalised new methods to tackle outstanding problems.

We need also to be aware that as a human endeavour we are not at a unique point in history where all aspects have been decided with certainty, and that human systems have triumphs and failings. We must have the courage of our convictions, and the sure analysis of our own and others thinking to develop this civilisation, and where necessary to have the resilience to deconstruct what has gone before, because even in mathematics we are not in utopia, and some things, promoted in the system from one generation to another, are wrong.

Mathematics, once constructed, is about the truth. It may be that this truth has no respect for high-ranking authority. I feel it is good to be aware of this possibility when we conduct our research.



## Processes of mathematical thinking

An objective in writing this work has been to make progress and to explain to the reader. To make progress in mathematics I have relied on what is to hand, my own understanding or misunderstanding, and my intuition.

My approach to understanding the subject is reliant on a feature of my personality, that I am not good at memorising a large number of facts, and therefore I must proceed by isolating a clear, distinct and small number of basic principles, and by applying rules of deduction to them which are recorded on some external medium, I can then reach finally a full set of results.

To isolate these principles, I often find they cannot be derived from studying only the latest mathematics texts, but to make sense of these basic building blocks requires backtracking through the mathematical literature, until the principles on which a sector of mathematics is based is simple and its future form of development is clear. This history is often a long one, because each new mathematical generation feels compelled to revolutionise the ideas of its predecessors, and indeed eventually we find in Western culture the language of discourse is no longer English, but mainly German and also French, with many other languages included, and before this the language of scientific communication is Latin, or if we want to go further back than this, it is Greek. It is my assertion that to follow this history is extremely important in isolating key concepts and the processes inherent in current thinking. This history is not, as is often presented by historians with standard mathematical culture, a history of personalities. In essence, it is the history of the development of ideas.

Once this subject is understood as a way of thinking, it is necessary to analyse it. Basic to the way I respond to this culture is the *insight* that I need. Perhaps some mathematicians have no intuition, just as some people are blind or deaf. Whatever the case, intuition is available to me, and I wish to apply it. Intuition may be thought of as an approach which grasps the whole of a subject at once, in an immediate understanding of all of its parts. It is my contention that in order to connect my intuition, which is what I feel about a result, to mathematical culture, I have to bring to the surface of my consciousness what is immediately grasped subconsciously, in order to analyse the contents of what I am thinking so that it is externally and explicitly expressed. Once this intuition has been transferred to written form, it is in a form in which it can be analysed. It is then possible to subject it to the mathematical culture which has arisen in describing logic, by devising methods of calculation in which ideas can be tested out. This initial stage is usually one in which the total system is not immediately expressed in an axiom system and regularised rules of deduction, but it allows further development in which these features are adopted.

The result of this process is that intuitive ideas are collected together and are compared with current mathematical culture. Apologising for ignoring necessary modesty here, in my opinion I have seen an explosion of new ideas and feel that this intensively creative process has given rise to significant mathematics.

We need to continue the process and explain this thinking, for two reasons. The first reason is that creative ideas are eventually subject to systemisation, so that they can be analysed using standard techniques adopted by the mathematical community. The second is so that jumps in reasoning which the author thinks are justified are often not obvious enough to a reader new to this thinking. Thus I must be forced to describe all stages in my thinking. Proper mathematical argument is about explicit reasoning, so that all processes are described and a clear argument can be followed without dispute.

The existence of disputes in mathematics is interesting. Thus, having introduced disputes in the first place by possibly controversial intuitive reasoning, an objective becomes to minimise or remove them by the introduction of explicit rules of deduction and symbolic manipulations together with a set of concrete meanings of these symbols which point to the world. This is the method I have chosen to develop new symbolic generalisations and create new mathematics.

The most significant directions of my life have been determined by the labour I wanted to do. Had it been that I were sent to a pig farm in the cultural revolution in China, on adjustment to the collapse of my excessive ambition, I would probably have become an exemplary worker exceeding all quotas, without a thought to my personal promotion. But that would have been a waste. I have received an excellent education that I did not request, and did not find myself succeeding particularly well within it, but having adjusted to the culture of science, I became fascinated by its ideas and content. The wonderful mathematician Ramanujan did not succeed fully at university because he was too interested in his investigations of the subject to follow properly the course work, and he was not interested in social promotion through the system. In the pursuit of my obsessive interests in science, in my life this has befallen me too.

Aesop's fable of the tortoise and the hare ends with the tortoise winning the race. In research a certain level of intelligence is useful, but what is important is persistent curiosity. I am the tortoise. In my research I have exceeded in duration, but despite my wishes, not in collective effort, many Long Marches. The result of this work is no longer mine, it belongs to human culture, but social promotion through the system as a consequence of this is of no interest to me, except where it might promote general social welfare.

At the start of my life my attitude was to ignore the rules and do the right thing. But now I have come to the conclusion that it is very useful to know what the rules are. They can be used to define acceptable and unacceptable behaviour, and to navigate through the social system to attain objectives.

I started my scientific career with an innate conservatism towards scientific conventional wisdom and an aspiration to reach the summit of its understandings. In this respect I treated science as a religion. But at university I began to have doubts as to the correctness of some theories, although I did not have the means or resources to question them properly.

In order to succeed well in the academic system, it is necessary to give the right answer to questions. Many mathematicians now teaching the subject freely admit that they crammed for examinations, and did not properly understand results, although they were concerned to memorise and replicate them. I believe some mathematicians writing on the subject today do so from the point of view that a proof has to be memorised, and if it is viewed abstractly it is not important to understand properly what is going on, provided the conclusion is reached rigorously.

In pursuit of these interests, time and again in mathematics I am confronted with the fact that I do not understand a proof. When I inspect my own proofs and come to the same conclusion, the answer is easy. However much I am attached to an idea behind a proof, which sometimes comes from experience of the material concerning it, if on reviewing what I am trying to say, the result is no nearer after all the effort, the theorem has to be jettisoned.

