

# The Concept of Branched Spaces

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*Abstract.* I describe certain spaces with unusual properties – branched spaces. The problem I solve is, given an abstract description of the properties of branched spaces, how do I provide a concrete model of these spaces? We describe the Euler-Poincaré characteristic and homological algebra for such spaces. Quantum chromodynamics may use branched spaces in the future. However the main subject matter of this article is conceptual pure mathematics and is not contingent on any model of physics.

## Part I – the Euler-Poincaré Characteristic

### Introduction.

A subset of branched spaces, which for the sake of contrast I will call ‘*familiar spaces*’, is well known. These familiar spaces contain objects like the square and the cube, the cylinder and the torus. I give a description of these particular objects, and then give a natural generalisation to describe branched space objects.

I show that the familiar space square is related to the polynomial  $\{x - 2\}^2$ , and the familiar cube has a mapping to  $\{x - 2\}^3$ , in particular the coefficients in the binomial expansion of  $\{x - 2\}^2$  give respectively the number of areas, edges and vertex points of the square, and the coefficients in the binomial expansion of  $\{x - 2\}^3$  give the number of volumes, areas, edges and vertex points of the cube. I show how this fits in with a description of the cylinder mapped to  $\{x - 1\}\{x - 2\}$ , and to the torus, mapped to  $\{x - 1\}^2$ .

I relate this to the Euler characteristic,  $\chi$ , which in familiar 3-space gives the number of ‘volumes – areas + edges – points’ in a ‘triangulation’ of the space – that is, more generally, the space is divided up into polygons, and  $\chi$  describes an invariant of the space – provided the topological shape remains the same – as a sphere or torus, etc.

We then introduce branched spaces via the polynomial  $\{x - m\}^n$ , where to begin with  $m$  is a whole number for  $n$ -dimensional such spaces. For example, the 3-branched cube maps on to the polynomial  $\{x - 3\}^3$ , and has 1 volume, 9 areas, 27 edges and 27 vertex points. I relate this idea to triangulation of branched space objects. I then generalise these space objects for  $n$  and  $m$  complex numbers.

We introduce as examples a conceptual model in the 1-dimensional case of what is meant by branched lines and points, describing this by what is known as generalised ‘Dedekind cuts’, and in the 2-dimensional case provide a model of a branched square.

We discuss branched deformation retracts, branched orientation,  $n$ -branched surgery with  $h$ -handles and  $h$ -crosscaps, and that  $\partial\partial = 0$  can fail for  $k$ -*explosions*.

## **Historical Synopsis.**

As mentioned in [5] and [7] the number of areas – edges + vertices

$$\chi = A - E + V$$

as an invariant of a simplicial decomposition of a polyhedron was first put in an equivalent form by Descartes.  $\chi$  itself was discovered by Euler [9].

It would be impossible to situate Riemann except in the middle of a long tradition, yet the paper [19] which defines the *genus*,  $g = 1 - \chi/2$  for a surface (this is the number of handles), is often taken as the starting point of our subject. The idea of connectivity given there was then extended to higher dimensions by Betti [6].

Non-oriented manifolds were introduced by Möbius [16], and first systematically classified by von Dyck [20], [21], [22].

The paper of Poincaré on Analysis Situs, and the five supplements to it has been translated into English by John Stillwell [18]. This is the beginning of our own considerations.

The work that developed in topology up to the mid 1930's was vast, particularly in Germany. For a bibliography of this period the reader could consult [25]. Of note is the work of Herman Weyl on Riemannian surfaces [28], and of Emmy Noether, who further developed the idea of homology groups [13].

## **The Euler-Poincaré characteristic.**

There are 3 main aspects to the usual formulation of the Euler-Poincaré characteristic.

- (1)  $\chi = \sum_i (-1)^i a_i$ ,  
where  $a_i$  is the number of  $i$ -dimensional faces.
- (2) From discussing the genus,  $\chi = \sum_i (-1)^i p_i$ ,  
where  $p_i$  are the Betti numbers of the space, defining the  $i$ -dimensional connectivity. The  $p_i$  are as defined by Poincaré, not Betti.
- (3)  $\chi =$  number of pits – number of passes + number of peaks of a surface as studied by Cayley [8]. This can be generalised to an  $n$ -dimensional manifold by considering a height function of the manifold immersed in  $R^{2n}$ .

