Intricate and hyperintricate numbers

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Abstract. In this pedagogic article I introduce a representation of 2 X 2 matrices called the *intricate* representation, which contains the complex numbers as a subalgebra, and for 2^n X 2^n matrices a corresponding representation called the *hyperintricate* representation. We develop some of their properties, such as formulae of $e^{i\theta}$ type, and a new proof of non-uniqueness of intricate factorisation. A frequently asked question is "what is the relationship between intricate numbers and quaternions?" I answer this. We also discuss roots of intricate numbers, which can be hyperintricate.

Keywords: matrices, hyperintricate number, Euler relation, division algebra

1 Intricate numbers.

A complex number, represented by $g = a1 + bi$, where $i = \sqrt{-1}$, can also be represented by matrices, where

$$1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \qquad i = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

This representation follows all the rules for a *field*, including the existence of a multiplicative inverse g^{-1} of a complex number, satisfying

$$g^{-1} = (a1 - bi)/(a^2 + b^2).$$

If we wish to extend this algebra to include all possible 2 X 2 matrices with real elements, then we can introduce two more *basis elements* – the *actual* matrix

$$\alpha = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$$

and the *phantom* matrix

$$\phi = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

Just as for complex numbers where we can represent the (a, b) pair of real and imaginary components as vectors in what is called an *Argand diagram*, we can also have a 4-dimensional diagram representing what I call an *intricate number*

$$h = a1 + bi + c\alpha + d\phi.$$

The linearly independent intricate basis elements satisfy

$$\begin{aligned} &1^2 = 1,\ i^2 = -1,\ \alpha^2 = 1,\ \phi^2 = 1, \\ &1i = i = i1,\ 1\alpha = \alpha = \alpha 1,\ 1\phi = \phi = \phi 1, \\ &i\alpha = -\phi = -\alpha i,\ i\phi = \alpha = -\phi i \text{ and } \alpha\phi = i = -\phi\alpha. \end{aligned} \tag{1}$$

An intricate number can represent uniquely any real 2 X 2 matrix.

A 2 X 2 real matrix

$$\begin{Vmatrix} p & q \\ r & s \end{Vmatrix}$$

does not have an inverse if its *determinant* $ps - rq = 0$, in which case it is called a *singular* matrix. For a complex number the basis elements 1 and i have determinant 1 – so all matrices except the zero matrix for a complex number have multiplicative inverses. We can see in contrast that non-zero intricate numbers may have no multiplicative inverse.

In more detail, the matrix above has the intricate representation

$$\begin{aligned} h &= a1 + bi + c\alpha + d\phi \\ &= \tfrac{1}{2}(p + s)1 + \tfrac{1}{2}(q - r)i + \tfrac{1}{2}(p - s)\alpha + \tfrac{1}{2}(q + r)\phi. \end{aligned}$$

The intricate conjugate is $(a1 - bi - c\alpha - d\phi)$. If the multiplicative inverse exists, it is

$$h^{-1} = (a1 - bi - c\alpha - d\phi)/(a^2 + b^2 - c^2 - d^2),$$

so the denominator (the determinant) is non-zero.

A matrix is called *nilpotent* if its nth power is zero, when it is necessarily singular, by determinant multiplication. The singular matrices $(\phi + i)$ and $(\alpha + i)$ have zero square.

2 The intricate Euler relations. [Ad12c]

Complex numbers satisfy the Euler relation

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which can be obtained using a Taylor series expansion, where

$$e^{\lambda} = 1 + \lambda + \lambda^2/2 + \lambda^3/3! + \ldots$$

For intricate basis elements α and ϕ, a similar argument gives

$$e^{\alpha\theta} = \cosh\theta + \alpha\sinh\theta,$$
$$e^{\phi\theta} = \cosh\theta + \phi\sinh\theta,$$

since

$$\cosh\lambda = 1 + \lambda^2/2 + \lambda^4/4! + \ldots$$
$$\sinh\lambda = \lambda + \lambda^3/3! + \lambda^5/5! + \ldots$$