There are standard proofs in mathematics which I have not understood after considerable effort at coming to grips with what they are saying. These proofs are sometimes technically erudite and sometimes extremely long. They can be surprising or counterintuitive. I do not believe the

majority of mathematicians spend time on inspecting them, but rely on the peer review process to check their validity. Doubts are allayed if the result is well-established and a considerable confirmatory literature surrounds it.

I have learnt that if, after considerable investigations, such a result is no clearer, that initial doubts are confirmed, and that the theorem is counterintuitive, a not unreasonable strategy is to assume that this does not arise from my own innate stupidity, but that the result is wrong. This at least allows an entry into other approaches to investigate what really might be the situation, since there is a logical mode of deduction which states that to prove a result, first assume the opposite and then prove a contradiction. Very often when I try this approach, the result is not a contradiction but an escape route, and if after much research an escape route cannot be closed off, persistent investigation has led to a refutation of current findings.

To begin with, this astonished me, and an implication going beyond the proclaimed rigorous methods of mathematics to subjects of weaker intent is that much human reasoning of the current day is suspect. But now in new work I have taken this conclusion not as an end view in my investigations, but its starting point.

As a word of encouragement, to make progress in mathematics it is best to assume that the truth can be arrived at by a process of successive corrections of theory. If I look at the work of Aristotle, whose deep thinking and wisdom ranged over history, logic, physics, biology and philosophy in a way that seems impossible because of intense specialisation today, we see someone who is concerned to find the truth when others of his generation were not. There is an erudite and difficult passage in his work where he deals with the motion of the moon. It is apparent 2200 years later that, not to put too fine a point on it, it is a load of codswallop. I feel that some intimidated students might have been examined on it, and heaven help them if they gave unapproved answers. It has been said that although Aristotle had many false ideas, that the history of much of scientific thought has consisted in correcting them. But it is not possible to make much progress when no prior thinking has been made of a topic. That progress has been made in mathematics has been contingent on the fact that those studying the subject had some wrong ideas but did not stay silent.

I wish to end this section with a remark on learning and unlearning. In general, I have found no problems in presenting my ideas to undergraduates in their twenties who are studying for degrees in a science subject, particularly mathematics and physics. These ideas are absorbed and accepted without trauma. The reason that revolution has its meaning is that Copernicus's book *On the revolutions of the heavenly spheres* could not be cognitively accepted by those who had been trained in a different outlook and had accepted it in their earlier years. I have found that those who have been taught the subject of mathematics in their youth, have reached maturity and maybe have been teaching this too, and have accepted this teaching cannot accommodate to the double burden of learning this new material which is at variance with what they have accepted, and at the same time unlearning what has been embedded in their way of thinking for very many years. A response from some younger people already trained is to find at every opportunity ways of undermining the ideas, always framed from the point of view of total acceptance of what they have been taught, and for such people who are older, of refusing to engage in any way with these ideas, and even the refusal to read anything about them. The task I face is a difficult one. I do not know enough about psychology to understand the proper techniques which people can use to unlearn a subject. I know that when learning a piece of music, if you introduce an error into the performance of it, it becomes necessary to unlearn the error, and this is difficult, because the error becomes embedded in the way the mind operates in producing automatic responses. Have I then been able to escape from this situation myself?

Perhaps I have an immediate readiness, frequently applied, of being able to change my conclusions in the face of evidence, together with the determination to find the facts and long term to maintain my stance or suspend my judgement when I believe the evidence is unproven. The only other way I can think that the mind uses to unlearn responses is to cry, but this is a speculation. I think this situation is about natural behaviour and needs to be viewed with compassion and understanding. In the long term, the logic of the viewpoint I am presenting will win out. That it may take a long time is unfortunate, but given the human condition this is understandable.

## **The author's responsibility**

This work is provided in the Mathematics section of the website [www.jimhadams.org](http://www.jimhadams.org), where a brief synopsis of each chapter is provided. It is the sequel to *Superexponential algebra* [Ad15], displayed after *Number, space and logic* in the Mathematics section. Nevertheless, I have made *Number, space and logic* depend less on [Ad15] by taking some material from it.

A central objective of this work is to encourage the graduate to produce mathematical ideas, and also to give assistance by providing a minimally fussy exposition of the research into number theory of the last fifty years. The creation of new mathematics can start from humble beginnings. Its process is too long delayed in the present academic system. We are seeking to foster insight so that the reader can pursue further investigations into the technical literature in a spirit of an understanding of its background. The reader will face a vast panorama of unified but simple new mathematics in this book. Its invention and discovery can be your work too.

Exercises are not provided to work through the text, the reason being that these are no more needed for the development of creative skills than the memorisation of words and standard literary works is necessary for an active participation in becoming an author. My hope is that the diligent reader will provide what is necessary for constructive work, and that it is no longer necessary to give direction.

Since I am aware that the first language of the reader may not be English, I have looked at the text and removed high literary style. Technical terms are I hope well explained, and examples are given.

I am surrounded by people who definitely dislike and do not relate to abstraction, particularly mathematical abstraction. Some writers in our subject use abstract methods of reasoning to obscure simple ideas and clothe them in a system of thought which is remote. But abstraction can also reveal, and even attain ends which cannot be achieved by other means. Mathematics is best written in a style appropriate to its audience. For a graduate text I am confronted with the need and indulgent necessity, if given enough time, of writing in a rigorous mathematical style that will appeal to a mature mathematical audience. I have attempted, after much effort in attaining results, to rewrite this text in a clear classical style that reveals the subject as well as developing it.

Except for marketing, our inclination is that a proof is only the final stage in the presentation of a mathematical idea. Proofs are necessary, but not sufficient.

I would advise that the academic system can refer without proof to topics containing accepted well-known but false results, or demand the application of wrong reasoning to reach standard and erroneous conclusions. Errors may propagate through the examination system from one

generation to another by selection of people who accommodate to them, with the authority structure too rigid to allow change. Much stress can be avoided on acknowledging this fact.