For the remainder of this article we will be discussing from a naive point of view aspect (1). Aspect (2) will be the subject of homological algebra in Part II.

## **The familiar square, cylinder, torus and cube.**

The area, number of edges and number of points of a square, each with sign given by the Euler characteristic  $\chi$ , are related by

$$\chi = A - E + V$$

for  $A$  the number of areas,  $E$  the number of edges and  $P$  the number of vertex points, and these are given in sequence by the coefficients of

$$\{x - 2\}^2 = x^2 - 4x + 4,$$

so  $A = 1$ ,  $-E = -4$  and  $V = 4$ .

For a cylinder, formed when two opposite edges and two opposite points of a square are identified, the values of A, -E and V are given in sequence by the coefficients of  $\{x - 1\}\{x - 2\} = x^2 - 3x + 2$ .

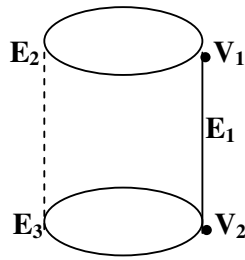


Figure. The familiar torus with one area, two edges and one vertex point is obtained from the familiar cylinder shown, by gluing the top and bottom edges together, so V1 and V2 coincide. This is a 'Δ-simplex' decomposition.

For a torus, where I identify edges E2 and E3 and points V1 and V2 above, A, -E and V are given in sequence by the coefficients of  $\{x - 1\}^2 = x^2 - 2x + 1$ .

For a cube, the volume, and the values of -A, E and -V are given by the coefficients of  $\{x - 2\}^3 = x^3 - 6x^2 + 12x - 8$ .

For the cube with two opposite faces identified, and the two sets of 4 points of those square faces identified, these are given by the coefficients of

$$\{x - 1\}\{x - 2\}^2 = x^3 - 5x^2 + 8x - 4,$$

with two sets of two opposite faces identified as

$$\{x - 1\}^2\{x - 2\} = x^3 - 4x^2 + 5x - 2,$$

etc., and for a 4-dimensional hypercube, by the coefficients of  $\{x - 2\}^4$ .

### Branched spaces.

I now generalise this idea to *branched* spaces.

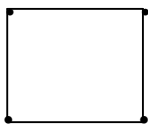
A 3-branched space, for example a 3-branched square, has A, -E and V given by the coefficients of  $\{x - 3\}^2$ . It therefore has:

1 area          6 sides    and          9 points.

The reader will with difficulty develop a visual model for this topology, but the idea is as consistent as  $\{x - 3\}^2$ , and I provide a model example at the end of this chapter. Recall that imaginary numbers were first thought of as not describing the 'real' world.

The question then arises, how do I compute the number of points etc., of a general branched simplex?

Consider a familiar square



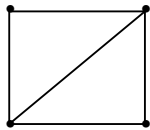
1 area, 4 edges, 4 points

and a 3-branched square



1 area, 6 edges, 9 points

I can triangulate the familiar square by forming a diagonal



I now have 2 areas, 5 edges and 4 points.

So I have for the untriangulated familiar square topology

$$\chi = 1 - 4 + 4 = 1,$$

for the cylinder

$$\chi = 1 - 3 + 2 = 0$$

and for the torus

$$\chi = 1 - 2 + 1 = 0.$$

These values of  $\chi$  are invariant under a change of triangulation that maintains the topological shape. Can I assume the same for branched simplexification?

If I do, then the branched Euler characteristics are

$$\{x - 3\}^2 \quad : \quad \chi = 1 - 6 + 9 = 4,$$

$$\{x - 1\}\{x - 3\} \quad : \quad \chi = 1 - 4 + 3 = 0,$$

$$\{x - 2\}\{x - 3\} \quad : \quad \chi = 1 - 5 + 6 = 2,$$

$$\text{and } \{x - 3\}^3 \quad : \quad \chi = 1 - 9 + 27 - 27 = -8, \text{ etc.}$$

Suppose for  $\{x - 3\}^3$  I add one edge, but keep the number of points constant. Then I must create an extra area to keep  $\chi$  the same. I can always add points and increment the number of edges correspondingly. Inductively, for any dimension I can add a hyper-area and add a hyper-edge whilst retaining  $\chi$  invariant.

