

# The Concept of Pathological Spaces

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## Introduction.

I have coined the name '*pathological space*', which is the subject of this article, to describe certain spaces with bizarre properties – they include shapes not met with in everyday life. The name also contains other word meanings – of 'path', which traces a line in space, and 'logical', describing operations concerned with mathematical truth.

The problem I solve is, given an abstract description of the properties of pathological spaces, how do I provide a concrete model of these spaces? I have also been asked 'what research topics can use pathological spaces in the future?', and immediately I replied 'quantum mechanics'. However the subject matter of this article is conceptual pure mathematics and is not contingent on any model of physics.

A subset of pathological spaces, which for the sake of contrast I will call '*familiar spaces*', is well known to mathematicians. These familiar spaces contain objects like the square and the cube, the cylinder and the torus. I give a description of these particular objects, and then give a natural generalisation to describe pathological space objects.

I show that the familiar space square is related (mapped) to the polynomial  $\{x - 2\}^2$ , and the familiar cube has a mapping to  $\{x - 2\}^3$ , in particular the coefficients in the binomial expansion of  $\{x - 2\}^2$  give respectively the number of areas, edges and vertex points of the square, and the coefficients in the binomial expansion of  $\{x - 2\}^3$  give the number of volumes, areas, edges and vertex points of the cube. I show how this fits in with a description of the cylinder mapped to  $\{x - 1\}\{x - 2\}$ , and to the torus, mapped to  $\{x - 1\}^2$ .

I relate this to the Euler characteristic,  $\chi$ , which in familiar 3-space gives the number of 'volumes – areas + edges – points' in a 'triangulation' of the space – that is, more generally, the space is divided up into polygons, and  $\chi$  describes an invariant of the space – provided the topological shape remains the same – as a sphere or torus, etc.

I then introduce pathological spaces via the polynomial  $\{x - m\}^n$ , where to begin with  $m$  is a whole number for  $n$ -dimensional such spaces. For example, the 3-pathological cube maps on to the polynomial  $\{x - 3\}^3$ , and has 1 volume, 9 areas, 27 edges and 27 vertex points. I relate this idea to triangulation of pathological space objects.

I then generalise these space objects for  $n$  and  $m$  complex numbers.

Finally, I introduce as examples, a conceptual model in the 1-dimensional case of what is meant by pathological lines and points, describing this by what is known as generalised 'Dedekind cuts', and in the 2-dimensional case provide a model of a pathological square.

**The familiar square, cylinder, torus and cube.**

Note that the area, number of edges and number of points of a square, each with sign given by the Euler characteristic  $\chi$ , are related by

$$\chi = A - E + P$$

for A the number of areas, E the number of edges and P the number of vertex points, and these are given in sequence by the coefficients of

$$\{x - 2\}^2 = x^2 - 4x + 4,$$

so A = 1, -E = -4 and P = 4.

For a cylinder, formed when two opposite edges and two opposite points of a square are identified, the values of A, -E and P are given in sequence by the coefficients of

$$\{x - 1\}\{x - 2\} = x^2 - 3x + 2.$$

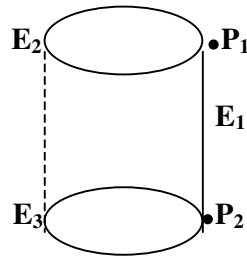


Figure. The familiar torus with one area, two edges and one vertex point is obtained from the familiar cylinder shown, by gluing the top and bottom edges together, so P<sub>1</sub> and P<sub>2</sub> coincide.

For a torus, where I identify edges E<sub>2</sub> and E<sub>3</sub> and points P<sub>1</sub> and P<sub>2</sub> above, A, -E and P are given in sequence by the coefficients of

$$\{x - 1\}^2 = x^2 - 2x + 1.$$

For a cube, the volume V, and the values of -A, E and -P are given by the coefficients of

$$\{x - 2\}^3 = x^3 - 6x^2 + 12x - 8.$$

For the cube with two opposite faces identified, and the two sets of 4 points of those square faces identified, these are given by the coefficients of

$$\{x - 1\}\{x - 2\}^2 = x^3 - 5x^2 + 8x - 4,$$

with two sets of two opposite faces identified as

$$\{x - 1\}^2\{x - 2\} = x^3 - 4x^2 + 5x - 2,$$

etc., and for a 4-dimensional hypercube, by the coefficients of

$$\{x - 2\}^4.$$

**Pathological spaces.**

I now generalise this idea to *pathological* spaces.

A 3-pathological space, for example a 3-pathological square, has A, -E and P given by the coefficients of  $\{x - 3\}^2$ . It therefore has:

$$1 \text{ area} \quad 6 \text{ sides} \quad \text{and} \quad 9 \text{ points.}$$

The reader will with difficulty develop a visual model for this topology, but the idea is as consistent as  $\{x - 3\}^2$ , and I provide a model example at the end of this paper. I recall that imaginary numbers were first thought of as not describing the ‘real’ world.

Our definition of the Euler-Poincaré characteristic,  $\chi$ , of a familiar  $m$ -dimensional hypercube, given as the sum of the coefficients of  $(x - 2)^m$ , corresponds with its assignation as a deformation retract with ends two  $(m - 1)$ -dimensional copies of a hypercube.

It is possible to amalgamate these two copies, with retracts in parallel or twisting – corresponding to two orientation types.

The question then arises whether this definition in terms of a deformation retract is extendable to pathological spaces. It is. In terms of connectivity a pathological retract is isomorphic to the inverse operation of what we will call later a ‘generalised Dedekind cut’.