Reflecting on my practical experience of the idea that major new understandings are never encountered in history without considerable resistance, I wish to issue a caution. Some such ideas have eventually been adopted by systems of authority not only because they represent the truth, but because their originator said so. I wish to repudiate such a notion. The authority is the truth. Period. Should it occur in the future that these ideas become the conventional wisdom, and if any one of them were in error, be aware that had I been there, my primary mathematical objective would be to deconstruct it.

Faced with my own condition, I love to promote an attitude: getting it wrong is the first stage in getting it right.

I thank Graham Ennis, who has sustained and inspired me with his encouragement. I have found John Baez's website beautifully presented and a treasure-trove of links to interesting and accessible accounts of the research literature. Very useful to my mathematical development have been the books *A first course in modular forms* by Fred Diamond and Jerry Shurman [DS05], and *Sphere packings, lattices and groups* by John H. Conway and Neil Sloane [CN98]. Research needs ideas, and also the people to pull them back to Earth. As well as the usual fulsome acknowledgement to Doly García in her criticism of this work, who cannot disagree with everything I say because sometimes I quite easily adopt her suggestions, I wish to express especially my gratitude to the stimulating interest of James Hamilton whose calculational checks amounted to an interactive collaboration on the chapter on polynomial wheel methods and comparison techniques. The contents of this work are my own, as is the responsibility for any errors.

As this work continued to develop, the conclusions have surprised and delighted me. I hope *Number, space and logic* will delight and surprise you too.

Jim Hamilton Adams  
Dublin  
September 2017

# **Table of contents**

## **Volume I Superstructures**

### **Foreword**

### **Table of contents**

### **Definitions of mathematical terms**

### **Chapter I**

#### **The meaning of the finite and the infinite**

- 1.1 Introduction
- 1.2 Formal language
- 1.3 Modified Zermelo-Fraenkel set theory (mZFC)
- 1.4 The Peano axioms for the transnatural numbers,  $\mathbb{M}_t$
- 1.5 A model for the real numbers
- 1.6 The capital  $\Xi$  function
- 1.7 The Euclidean algorithm
- 1.8 Fermat's little theorem
- 1.9 Euler's totient formula
- 1.10 Transp-adic numbers
- 1.11 The inconsistency of the uncountable continuum assumption for  $\mathbb{N}^{\mathbb{N}}$
- 1.12 Ordinal (ladder) arithmetic in  $\mathbb{M}_t$

### **Chapter II**

#### **The meaning of branched spaces**

- 2.1 Introduction
- 2.2 Vector spaces, scalar products and modules
- 2.3 Boxes and box scalar products
- 2.4 The probability logic of dependent and independent events
- 2.5 Posets and lattices
- 2.6 Finished and unfinished sets
- 2.7 Twisted and untwisted logic
- 2.8 The Euler-Poincaré characteristic
- 2.9 The familiar square, cylinder, torus and cube
- 2.10 Branched spaces
- 2.11 Models for branched lines and areas
- 2.12 Models for multiobjects
- 2.13 Deformation retracts and orientation
- 2.14 Branched handles, crosscaps and surgery of branched spaces
- 2.15 Explosion boundaries
- 2.16 Trees and amalgams
- 2.17 The general polynomial

## **Chapter III**

### **The meaning of superstructures**

- 3.1 Introduction
- 3.2 Notation
- 3.3 Intricate and hyperintricate numbers
- 3.4 The j-abelian property
- 3.5 Dw exponential structures
- 3.6 Dw suoperator superstructures
- 3.7 Sunorms and branching
- 3.8 Nonassociative representations of superstructures
- 3.9 Interaction and descent of suoperator expressions
- 3.9 Substructures and singularities
- 3.10 Suderivatives
- 3.11 The meaning and ideas of category and superstructure theory

## **Chapter IV**

### **Zargonions**

- 4.1 Introduction
- 4.2 Division algebras
- 4.3 Zargon algebras
- 4.4 The quaternions
- 4.5 The nonassociative octonion division algebra
- 4.6 T-junctions, fans and vulcannions
- 4.7 The classification of novanions
- 4.8 Eigenvalues
- 4.9 The 10-novanions
- 4.10 n-novanions
- 4.11 The search for other novanion algebras
- 4.12 The 64-novanions
- 4.13 The García classification problem
- 4.14 The generation and classification of zargon algebras
- 4.15 Zargon algebras of negative dimension
- 4.16 Zargon brackets
- 4.17 Discreteness and zargon rings
- 4.18 Tharlonions and tribbles
- 4.19 The origin of the discrepancy of our results with K theory

## **Chapter V**

### **Groups**

- 5.1 Introduction
- 5.2 Basic ideas in group theory
- 5.3 Normal subgroups
- 5.4 The isomorphism theorems
- 5.5 The Schur multiplier
- 5.6 The standard classification of simple groups
- 5.7 The orbit-stabiliser theorem

- 5.8 Sylow's theorems
- 5.9 Composition series
- 5.10 Continuous groups
- 5.11 Zargon groups

## **Chapter VI**

### **The discovery of the polynomial wheel**

- 6.1 Introduction
- 6.2 Galois representation theory, Galois solvability and proofs
- 6.3 The incorrectness of the Galois solvability hypothesis
- 6.4 A geometric realisation of the cubic
- 6.5 Discriminants, the Sylvester determinant and Bring-Jerrard form
- 6.6 The comparison method for the quintic and its elliptic curve
- 6.7 The solution of the sextic given the solution of the quintic
- 6.8 Polynomial wheel methods solving general polynomials in radicals
- 6.9 Solutions for Gaussian integer degrees

## **Chapter VII**

### **Matrix and zargonion varieties**

- 7.1 Introduction
- 7.2 Elliptic curves
- 7.3 Rational points on derived elliptic curves
- 7.4 Eigenvalues of matrix polynomials
- 7.5 Homogenous matrix varieties in two variables.
- 7.6 Extension maps and contraction maps.
- 7.7 Zargon suboxes
- 7.8 Zargonion varieties
- 7.9 Zargonion twisted manifolds