I note that for  $\{x + 3\}^2$ , (with a *plus* sign) if I add a hyper-area I must subtract a hyper-edge to retain  $\chi$  invariance, likewise for  $\{x + k\}^n$ ,  $k$  a complex number.

For complex hyper-volumes, take the example of adding a *semi-point*, say half a point, then the addition of the corresponding compensating semi-edge must be adjusted to leave  $\chi$  invariant.

### **Models for branched lines and areas.**

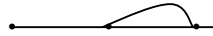
I now provide model examples of branched spaces, firstly in one dimension.

To begin with, consider  $\{x - 1\}$ , which represents a circle. If the circle consists of real numbers, then a 'Dedekind cut' – a removal of one point – leaves the resulting 'manifold' in one piece.

If I consider  $\{x - 2\}$ , representing a real line with two end points, each end point of which is connected in only one way with the rest of the interval – in other words the line is a *closed* interval, then removal of an interior point leaves the resulting manifold in two pieces.

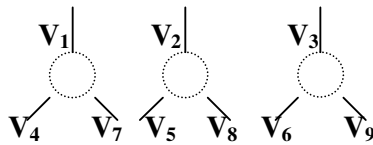
Now look at  $\{x - 3\}$ . I consider three end points, each end point of which is connected in only one way with the rest of the interval, so by analogy with the previous case I will call this interval again closed. Then a Dedekind cut – the removal of one interior point – leaves the resulting manifold in *three* pieces. Thus a branched line represented by  $\{x - n\}$  with  $n$  a variable, under removal of a unique interior point, divides the line into  $n$  pieces. Normally, if the point were not unique, there would be more than  $n$  ends. An alternative is that the branched line is considered *affine*, so that always the removal of the first selected point divides the line into  $n$  pieces and there are  $n$  ends. These are extended meanings of line or ‘edge’.

The sequence of points in this interval can be reconnected in its interior, for example:



where I have shown three such points. All such points can be reconnected in this way. If there are no interior reconnections, so that all the points are connected in an expanding tree, I call the resulting analogue of a real number interval an *explosion*.

The next model example, of a 3-branched square, was first developed by Doly García. All sets of interior edges except for one are reconnected, or the space is affine. I represent 3 sets of ‘3 vertex points and one edge’ as follows:



I then connect vertex point  $V_1$  with an ‘edge’ to simultaneously  $V_2$  and  $V_3$ , then  $V_4$  with an edge to  $V_5$  and  $V_6$ , and  $V_7$  to  $V_8$  and  $V_9$ , making 1 area, 6 edges and 9 points. The closed end points are here connected as a 2-branch.

Further we note that removing an edge from the 3-branched square reduces the dimension by one – the area dimension disappears. Reversibly, in the process of adding an edge, the number of areas is increased by one, thus retaining the Euler-Poincaré characteristic.

### **Deformation retracts and orientation.**

Our definition of the Euler-Poincaré characteristic,  $\chi$ , of a familiar  $m$ -dimensional hypercube, given as the sum of the coefficients of  $(x - 2)^m$ , corresponds with its assignation as a deformation retract with ends two  $(m - 1)$ -dimensional copies of a hypercube.

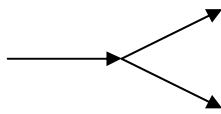
It is possible to amalgamate these two copies by gluing at both ends an opposite retract, with glued retracts corresponding to two orientation types – with the same or reversed orientation.

The question then arises whether this definition in terms of a deformation retract is extendable to branched spaces. It is. We treat this as a global phenomenon, where a

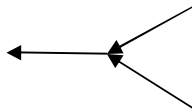
localisation of this is visible at the ends of the retract. Since our philosophy is that the retract is built out of objects which are not necessarily real numbers, the question is evident as to how the localisation is manifest in the interior of the retract. In terms of connectivity a branched retract is isomorphic to the inverse operation of what we previously called a generalised Dedekind cut.