Consider  $(x - n)^m$  for a pathological space. Its  $(m - 1)$  dimensional subobjects exist in  $n$  copies at the ends of the retract. If these  $n$  copies are given an order  $(1, \dots n)$ , then to amalgamate them after a pathological retract corresponds to assigning a permutation of  $(1, \dots n)$  to this. In other words, the group of pathological orientation transformations for this retract is the symmetric group on  $n$  objects.

I can consider such pathological geometries given by the coefficients of an  $n$ th degree polynomial (here  $\prod$  indicates multiplication, from  $i = 1$  to  $n$ )

$$\prod_{i=1}^n (x - m_i),$$

and indeed I can consider negative, rational, algebraic, transcendental and complex numbers of hypervolumes, volumes, areas, edges and points.

We now extend the idea of the pathological representation

$$(x - n)^m,$$

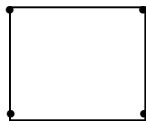
where we had  $x$ ,  $n$  and  $m$  at most as complex numbers, to  $x$ ,  $n$  and  $m$  *matrices*.

We have detailed in another paper [1] the hyperintricate representation of matrices, in which the complex numbers occur as subobjects of intricate numbers – representable by  $2 \times 2$  real matrices. In this formalism, the pathological representation now becomes expandable in terms of a hyperintricate binomial theorem.

The question then arises, how do I compute the number of points etc., of a general pathological simplex?

Consider a familiar square

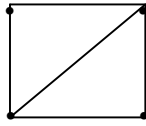
and a 3-pathological square



1 area, 4 edges, 4 points

1 area, 6 edges, 9 points

I can triangulate the familiar square by forming a diagonal



I now have 2 areas, 5 edges and 4 points.

So I have for the untriangulated familiar square topology

$$\chi = 1 - 4 + 4 = 1,$$

for the cylinder

$$\chi = 1 - 3 + 2 = 0$$

and for the torus

$$\chi = 1 - 2 + 1 = 0.$$

These values of  $\chi$  are invariant under a change of triangulation that maintains the topological shape. Can I assume the same for pathological simplexification?

If I do, then the pathological Euler characteristics are

$$\{x - 3\}^2 \quad : \quad \chi = 1 - 6 + 9 = 4,$$

$$\{x - 1\}\{x - 3\} \quad : \quad \chi = 1 - 4 + 3 = 0,$$

$$\{x - 2\}\{x - 3\} \quad : \quad \chi = 1 - 5 + 6 = 2,$$

$$\text{and } \{x - 3\}^3 \quad : \quad \chi = 1 - 9 + 27 - 27 = -8, \text{ etc.}$$

Suppose for  $\{x - 3\}^3$  I add one edge, but keep the number of points constant. Then I must create an extra area to keep  $\chi$  the same. I can always add points and increment the number of edges correspondingly. Inductively, for any dimension I can add a hyper-area and add a hyper-edge whilst retaining  $\chi$  invariant.

I note that for  $\{x + 3\}^2$ , (with a *plus* sign) if I add a hyper-area I must subtract a hyper-edge to retain  $\chi$  invariance, likewise for  $\{x + k\}^n$ ,  $k$  a complex number.

For complex hyper-volumes, take the example of adding a *semi-point*, say half a point, then the addition of the corresponding compensating semi-edge must be adjusted to leave  $\chi$  invariant.

### **Models for pathological lines and areas.**

I now provide model examples of pathological spaces, firstly in one dimension.

To begin with, consider  $\{x - 1\}$ , which represents a circle. If the circle consists of real numbers, then a Dedekind cut – a removal of one point – leaves the resulting ‘manifold’ in one piece.

If I consider  $\{x - 2\}$ , representing a real line with two end points, each end point of which is connected in only one way with the rest of the interval – in other words the line is a *closed* interval, then removal of an interior point leaves the resulting manifold in two pieces.

Now look at  $\{x - 3\}$ . I consider three end points, each end point of which is connected in only one way with the rest of the interval, so by analogy with the previous case I

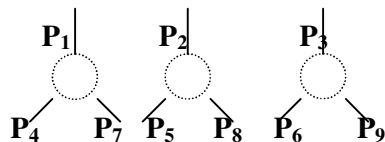
will call this interval again closed. Then a Dedekind cut – the removal of one interior point – leaves the resulting manifold in *three* pieces. Thus a pathological line represented by  $\{x - m\}$  with  $m$  a variable, under removal of an interior point, divides the line into  $m$  pieces. This is the extended meaning of line or ‘edge’.

The sequence of points in this interval can be reconnected in its interior, for example:



where I have shown three such points. All such points can be reconnected in this way. If there are no interior reconnections, so that all the points are connected in an expanding tree, I call the resulting analogue of a real number interval an *explosion*.

The next model example, of a 3-pathological square, was first developed by Doly García. All sets of interior edges except for one are reconnected. I represent 3 sets of ‘3 vertex points and one edge’ as follows:



I then connect vertex point  $P_1$  with an ‘edge’ to simultaneously  $P_2$  and  $P_3$ , then  $P_4$  with an edge to  $P_5$  and  $P_6$ , and  $P_7$  to  $P_8$  and  $P_9$ , making 1 area, 6 edges and 9 points.

**Summary.**

I have given a sketch of an unusual and general type of space, which describes objects characterised by pathological simplexes. This includes familiar objects in spaces described in various standard homology and cohomology theories.

**Reference.**

[1] J.Adams *Intricate and Hyperintricate Numbers*, 2010.