## **Chapter VIII**

### **Modular forms**

- 8.1 Introduction
- 8.2 Galois representation theory
- 8.3 Quadratic and general totient reciprocity
- 8.4 Hyperintricate and hyperpolyticate numbers
- 8.5 Eigenvalues and hyperactual numbers are J-abelian
- 8.6 Two dimensional lattices and Heegner numbers
- 8.7 Two dimensional lattices and abelian groups
- 8.8 Two dimensional lattices, topology and divisors
- 8.9 Two dimensional lattices and elliptic curves
- 8.10 Two dimensional lattices and theta functions



## **Chapter IX**

### **Zargonion lattices**

- 9.1 Invitation to the reader
- 9.2 Introduction and history
- 9.3 The  $E_8$  lattice
- 9.4 The Leech lattice
- 9.5 The standard and novanion group constructions from octonions
- 9.6 Zargonion simple group constructions

## **Chapter X**

### **Conformal and nonconformal analysis**

- 10.1 Introduction
- 10.2 Fourier series
- 10.3 Convolutions
- 10.4 Good kernels, the Cesàro mean and Fejér's theorem
- 10.5 The Maurer-Cartan equation
- 10.6 The intricate analytic Cauchy-Riemann equations
- 10.7 J-diffeomorphisms and non-fixed J
- 10.8 Intricate components and relative orientation
- 10.9 The intricate Cauchy-Riemann equations are nonconformal
- 10.10 The zargonion analytic Cauchy-Riemann equations
- 10.11 The complex Cauchy integral formula
- 10.12 The zargonion Cauchy integral formula
- 10.13 Picard's theorems
- 10.14 Singularities of mixed twisted and untwisted structures
- 10.15 Nonconformal structures are mixed and singular

## **Chapter XI**

### **Conformal and nonconformal suanalysis**

- 11.1 Introduction
- 11.2 Dw superstructures
- 11.3 Stepping down and stepping up
- 11.4 Sudifferentiation
- 11.5 Suintegration

**Volume II**  
**Trees and amalgams**

**Foreword**

**Table of contents**

**Definitions of mathematical terms**

**Chapter I**

**Trees and amalgams**

- 1.1 Introduction
- 1.2 Undirected graphs
- 1.3 Cubic graphs
- 1.4 Frucht's theorem
- 1.5 Directed graphs
- 1.6 Trees
- 1.7 Free groups
- 1.8 The Nielsen-Schreier theorem
- 1.9 Branched retracts
- 1.10 Amalgams

**Chapter II**

**Branched spaces and explosions**

- 2.1 Introduction
- 2.2 The suoperational model
- 2.3 Topological branching and explosions
- 2.4 The Riemann-Roch theorem
- 2.5 Diffeomorphisms
- 2.6 Bézout's theorem
- 2.7 The Gauss-Bonnet theorem
- 2.8 The Atiyah-Singer index theorem
- 2.9 Embeddings and projections
- 2.10 The Lefschetz fixed point theorem

**Chapter III**

**Surgery**

- 3.1 Introduction
- 3.2 Surgery on unbranched spaces
- 3.3 The idea of surgery on branched spaces
- 3.4 Surgery as additive maps
- 3.5 Multiplicative maps
- 3.6 Superstructural maps

**Chapter IV**  
**Sequent calculus and colour logic**

- 4.1 Introduction
- 4.2 Introduction to colour sets and other set theories
- 4.3 Supervarieties and colour logic
- 4.4 Classical sequent calculus and Gentzen normal form
- 4.5 A colour model for modal logics
- 4.6 Colour sequents

**Chapter V**  
**Probability logic, distributions and Zargonion fractals**

- 5.1 Introduction
- 5.2 The propositional probability mapping to predicate logic
- 5.3 Distributions
- 5.4 Polynomial fit
- 5.5 Weighted distributions
- 5.6 The maximum significance theorem
- 5.7 Colour probability
- 5.8 Zargonion probability logic
- 5.9 Zargonion dimension and fractals

**Chapter VI**  
**Homology superstructures**

- 6.1 Introduction
- 6.2 Orientation and antiorientation
- 6.3 Singular homology and cohomology
- 6.4 Ext and Tor

**Chapter VII**  
**Cohomology superstructures**

- 7.1 Introduction
- 7.2 Our view on K theory
- 7.3 Generalised Steenrod squares
- 7.4 Characteristic classes
- 7.5 Étale cohomology
- 7.6 Crystalline cohomology
- 7.7 Some comments on motivic cohomology

**Chapter VIII**  
**Homotopy superstructures**

- 8.1 Introduction
- 8.2 Rotations, reflections and gluing
- 8.3 Circles, ellipses and  $D_w$  superstructures
- 8.4 Zargonion rotations as knots and twistings

- 8.5 Epicycle knots
- 8.6 Quillen homotopy
- 8.7 Zargonions and Bott periodicity
- 8.8 Mixed structures and the Poincaré conjecture

**Volume III**  
**The finite and the infinite**

**Foreword**  
**Table of contents**  
**Definitions of mathematical terms**

**Chapter I**  
**Number theory**

- 1.1 Introduction
- 1.2 Finite arithmetic
- 1.3 Ladder algebra and transcendence
- 1.4 The Goodwin bijection
- 1.4 l-adic numbers

**Chapter II**  
**Class field theory**

- 2.1 Introduction
- 2.2 Algebraic congruence arithmetic
- 2.3 l-adic Fourier series
- 2.4 Gauss, Ramanujan and Kloosterman sums
- 2.5 The Rogers-Ramanujan identities
- 2.6 Theta functions

**Chapter III**  
**The consistency of analysis**

- 3.1 Introduction
- 3.2  $V = L$
- 3.3 Forcing
- 3.4 Gentzen's consistency theorem for arithmetic
- 3.4 A Gentzen-type proof of the consistency of analysis