If we choose to represent

$$e^h = e^{a1 + bi + c\alpha + d\phi} = e^{a1}e^{bi}e^{c\alpha}e^{d\phi}, \qquad (2)$$

then multiplicative non-commutativity, e.g.

$$e^{bi}e^{c\alpha} \neq e^{c\alpha}e^{bi}$$

from the Taylor series expansions, immediately tells us that for abelian addition (2) cannot hold, although a1 commutes. In the example above the non-commutation expressed as $e^{bi}e^{c\alpha} - e^{c\alpha}e^{bi}$ depends on ϕ, b and c only.

The square of $J = (bi + c\alpha + d\phi)$ is $(-b^2 + c^2 + d^2)$. Setting $z = a1 + JK$, for K real, when $J^2 = -1$ the Taylor expansion gives

$$e^z = e^{a1 + (bi + c\alpha + d\phi)K} = e^{a1 + JK}$$
$$= e^{a1}(\cos K + J\sin K), \qquad (3)$$

when $J^2 = +1$

$$e^z = e^{a1}(\cosh K + J\sinh K) \qquad (4)$$

and when $J^2 = 0$

$$e^z = e^{a1}(1 + JK). \qquad (5)$$

We note for representations of this form that

$$e^{a1 + JL + JM} = e^{a1}e^{JL}e^{JM}. \qquad (6)$$

That these solutions are not the most general may be deduced from the observation that the determinant of $e^{a1 + (bi + c\alpha + d\phi)K}$ is given by an expression of the form (3), (4) or (5) multiplied by its intricate conjugate, and in each case this is e^{2a1}, which cannot be zero or negative.

Since a determinant is a multiplicative function, that is for matrices C and D

$$\det CD = \det C \det D,$$

a representation with zero or negative determinant may be obtained on multiplying by an intricate number which itself has a zero or negative determinant. These issues are discussed in [Ad12c]. A non-unique intricate representation extending this type is

$$W = e^{x1 + J_1L} + \Delta . e^{y1 + J_2M},$$

with $\det \Delta = -1$.

3 Hyperintricate numbers.

We can define n-*hyperintricate* numbers recursively, by building up starting from intricate ones. Consider a $2^n \times 2^n$ matrix. Let "+" be a chosen $2^{n-1} \times 2^{n-1}$ matrix which is a hyperintricate basis element of lower dimension, for example an intricate basis element 1, i, α or ϕ. Let "–" be the corresponding matrix with all negative entries from "+". Consider the set of $2^n \times 2^n$ hyperintricate basis elements

$$\begin{Vmatrix} + & 0 \\ 0 & + \end{Vmatrix} \qquad \begin{Vmatrix} 0 & + \\ - & 0 \end{Vmatrix} \qquad \begin{Vmatrix} + & 0 \\ 0 & - \end{Vmatrix} \qquad \begin{Vmatrix} 0 & + \\ + & 0 \end{Vmatrix}$$

Any $2^n \times 2^n$ matrix can be represented uniquely by a linear combination of these.

All n-hyperintricate basis elements beyond the intricate have determinant +1, since the "+" and "–" components both have the same determinant, ±1. The n-indexed determinant is derived from the product of two (n – 1)-hyperintricate determinants, which multiplied together to form the higher dimensional one, has value always +1. We caution that for n > 1, general determinants are *not* additive functions.

A $j \times j$ matrix may be extended both right and below with zero entries to give a larger $2^n \times 2^n$ matrix, or main diagonal entries of 1 may be substituted here to maintain determinants. By this means matrix theorems may be expressed hyperintricately.

I now introduce some notation. I will do this by giving examples of 4×4 matrices. Write

$$1_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \qquad \alpha_i = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

$$i_1 = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix} \qquad \phi_i = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{Vmatrix}$$

So "+" corresponds with the subscript, which will be described as an example of an index, for example in α_i. Mnemonically 'subscripts are the little part'.