Consider  $(x - n)^m$  for a branched space. We will call  $n$  the *branch root* and  $m$  the *branch degree*. We will show the branch root for an  $n$ -edge differs in general from the number of orientations of the  $n$ -edge.

To give an example, consider the García diagram for a 3-branched square given above. Its 1-dimensional subobjects are the 3-edges of which we have displayed 3. The orientation of each 3-edge may be represented by



which can be subject to a threefold rotation or a reflection. The opposite orientation may be represented by



again with a threefold rotational symmetry, or combined with a reflection about the horizontal axis.

If we consider the reflections as equivalent orientations then the total number orientations for a 3-edge is 6. The three 3-edges in the García diagram are free to have each of the six possibilities.

If we select a set of these, then the orientation of a new connection between  $V_1$ ,  $V_2$  and  $V_3$  etc. to flow as a continuation in the same direction, is fixed. Thus  $V_1$  is connected to three possibilities:  $V_2$ ,  $V_5$  and  $V_8$ , then to a further 3 possibilities:  $V_3$ ,  $V_6$  and  $V_9$ , making 9 possible connections with  $V_1$ .

The number of connections with  $V_2$  is then reduced, since, say,  $V_1$  has been already selected with  $V_2$ , making two possibilities with  $V_5$  and  $V_8$ , and again 2 possibilities with, say,  $V_6$  and  $V_9$ .

Finally,  $V_3$  has only one set of connections available. Thus the number of orientations for a 3-branched square is  $3 \times 6 + 3^2 + 2^2 + 1^2 = 32$ .

For a line segment, the number of orientations corresponds with its number of end-points. We have seen this is not the case for an  $n$ -edge. Thus what in former considerations was isomorphic has become distinct.

**Branched handles, crosscaps and surgery of branched spaces.**

In order to understand how we can extend the idea of gluing handles and Möbius strips to holes in 2-branched spaces (manifolds), to encompass n-branched spaces, we need a formulation that is first of all compatible with our previous considerations relating  $\chi$  from  $(x - 2)^3$ , topologically a ball, to  $\chi$  for  $(x - 1)^3$ , a handlebody – which can be pictured as a torus surface in classical 3-space swept out and reconnected along a fourth dimension, and likewise a disk,  $(x - 2)^2$ , to a handle,  $(x - 1)^2$ .

We will define an analogy between

$$(x - 2)^m \rightarrow (x - 1)^m$$

for such identification or surgery and

$$(x - n)^m \rightarrow (x - n + 1)^m$$

in the n-branched space. More generally, the analogy extends to

$$(x - n)^m \rightarrow (x - n + u)^m,$$

for which we can characterise this mapping by a difference

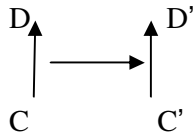
$$\Delta_u(x - n)^m = [(x - n)^m - (x - n + u)^m],$$

a type of construction that occurs in differential calculus.

In our model, an n-branched object will be called *closed* when its boundary (of say vertices) is present, and *open* when it is absent.

There are two basic modes of construction we can perform. The first is, having been provided with a ready-made n-object with boundary, to identify parts of this boundary, possibly via other objects. The second is to perform surgery to remove a number of n-object copies and then glue other derived n-objects. To do this we need a concept of the interior of an n-object, and in order to introduce this, it will be useful to describe the abutment of n-objects to create an extended n-object. We discuss to begin with the first of these ideas, then for a 3-branched object we are interested in surgery involving  $u = 1$  and 2.

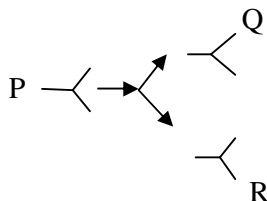
To generate a 2-branched torus from a 2-branched square



identify C and C', D and D' along the entirety of the retracts CD and C'D', and then identify at the C, C', D and D' boundaries the 2-edges CC' and DD'.

To generate a Möbius strip, identify C and D', D and C' along the entirety of the retracts CD and C'D'.

For the formation of the 3-branched square we have inserted and connected three more 3-edges from those at the vertices of P, Q and R shown below, where the arrows are the retract. There are six 3-edges.