**Chapter IV**  
**Fermat's last theorem**

- 4.1 Introduction
- 4.2 Fermat's last theorem and elementary methods – first method
- 4.3 Fermat's last theorem and elementary methods – second method
- 4.4 The proof using the modularity theorem

## **Chapter V**

### **Global zeta functions**

- 5.1 Introduction
- 5.2  $\Gamma$ -functions
- 5.3 The basic problem
- 5.4 Complex Dirichlet exponential algebras
- 5.5 The first proof of the Riemann hypothesis
- 5.6 L-series

## **Chapter VI**

### **Local zeta functions**

- 6.1 Introduction
- 6.2 Background and history of the Weil conjectures
- 6.3 The statement of the Weil conjectures
- 6.4 The projective line
- 6.5 Projective space
- 6.6 Elliptic curves
- 6.7 Weil cohomology
- 6.8 Grothendieck's proof of three of the four conjectures
- 6.9 Deligne's first proof of the Riemann hypothesis conjecture
- 6.10 Deligne's second proof

## **Chapter VII**

### **The Goldbach conjecture**

- 7.1 Introduction
- 7.2 The Helfgott proof of the weak Goldbach theorem
- 7.3 The proof via the Riemann hypothesis
- 7.4 Conclusion

## **References and Index**

### **References**

### **Index**

## Definitions of mathematical terms

The following terms, symbols, ideas and definitions are used in the text. The arrangement is by ideas rather than alphabetic. This may be scanned as a more technical alternative to the details of the contents in the Foreword, or as a further introduction to the contents which follow in the eBook. The notation, for example (SA III), refers to *Superexponential algebra* chapter III, and (NSL I, IV) refers to this work, volume I, chapter IV.

### 1. Sets S, T (SA III).

$\emptyset$  (*the empty set*). The set with no members.

$\odot$  (*the void set*). The set satisfying a false condition.

$\in$  (*belongs to*). If  $x$  is a member of a set  $S$  then  $x \in S$ .

$\subset$  (*properly included in*). If a set  $S$  is included in a set  $T$  and  $S$  does not equal  $T$ .

$\subseteq$  (*included in*). Inclusion, when  $S = T$  is possible.

$CS_T$  (*complement of S in T*). Those  $x$  not belonging to  $S$  but that belong to  $T$ , and  $S \subset T$ .

$\cup$  (*union of sets*). If  $x$  belongs to  $S$  or  $x$  belongs to  $T$  then  $x$  belongs to  $S \cup T$ .

$\cap$  (*intersection of sets*). If  $x \in S$  and  $x \in T$  then  $x \in S \cap T$ .

2.  $\mathbb{N}$  is the set  $\{1, 2, 3 \dots\}$  of *positive whole numbers*, also called *natural numbers*. If this set contains the element 0, we denote it in this eBook by  $\mathbb{N}_{\cup 0}$ . If we wish to emphasise that it does not contain zero, we use  $\mathbb{N}_{\neq 0}$ .

$\mathbb{Z}$  (from the German Zahl for number) is the set  $\{\dots, -3, -2, -1, 0, 1, 2, \dots\}$  of negative, zero or positive *integers*.

$\mathbb{Q}$  is the set of *rational numbers*  $m/n$ , where  $m \in \mathbb{Z}$  and  $n \in \mathbb{N}$ , for example  $1/2$ .

$\mathbb{A}$  is used in this eBook as the set of *algebraic numbers*, sums and differences of numbers of the form  $p^q$ , where  $p, q \in \mathbb{Q}$ , but  $p$  and  $q$  together are not both zero, for example  $1 + 2^3\sqrt{\frac{1}{5}}$ .

$\mathbb{M}_t$  is the set of *transnatural numbers* for index  $t$ , satisfying the rules for  $\mathbb{N}$ .  $\mathbb{M}_1 = \mathbb{N}$ , with proper injections  $\mathbb{M}_t \rightarrow \mathbb{M}_{t+1}$ ,  $\mathbb{M}_t \rightarrow \mathbb{M}'$  and surjections  $\mathbb{M}' \rightarrow \mathbb{M}_{t+1}$  for distinct  $\mathbb{M}'$ .

$\mathbb{Z}_t$  *Transintegers* are positive, zero or negative transnatural numbers.

$\mathbb{Q}_t$  *Transrational numbers* are of the form  $m/n$  where  $m \in \mathbb{Z}_t$  and  $n \in \mathbb{M}_t$ .

$\mathbb{A}_t$  *Transalgebraic numbers* are sums and differences of the form  $p^q$ , where  $p, q \in \mathbb{Q}_t$ , or alternatively formed in a similar way from superexponential operations.

$\mathbb{R}$  *Real numbers* have no imaginary component but are possibly not transalgebraic.

*p-adic number*. Arithmetic with a nonstandard idea of closeness between numbers.

3. *Peano axioms*. The rules for natural number arithmetic.

*UCH*. The false *uncountable continuum hypothesis* that  $\mathbb{R}$  is bijective to  $2^{\mathbb{N}}$ .

*ZFC*. *Zermelo-Fraenkel* standard set theory with the axiom of choice.

*mZFC*. *Modified ZFC*, allowing a set satisfying a false condition (but not invalid choice).

*Propositional calculus*. The logic of truth tables for true and false.

*Predicate calculus*. Propositional calculus allowing the statement 'there exists'.

*Untwisted logic*. Logic for a set embedded in an oriented manifold.

*Twisted logic*. Logic for a set embedded in a twisted manifold.

**4. Congruence arithmetic (mod n).** Finite or ‘clock’ arithmetic where transnatural numbers come back to themselves, so its set is  $\{0, 1, \dots, (n - 1)\}$  and  $n = 0$ .

*Prime number.* A transnatural number which only when divided by 1 and itself gives a transnatural number. Example: 7.

*Totient ( $\varphi(s)$ ).* For a transnatural number  $s$  as a product of primes  $p, q, \dots, r$  to powers  $j, k, \dots, m$ , if  $s = (p^j)(q^k)\dots(r^m)$  then  $\varphi(s) = s \left[ \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{q}\right) \dots \left(1 - \frac{1}{r}\right) \right]$ . Example:  $\varphi(20) = 8$ .