If in general each of the 16 real 4×4 matrices are represented by e.g. $\alpha_i = A_B$, then

$$(A_B) + (A_C) = A_{(B+C)},$$
$$(A_B) + (C_B) = (A + C)_B,$$
$$(A_B)(C_D) = (AC)_{BD},$$
$$A_{-B} = -(A_B) = (-A)_B.$$

For further nesting of matrices, consider instead of stepping down a further level, introducing (possibly) a comma, thus: $A_{B,C}$, so that matrix multiplication becomes

$$(AB)_{CD,EF} = (A_{C,E})(B_{D,F}).$$

The *indices* of a basis element $m_{n \ldots p}$, are the vectors m, n, ... p, and its *index dimension* is the number of indices.

Intricate and hyperintricate numbers appear in four avatars – as scalars, satisfying a non-commutative algebra, as vectors with a linearly independent basis, as matrices – where the first instance is intricate numbers, and in the hyperintricate case, say as the tensor $m_{n,p}$, where m, n and p are vectors.

Indices may be *permuted*, and except for interior coefficient algebras, compression and expansion studied later, uniformly applied the resulting algebraic relations under addition and multiplication are the same.

We define, in violation of normal usage, an n-*hypercomplex* number to be an n-hyperintricate number with each index restricted to the set {1, i}. We can also define *hyperactual* numbers, containing elements of {1, α} in all indices and *hyperphantom* numbers for which every index $\in$ {1, ϕ}. A (hyper)actual or (hyper)phantom number is not a member of a field. Another way of putting this is that complex numbers constitute the only algebra of the three which is analytic. This arises because $(1 + \alpha)$ and $(1 + \phi)$ have determinant zero, and so are singular with no inverse and $(a1_1 + bi_i)$ has inverse $(a1_1 – bi_i)/(a^2 – b^2)$, which does not exist for a = b.

4 Exterior, interior and relative coefficient algebras.

A real number, r, multiplied by a hyperintricate basis element A_B multiplies each element of the matrix by r. Then

$$rA_B = (aA)_{(bB)} \tag{7}$$

where ab = r.

Hyperintricate basis elements may have coefficients acting on the left or right (or both) which are themselves hyperintricate. These coefficients may be considered as a sum of terms of real values multiplied by hyperintricate basis elements.

Generally speaking, there is more than one type of algebra in which the coefficients are multiplied by hyperintricate basis elements. In all cases real components of the coefficient bases are treated as in (7).

An *exterior coefficient algebra* takes the index dimension, n, of the coefficients and to the m-hyperintricate basis element to which it is attached, appends a basis element of the coefficient to the (say) trailing 1 indices of the m-hyperintricate basis element. The exterior coefficient algebra is commutative with respect to a coefficient and an m-hyperintricate basis element.

The *interior coefficient algebra* extracts indices from the basis element of the coefficient, permutes them in a uniform way and multiplies corresponding indices to those in the m-hyperintricate basis to which it is attached. For identity permutations and coefficients with index dimension equal to that of basis elements, this corresponds to normal matrix multiplication. The interior coefficient algebra is not commutative in general.

The exterior coefficient algebra may be considered as a special case of the interior coefficient algebra, in which the coefficient is multiplied by trailing indices of 1 in the m-hyperintricate basis element.

The *relative coefficient algebra* operates on all indices rather than selective ones and treats r, a and b in (7) as intricate or hyperintricate numbers. Consider the example $A_B = 1_1$, r = i, a = α, b = ϕ, and real numbers t = uv. We would have

$$tiA_B = (u\alpha A)_{(v\phi B)} = t\alpha_\phi$$
$$= -(u\phi A)_{(v\alpha B)} = -t\phi_\alpha,$$

which is not the case. A solution not directly involving equivalence classes is to treat r, a and b as conforming to the scalar algebra given in (1) and to perform operations relative to the basis A_B. We write to indicate this

$$tiA_B = (uaA)_{(vbB)} \qquad (\text{rel } A_B),$$

then we have for example

$$ti1_1 = (u\alpha 1)_{(v\phi 1)} \qquad (\text{rel } 1_1),$$
$$ti1_1 = -(u\phi 1)_{(v\alpha 1)} \qquad (\text{rel } 1_1).$$

The relative coefficient algebra is not commutative in general.