Now identify the three 3-edges P, Q and R, corresponding to the initial retract, which are to be amalgamated at the retraction of their vertices and to a common 3-edge. If we allocate these vertices in the order they are connected by the remaining three 3-edges, and then amalgamate these remaining three 3-edges, this is the 3-branched torus.

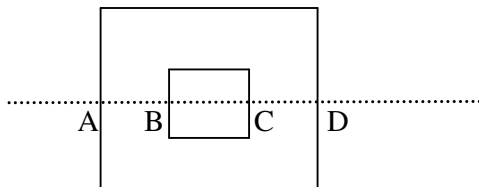
If the 3-edges, P, Q and R, are amalgamated at their vertices in an order that is different than the initial retract, this is a 3-branched Möbius strip. The amalgamation can be cyclic, in which case there is one boundary, or a swap, in which case there are two boundaries, one corresponding to the swap and one corresponding to the identity.

In general the amalgamation is given by the group of permutations on  $n$  objects, called the symmetric group, and the number of boundaries is equal to the number of cycles, including individual retract identities.

The 3-branched retract we have been considering has an  $(m - 1)$ -dimensional 3-object on the left and  $(3^m - 1)$  3-objects on the right. To form an abutment of these  $(3^m - 1)$  3-objects on the right, for each of these amalgamate a 3-object on the left associated with  $(3^m - 1)$  3-objects on the right. Then for  $k$  such iterated abutments, there will exist  $(3^m - 1)^{k+1}$  3-objects each of dimension  $(m - 1)$  on the right.

It is possible to form  $h - 1$  further copies of this abutted object and amalgamate the retracted part of the boundaries of the  $h$  versions. *Object A* will leave the left hand of these retracts unamalgamated.

In order to deal with surgery, we first need to explore its simplest instances. For the square with a hole



the hole can be considered as the removal of a subobject of the same type as the containing square. For the 1-dimensional subobject given by the horizontal line we can also consider this as three retracts (synonymous in this case with two abutments) given by

$$A \text{ --- } B \quad C \text{ --- } D$$

the (point) retract AB, the surgery subobject BC, and the retract CD. The subobject classifier here is defined as Boolean. Extensions beyond the Boolean are given in [4]. Then as probabilities AB and CD map to  $\tau$ , or *certain*, and BC to  $\upsilon$  as *impossible*.

There are possible three types of horizontal line in the above diagram, as above the hole, where the standard retract holds, intersecting with the hole, as given, and below the hole. Correspondingly there are 3 vertical lines, under the designation of the squares as Cartesian products.

For a 3-square, consider 2 further abutments. Let surgery be performed, represented along a *horizontal* 3-edge, as an allocation of a  $\tau$  or  $\upsilon$  classifier, and a  $\tau$  classifier above and below. To allow this existence of the unimpeded retract above and below,

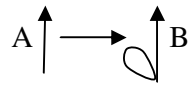
form two further *vertical* abutments of the already abutted object. Then the new object has an interior hole which is classified by  $\upsilon$  as four 3-branched squares, two for each of the horizontal and vertical assignments.

That there are *two* such assignments, horizontal and vertical, follows from the *two* pairs of three 3-edges for each 3-branched square.

The classification of derived objects can be developed further. We have mentioned only object A. Interior holes as already described can be glued to objects of type A.

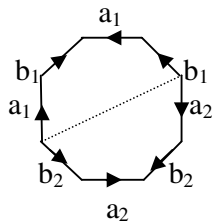
For a normal handle, it would not seem reasonable to glue  $h$  copies of a cylinder to a single hole. The conceptual model of  $n$ -branched spaces liberates us from that constraint.

A construction which generates from the 2-branched square a 2-branched torus with an extra handle is shown below.



We have obtained the 2-branched square, by inserting and connecting two more 2-edges from those at the vertices of A and B, where the horizontal arrow is the retract, and produced a torus from this, except for the loop shown at the bottom vertex of B, by identifying the four 2-edges in pairs with the A retracted to B pair matched. The loop may be detached at the base, but is reconnected under the identification of the base vertex of B with the base vertex of A. This now forms a hole in the torus, which can be glued to the hole of a copy of that torus with a hole.