**5. Abelian.** Occurs for a set with a general operation  $+$  (not necessarily addition as usually understood) when  $a + b = b + a$  always.

*Commutative.* Abelian, but generally written for  $\times$  rather than  $+$ .

*Associative.* Satisfying  $a + (b + c) = (a + b) + c$ , or  $a(bc) = (ab)c$ , etc.

**6. Eudoxus numbers,  $\mathbb{U}$ .** Any number which multiplied by some integer can have a size within a range of finite positive natural numbers.

*Complex numbers,  $\mathbb{C}$ .* Numbers of the form  $a + bi$ , where  $a, b \in \mathbb{U}$  and  $i = \sqrt{-1}$ .

*Gaussian integers.* Complex numbers where  $a$  and  $b$  above are integers.

$\mathbb{F}$  is a *field* (SA III). It contains axioms (rules) for addition and multiplication. Examples could be the Eudoxus numbers  $\mathbb{U}$  and complex numbers  $\mathbb{C}$ .

$\mathbb{Y}$  is a *zero algebra* (SA III). This is similar to a field except for the existence of multizeros.

*Exponential algebra,* (SA XII and XIII). Contains axioms for exponentiation.

*$\mathbb{D}w$  exponential algebra,* (SA XIII). A nonstandard exponential algebra.

*Suoperator* (chapter III and SA XV). An operation, of which the first three are addition, multiplication and exponentiation, where the  $n$ th is found by repeating the  $(n - 1)$ th.

*Superstructure.* Contains axioms connecting suoperator  $n$ th operations for various  $n$ .

**7. Standard protocol.** (chapter I). The ordinal infinity  $\Omega_{\mathbb{N}} = \sum_{\text{all } \mathbb{N}} 1$ . This is not a natural number, and is treated as being irreducible.

*Ladder number.* A suoperator expression in  $\Omega_{\mathbb{N}}$ , with Eudoxus coefficients.

*Strict transfer principle.* The axioms for variables in a suoperator algebra also hold for the variable  $\Omega_{\mathbb{N}}$ .

*Capital  $\Xi$  function.* Satisfies  $1^{\Xi(a)} = a$ .

**8. Implies.** (exercise, SA XI). For statements  $A$  and  $B$ ,  $A$  implies  $B$  is only false when  $A$  is true and  $B$  is false.

*Sufficient.*  $A$  is sufficient for  $B$  means  $A$  implies  $B$ .

*Necessary.*  $A$  is necessary for  $B$  means  $A$  is implied by  $B$  (the same as  $B$  implies  $A$ ).

**9. Function** (SA III). A set of pairs,  $\{x, f(x)\}$ , all  $x$  of which have a value  $f(x)$ .

*Injection.* A mapping from all the sets  $\{a, b\}$  to  $\{f(a), f(b)\}$ , where  $f(a) \neq f(b)$  if  $a \neq b$ .

*Surjection.* A mapping where every  $f(x)$  in the set  $\{f(x)\}$  has a value from an  $x$ .

*Bijection.* A mapping which is simultaneously injective and surjective.

**10. Magma,  $\mathbb{M}$**  (SA III). A set with one binary operation, with no other properties specified.

*Polymagma* (SA XV). Maps a number of copies of a set to itself.



*Group, G* (SA III). This satisfies the multiplicative axioms for a field, except multiplication may be noncommutative:  $ab \neq ba$ .

*Subgroup*. A set of elements in a group which satisfies within itself all the properties of the containing group.

*Order of a group*. The number of elements (or members) in a group.

*Homomorphism of groups* is a surjective map  $h: G \rightarrow G'$  of groups with  $h(ab) = h(a)h(b)$ .

*Isomorphism of groups*. A bijective homomorphism.

*Automorphism of a group* is an isomorphism of a group to itself.

*Inner automorphism of a group* is an automorphism of the form  $x \leftrightarrow a^{-1}xa$ .

*Outer automorphism of a group*. An automorphism which is not inner.

*Normal subgroup* is invariant under all inner automorphisms of the containing group  $G$ .

*Simple group* has no normal subgroups other than itself and 1.

**11. Ring, A**. Satisfies the additive and multiplicative axioms of a field, except there is no general division and multiplication may be noncommutative. Example: matrices.

*Unital ring*. A ring with a multiplicative identity, 1. We assume rings are unital.

*Automorphism of a ring*. (SA X). A bijective map of a ring  $A$ ,  $H: A \leftrightarrow A$ , where  $H(ab) = H(a)H(b)$  and  $H(a + b) = H(a) + H(b)$ .

**12. Vector,  $\mathbf{v}$**  (in bold). A matrix as one row (a row vector), or as one column (a column vector). Example: the row vector  $(x, y, z)$ .

*Vector space*. Contains vectors with magnitude and direction, which can be added together and multiplied by scalars in a field.

*Base point*. The origin for a vector space.

*Module*. A module over a ring is a generalisation of a vector space over a field, being an additive abelian group like a vector space, where the scalars are the elements of a ring.

*Scalar product of two vectors*. The matrix product of multiplying each element of a row vector in turn with the corresponding elements of a column vector. Example:  $x^2 + y^2 + z^2$ .

*Eigenvector*. A vector  $\mathbf{x}$  satisfying for matrix  $B$ ,  $B\mathbf{x} = \lambda\mathbf{x}$ .

*Eigenvalue*. A value  $\lambda$  for the eigenvector  $\mathbf{x}$  above. Example:  $\lambda$  is a complex root value.

**13. Matrix (plural matrices)**. An array of numbers  $B = b_{jk}$ , where the element  $b_{ij}$  exists in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. (SA I and II).

*Symmetric matrix*.  $U = u_{jk} = u_{kj}$ .

*Antisymmetric matrix*.  $V = v_{jk} = -v_{kj}$ .