5 Compression and expansion.

The *compression* of a v-hyperintricate number from $2^v \times 2^v$ matrix basis elements to $2^w \times 2^w$ basis elements, where we are compressing v – w + 1 vectors, consists in multiplying together in order the vectors to be compressed in the v-hyperintricate algebra.

The compression homomorphism, κ, with abelian addition and non-commutative multiplication, satisfies for basis elements, and correspondingly for composites (we may use here real numbers r and s, although we can incorporate r and s as intricate numbers via an interior or relative coefficient algebra)

$$\kappa(rA_B) = r^2AB,$$
$$\kappa(rA_B + sC_D) = (r^2AB) + (s^2CD),$$

as may be verified using basis element universals.

Where B or C are 1 or B = C, we connect compression with matrix multiplication via

$$\kappa(rA_BsC_D) = \kappa(rA_B)\kappa(sC_D) \qquad (8)$$

otherwise for distinct non-real B and C, by non-commutation of basis elements we obtain

$$\kappa(rA_BsC_D) = -\kappa(rA_B)\kappa(sC_D),$$

for example

$$\kappa[(\alpha_\phi)^2] = -\kappa(\alpha_\phi)\kappa(\alpha_\phi) = -i^2 = 1.$$

Thus the zero matrix is compressed to a zero matrix, and the unit matrix to a unit matrix. However $\kappa(rA_0) = 0$, and A_0 is 0, but $\kappa(\alpha_\alpha) = 1$ and $\alpha \neq 1$. Compression is an epimorphism from the v-hyperintricate algebra to the w-hyperintricate algebra.

The hypercomplex, hyperactual and hyperphantom algebras commute, so for hypercomplex, hyperactual and hyperphantom numbers, κ is commutative, and (8) always holds.

There is an opposite operation, κ^{op}, called *expansion*, so that for expansion

$$\kappa^{op}(r^2AB) = rA_B,$$
$$\kappa^{op}[(r^2AB) + (s^2CD)] = (rA_B) + (sC_D).$$

We can extend this type of notion of compression *for intricates down to reals* by taking the determinant of the matrix basis elements. First note that the number 1 we have been using is in fact a diagonal 2 X 2 matrix. To distinguish this from its real value elements, denote the latter occasionally by 1~.

We can now compress intricate basis elements down to ±1~. We have the mappings

$$1 \to 1\sim,\ i \to 1\sim,\ \alpha \to -1\sim \text{ and } \phi \to -1\sim,$$

where we denote this compression mapping by λ, so that the determinant

$$\lambda(r1 + s\alpha + t\phi + ui) = [(r + s)(r - s) - (t + u)(t - u)1\sim$$
$$= (r^2 - s^2 - t^2 + u^2)1\sim,$$

and therefore we have proved

$$\lambda(r1 + s\alpha + t\phi + ui) = \lambda(r1) + \lambda(s\alpha) + \lambda(t\phi) + \lambda(ui).$$

However, for the real component, and except possibly for sign similarly for i, α and ϕ

$$\lambda[(r_1 + r_2)1] = \lambda(r_11) + \lambda(r_21) + 2r_1r_21\sim = (r_1 + r_2)^21\sim.$$

The expansion λ^{op} is defined in like manner to κ^{op}.

6 Symmetric, antisymmetric and upper triangular matrices.

A matrix U is *symmetric* when its elements satisfy $u_{jk} = u_{kj}$, and the elements of a matrix V are *antisymmetric* when $v_{jk} = -v_{kj}$. Any matrix W may be represented uniquely as W = U + V.

