Xenan Zeroes

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Recent conversations with inhabitants of the planet Xena reveal they have a different perspective on division by zero!

The Xenans set special store by the sequence

$0^{-n}, \ldots 0^{-1}, 0^{0}, 0^{1}, 0^{2}, \ldots 0^{n}$

of exponents of zero. In Xenan arithmetic,

$0^{0} \neq 1$ or 0.

Indeed, they also say that if r is a non-positive real number, then

0^{r} is not a real number.

The Xenans work with exponents of zeroes in much the same way as we deal with non-zero real numbers, so

$0^{s}0^{t} = 0^{s + t}$

for s and t real numbers (in Xenan arithmetic, s and t can also be Xenan numbers, which we will sometimes denote by x). However, in Xenan arithmetic, if $s \neq t$ then

$0^{s} \neq 0^{t}$.

The whole approach of the Xenans is to treat 0^{r}, $r > 0$, as an infinitesimal would be in non-standard analysis in Earth arithmetic, so

$0^{r} + u = u + 0^{r}$,

which cannot (except for the case $0^{(0^{r})} + u$, to be dealt with later) be simplified any further. The Xenans also consider sometimes an infinitesimal non-commutative arithmetic with their 0^{r}.

Thus the Earth rules for arithmetic (the standard rules of addition and multiplication, called field axioms, in which Earth 0 can for instance be set to the Xenan $0^{(0^{r})}$, for a real number $r < 0$, together with standard rules for exponentiation with real powers) also hold in Xenan arithmetic, with the exceptions

$0^{1}0^{1} = 0^{2} \neq 0^{1}$,

as we have already stated, also

if $r \leq 0$, it is possible to add or multiply by 0^{r}, but the result is not a real number

and

the manipulation of 0^{r}, for $r < 0$, is similar to Earth arithmetic for ordinal infinities [1]. Thus commutativity does not hold:

$1 + 0^{r} = 0^{r} \neq 0^{r} + 1$

and

$2.0^{r} = 0^{r} \neq 0^{r}.2 = 0^{r} + 0^{r}$.

The Xenan $0^{(0^{r})}$, $r < 0$, is treated in the same way as the left hand side above, as $0^{(2.0^{r})} = 0^{(1 + 0^{r})}$, so

$0^{(0^{r})}0^{(0^{r})} = 0^{(2.0^{r})} = 0^{(0^{r})}$

and there are further axioms

$u + 0^{(0^{r})} = 0^{(0^{r})} + u = u$

and for u positive

$u0^{(0^{r})} = 0^{(0^{r})}u = 0^{(0^{r})}$.

There is a *higher order* Xenan arithmetic, where the properties of $0^{(0^r)}$ are taken instead to be the same as 0^r above, and the x in the new Xenan higher order zero $0^{(0^x)}$ is given by $x = -0^r = -1.0^r$, with the higher order zero having the same properties as we had previously with $0^{(0^r)}$. This can be generalised to even higher orders.

The consequences are rather similar to non-standard analysis [2] and synthetic geometry [3] in Earth mathematics.

It is possible, having done a calculation in Xenan arithmetic, to 'collapse' to ordinary Earth standard arithmetic, under the transformation of 0^x to 0, for x a positive Xenan number, under the understanding that the transformation of 0^x, $x \leq 0$, cannot be reduced to any real number.

Further reading.

[1] T. Jech, *Set Theory,* Springer (2002).
[2] I. Moerdijk and G. E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer, (1991).
[3] A. Kock, *Synthetic Differential Geometry*, Cambridge University Press, (1981).

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