*Matrix transpose*. If  $W = w_{jk}$ , then the transpose  $W^T = w_{kj}$ .

*Unit diagonal matrix*. Denoted by  $I = b_{jk}$ , where  $b_{jk} = 1$  when  $j = k$ , otherwise  $b_{jk} = 0$ .

*Trace of a matrix*. The sum of all (main) diagonal entries  $b_{jk}$ , where  $j = k$ .

*Determinant (or hypervolume) of a matrix, ( $\det B$ )*. (SA I and II).

*Singular matrix, D*. Satisfies  $\det D = 0$ .

*Units,  $K^*$* . The invertible elements of a ring, for example giving  $\det \neq 0$ .

*Box*. An  $n$ -dimensional array of numbers.

*Box scalar product of two boxes*. A scalar value obtained from boxes using scalar products of vectors and determinants of matrices.

**14. Intricate number.** A representation of  $2 \times 2$  matrices, that is, with two rows and two columns, given by  $a1 + bi + c\alpha + d\phi$ . (SA I).

*Intricate basis element.* One of the vectors  $1, i, \alpha$  or  $\phi$  above.

*Real basis element.* The number  $1$  in its intricate representation.

*Imaginary basis element.* The number  $i$  in its intricate representation.

*Actual basis element.* The number  $\alpha$  in its intricate representation.

*Phantom basis element.* The number  $\phi$  in its intricate representation.

*Intricate conjugate.* The number  $a1 - bi - c\alpha - d\phi$ .

*J.*  $J = bi + c\alpha + d\phi$  in which  $J^2 = 0$  or  $\pm 1$ .

*JAF.* A changed basis for  $i, \alpha$  and  $\phi$ .

**15. Hyperintricate number.** A representation of  $2^n \times 2^n$  matrices. (SA II).

*Layer.* For example, a hyperintricate number with a component in 3 layers is  $A_{B,C}$  where  $A, B$  and  $C$  are intricate numbers, possibly intricate basis elements.

*n-hyperintricate number.* A hyperintricate number representable by sums of components in  $n$  layers. Sometimes denoted by  $\mathfrak{Y}_n$ .

*n-hyperintricate conjugate,  $\mathfrak{Y}_n^*$ .* Satisfies  $\mathfrak{Y}_n^* \mathfrak{Y}_n = \det \mathfrak{Y}_n$ .

*J-abelian hyperintricate number.* A number giving the example  $A_B + \dots + D_E$ , where  $A = p1 + qJ, B = p'1 + q'J', \dots, D = t1 + uJ, E = t'1 + u'J'$ . Two such hyperintricate numbers with identical  $J$  and  $J'$  commute.

**16. Division algebra.** (SA III and V). A division ring where multiplication might be nonassociative. Multiplying two elements of such an algebra cannot give zero unless one of them is zero.

*Zargon algebra.* A division algebra except possibly for zero scalar components.

*Quaternions.* (SA III and V). A type of associative division algebra.

*Octonions,  $\mathbb{O}$ .* (SA V). A nonassociative division algebra.

*n-vulcannions.* General nonassociative division algebras of dimension  $n = 6k + 2$ .

*Vulcan number,  $v$ .* The dimension of a vulcannion minus one – the number of its space components.

*T-junction.* A diagram used to classify vulcannions.

*n-novanions.* (SA V). An  $n$  dimensional nonassociative division algebra, but not when both the real parts in a multiplication are zero.

*Zargonion.* A combination of algebras obtained from vulcannions and novanions.

*Tribble.* A zargonion except space components have square 0. It may not have a divisor.

*Tharlonion.* A zargonion except space components have square 1. It may not have a divisor.

**17. Norm.** Applied to complex numbers  $a + bi$ , the norm is  $\sqrt{a^2 + b^2}$ . For intricate numbers  $a1 + bi + c\alpha + d\phi$  the norm squared is  $a^2 + b^2 - c^2 - d^2$ . Applied to a  $n \times n$  matrix  $B$ , the norm is the positive  $n$ th root of  $\det B$ . Applied to  $n$ -zargonions  $a1 + bi + c\alpha + d\phi + b'i' + c'\alpha' + d'\phi' + \dots$ , the norm is  $\sqrt{(a^2 + b^2 + c^2 + d^2 + b'^2 + c'^2 + d'^2 + \dots)}$ .

**18. Additive format of a polynomial equation.** The form  $ax^n + bx^{n-1} + \dots + d = 0$ . (SA VII and VIII).

*Monic polynomial.* Example in the case of a polynomial equation: when a above = 1.

*Fundamental theorem of algebra.* The complex polynomial in additive format given by  $ax^n + bx^{n-1} + \dots + d$  always has some values which are zero.

*Multiplicative format of a polynomial equation.* The form  $(x - p)(x - q) \dots (x - t) = 0$ .

*Zero of a polynomial.* A value of a polynomial  $f(x) = ax^n + bx^{n-1} + \dots + d$  so that  $f(x) = 0$ .

*Root of a polynomial equation.* The roots of a polynomial equation  $f(x) = 0$  are the values of  $x$  satisfying this.

*Equaliser of two polynomials.* The intersection of their values.

*Degree of a polynomial.* The value of  $n$  for  $f(x)$ .

*Duplicate root.* A root of the equation  $(x + a)^2 = 0$ .

*Antiduplicate root.* A root of the equation  $(x + a)(x - a) = 0$ .

*Independent roots.* Occur when no known dependency relation is used in the solution of a polynomial equation.

*Dependent roots.* Occur when a known dependency relation is used in the solution of a polynomial equation.

*Multivariate polynomial.* A polynomial in a number of variables.

*Variety.* A polynomial equation in a number of variables. Example:  $3x^2y + xyz + 4x^2z^2 = 0$ .

**19.** *Equivalence relation*  $\equiv$  in a set  $S$ . Satisfies  $a \equiv a$  (*reflexive*), if  $a \equiv b$  then  $b \equiv a$  (*symmetric*) and if  $a \equiv b$  and  $b \equiv c$  then  $a \equiv c$  (*transitive*), for  $a, b, c \in S$ .