This torus with one handle, in other words a sphere with two handles, can be represented by the diagram below



where the  $a_1$ 's,  $b_1$ 's,  $a_2$ 's and  $b_2$ 's are identified by gluing in matched directions and the identified hole is given by the dashed line.

Analogously, for a 3-branched torus with one 3-branched handle, we need the equivalent of a loop. This is shown below as a self-attached 3-edge,



R

so the 3-branched loop is all reconnected at R.

To the 3-branched torus, now attach a 3-branched loop at R. The loop can be detached at R, but is reconnected under the identification of vertices for the 3-branched torus,

so it forms a 3-branched hole. To form a 3-branched h-handle, identify by gluing onto the 3-branched hole  $h$  copies of this 3-branched torus with hole.

Thus a 3-branched torus with hole can be identified by gluing with a 3-branched Möbius strip. This is the crosscap construction for 2-branched spaces to produce non-orientable manifolds.

The constructions we have mentioned can be extended in a natural manner to  $n$ -branched spaces.

### **Explosion boundaries.**

The analogue of a real line is an *explosion*, for which we include next a discussion in terms of explosion analysis. Note that, say, a 3-explosion has an infinity of ends, which may themselves be reassembled to form a manifold in the usual sense. Thus we open ourselves to the possibility of a triple boundary  $\partial\partial\partial = 0$ , more generally of a  $k$ -branched explosion with boundary a  $k - 1$  branched explosion, etc., so  $\partial^k = 0$ . Here is an example of  $\partial^k = 0$  and  $\partial^{k-1} \neq 0$ ,  $k > 2$ .

Let  $\mathbb{H}$  be a  $k$ -explosion,  $k > 2$ . Consider a real interval (2-explosion),  $R$ , within it. Let there be a metric on this real line, and let the total length of the interval be  $t$ . For each point  $p_i$  of  $R$ , select a further 2-explosion not belonging to  $R$  except at  $p_i$ , with length  $u(p_i)$  from all  $p_i$ . Let the end point of this line be at  $q_i$ . Then the boundary of  $R$  is the end-points of  $R$  together with all  $q_i$ .

For each  $p_i, p_j \in R$  with distance interval  $t_{ij}$ , consider an *induced metric* on  $q_i, q_j$  with length also  $t_{ij}$ . Then the boundary of  $R$  includes the  $q_i$ , and the  $q_i$  have induced the structure of a real line, which itself will have two boundary points, the boundary of which is zero. Thus if this is the only line selected  $\partial\partial\partial = 0$ , but  $\partial\partial \neq 0$ .

The description of homology and cohomology in terms of exact sequences is an encapsulation of  $\partial\partial = 0$ . We have provided a counterexample, and we will see in Part II that the Hom functor can be used to describe branched spaces. This already gives a description of Ext as an extended exact sequence. The tensor product is also characterised by an exact sequence involving Tor, and the tensor description of hyperintricate numbers [2] can be encapsulated by this means.

### **The general polynomial.**

The branched spaces given by the coefficients of an  $m$ th degree polynomial (here  $\Pi$  indicates multiplication, from  $i = 1$  to  $m$ ) are represented by

$$\Pi(i = 1 \text{ to } m)\{x - n_i\},$$

and indeed  $I$  can consider negative, rational, algebraic, transcendental and complex numbers of hypervolumes, volumes, areas, edges and points. This assignation is consistent with the idea of a topos in which its morphisms lie in a category, in particular when the union of an element and a negative element is the initial object. Further examples are Grothendieck groups [23].

We now extend the idea of the branched representation where we had  $x$ ,  $n$  and  $m$  at most as complex numbers, to  $x$ ,  $n$  and  $m$  *matrices*.

We have detailed in other papers [1], [3] the hyperintricate representation of matrices, in which the complex numbers occur as subobjects of intricate numbers – representable by  $2 \times 2$  real matrices. In this formalism, the branched representation now becomes expandable in terms of a hyperintricate binomial theorem.

Thus having described the branched Euler characteristic in terms of a polynomial, questions of Galois representations obtrude. In particular, there is a Galois cohomology corresponding to the Galois group of such polynomials.

### Summary.

I have given a sketch of an unusual and general type of space, which describes objects characterised by branched simplexes. This includes familiar objects in spaces described in various standard homology and cohomology theories.

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