*Equivalence class.* A partition of a set where an equivalence relation between elements defines membership of the partition.

*Partial order*  $\leq$  of a set  $S$ . Satisfies  $a \leq a$ , if  $a \leq b$  and  $b \leq a$  then  $a = b$  (*antisymmetric*) and if  $a \leq b$  and  $b \leq c$  then  $a \leq c$ , for  $a, b, c \in S$ .

*Poset.* A partially ordered set.

*Total order*  $\leq$  of a set  $S$  is a partial order existing for all  $a, b, c \in S$ .

*Well-ordering*  $\leq$  of a set  $S$ . A total order where every nonempty subset has a least element.

**20.** *Left (or right) coset* of a subgroup  $S$  of  $G$  is the set of elements  $aS$  (or respectively  $Sa$ ), with  $s \in S$  and  $a \in G$ .

*Quotient group*  $G/S$  of  $G \bmod S$ . The family of left cosets of the group  $G$  with subgroup  $S$ ,  $sG$ ,  $s \in S$ .

**21.** *Ideal*,  $C$ . (SA III and XI). A subset of a ring,  $A$ , with the rule that  $\{c, d\} \in C$  and  $a \in A$  implies  $(c - d) \in C$  and both  $ac$  and  $ca \in C$ .

*Principal ideal*,  $(a)$ . The ideal generated by one element,  $a$ , of the ring  $A$ . For every  $r \in A$ ,  $(a)$  is  $ra$ . Example: for  $a \neq 0$  belonging to the integers  $\mathbb{Z}$ ,  $(3a) \subset (a) \subset \mathbb{Z}$ .

*Prime ideal*,  $P$ . If  $a$  and  $b$  are two elements of  $A$  such that their product  $ab$  is an element of  $P$ , then  $a$  or  $b$  is in  $P$ , and  $P$  is not equal to the whole ring  $A$ . Example: integers containing all the multiples of a given prime number, together with zero. Example: the zero ideal  $(0)$ .

*Maximal ideal*,  $M$ . In any ring  $A$ , this is an ideal  $M$  contained in just two ideals of  $A$ ,  $M$  itself and the entire ring  $A$ . Every maximal ideal is prime. Nonexistence: the zero ideal  $(0)$  is not a maximal ideal of  $\mathbb{Z}$  because  $(0) \subset (2) \subset \mathbb{Z}$ , nor is the ideal  $(6)$ , since  $(6) \subset (2) \subset \mathbb{Z}$ .

*Nilradical*,  $N(A)$ . The intersection of all prime ideals of a ring.

**22. Unfinished set.** Example: the interval  $a < x < b$  with the end points  $a$  and  $b$  removed.

*Finished set.* Example: the interval  $a \leq x \leq b$  with the end points  $a$  and  $b$  present.

*Topology.* A theory of space using finished and unfinished sets.

**23. Exact sequence.** (SA III).

*Homology.* A theory of holes. The dimension of the  $n$ th homology is the number of holes in a space for dimension  $n$ .

**24. Euler-Poincaré characteristic.** In 3-space the number of vertex points – edges + faces of a space divided into  $n$ -dimensional polygons.

*Möbius strip.* A reconnected rectangle with a twist.

*Handle.* Obtained on a surface by cutting out two holes and gluing in a cylinder.

*Crosscap.* Obtained on a surface by cutting out a hole and gluing in a Möbius strip.

**25. Graph.** A set of vertices and arrows (or edges) with a mapping from origin to terminus, and provided with a reverse mapping changing orientation.

*Path.* A finite sequence of edges with the terminus of each edge connecting to an origin of the next edge.

*Circuit.* A path with its end vertices connected together (start origin = end terminus vertex).

A *graph is connected* if all vertices are contained in a path.

*Tree.* A connected nonempty graph without circuits.

*Node.* An origin or terminus in a tree.

*Parent node.* An origin in a tree.

*Child node.* A terminus in a tree.

*Root of a tree.* A child with no parent.

*Leaf of a tree.* A parent with no child.

**26. Homotopy.** A theory of paths through a topological space.

*Winding number.* The number of times a loop winds round a point.

**27.  $n$ -branched space.** A space where the removal of a point disconnects the space into  $n$  pieces.

*Explosion.* An  $n$ -branched space with an infinite or transfinite number of points.

*Supernorm.* An evaluation of the magnitude of an explosion.

*Branch number.* An evaluation of the number of branches in an explosion.

**28. Deformation retract.** The set of all occurrences of a vector transported along another vector.

*Branched vector.* A directed tree, with arrows proceeding from its root and splitting to its branches.

*Branched deformation retract.* A branched vector transported and split into copies along another branched vector.

*Amalgam.* A branched retract with some of its nodes connected.

**29. Morphism.** An associative mapping with identity in category theory.

*Category.* A description of mathematics in terms of morphisms.

*Object.* The element where a morphism comes from or goes to.

*Arrow.* A morphism considered as a directed mapping between objects.

*Hom-set.* The description of categories as a collection of arrows.

*Functor (covariant).* Describes states and transformations as if they were on the same footing. If  $f$  and  $g$  are morphisms, and  $T$  is a functor, it is covariant if  $T(f \circ g) = T(f) \circ T(g)$ .

*Contravariant functor.* A functor reversing the order of composition:  $T(f \circ g) = T(g) \circ T(f)$ .

*Morphism of functors (natural transformation).* An example is the determinant, as a transformation from commutative rings to groups.

*Topos.* A categorical description of a set.

*Adjoint functor.* A specific type of interrelationship between functors, of general use in mathematics.

*Kan extension.* The combinations of mappings between two sets are described in terms of hom-sets by an exponential, and this can be differentiated or integrated. Kan extensions implement this idea.

**30. Explanation.** A theory or theorem using matrices or other combinatorial means.

*Charade.* The image of an explanation in homological algebra.

*Deconstruction.* The mathematical refutation of a generally accepted result.