A 2^n X 2^n matrix V is antisymmetric when all of its hyperintricate components are antisymmetric. This follows from the uniqueness of the hyperintricate representation. The square of a symmetric basis element is 1 and of an antisymmetric basis element is -1. So this antisymmetry happens when a search of the indices of a basis element finds an odd number of i's, otherwise the basis element is symmetric.

A 2^n X 2^n *upper triangular* matrix has all zero entries below the main diagonal. This diagonal is represented entirely by hyperactual components.

A necessary condition for the remainder outside the diagonal to be upper triangular is that each antisymmetric component where a ϕ or i index is ranked earliest is summed equally with the symmetric component in which this ϕ or i is interchanged.

This condition is sufficient. Main diagonal symmetries, including antisymmetries, are symmetries with the largest such scope, only determined by the leading ϕ or i index. The lower triangular region will then only be zero if a leading ϕ or i index for each component is summed with an interchanged i or ϕ index of equal value.

7 Factorisation of intricate numbers. [Ad12e]

An intricate number may be represented as $1(p1 + qi + r\alpha + s\phi)$, and 1 may be factorised intricately in an infinite number of distinct ways.

All integers have integer valued intricate factorisations in an infinite number of ways. This follows because

$$(a^2 + b^2 - c^2 - d^2) = (a1 + bi + c\alpha + d\phi)(a1 - bi - c\alpha - d\phi)$$

and any integer may be represented for integers a, b, c, d by $(a^2 + b^2 - c^2 - d^2)$, since

$$(a^2 - c^2) = (a + c)(a - c)$$

and if $(a - c) = 1$, $(a + c)$ can have any odd value, thus considering $(b^2 - d^2)$, this can have any odd value, and varying over $(a + c)$ any even integer can be accommodated. Likewise, if $(a - c) = 2$, the product with $(a + c)$ forms an arbitrary multiple of 4, and keeping $(b^2 - d^2)$ odd, any odd integer can be accommodated.

8 Intricate products under non-commutation.

Say we wished to evaluate

$$(a + bi + c\alpha + d\phi)(p + qi + r\alpha + s\phi) = (t + ui + v\alpha + w\phi)(a + bi + c\alpha + d\phi).$$

Then multiplying the right hand side by the intricate conjugate $(a - bi - c\alpha - d\phi)$ gives, on putting $E = (a^2 + b^2 - c^2 - d^2)$ and equating intricate parts,

$$t = p,$$
$$Eu = q(a^2 + b^2 + c^2 + d^2) + 2[r(-bc - ad) + s(ac - bd)],$$
$$Ev = r(a^2 - b^2 - c^2 + d^2) + 2[q(bc - ad) + s(ab - cd)]$$

and

$$Ew = s(a^2 - b^2 + c^2 - d^2) + 2[q(ac + bd) - r(ab + cd)].$$

9 Representations of quaternions by hyperintricate numbers.

The quaternions are extensions of the complex numbers with 3 'imaginary' – or quaternionic – parts. So we can represent a quaternion by

$$a1 + bi + cj + dk$$

where

$$1^2 = 1,\ i^2 = j^2 = k^2 = -1,$$
$$1i = i = i1,\ 1j = j = j1,\ 1k = k = k1,$$
$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik$$

and the inverse is

$$(a1 - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2).$$

This (1, i, j, k) basis is representable by four hyperintricate numbers – in fact the four given previously – 1_1, α_i, i_1 and ϕ_i. An alternative is 1_1, i_α, 1_i and i_ϕ. There are others, including 1_{11}, $i_{\alpha\phi}$, $\alpha_{\phi i}$ and $\phi_{i\alpha}$.

Reference [Ad12a] contains a hyperintricate proof that the only standard associative division algebras are the reals, complex numbers and quaternions. As defined in [Ad12a], where there is more than one basis element with square 1, there exist non-standard associative algebras of dimension 2^n, for n any whole number.

10 Roots of intricate basis elements. [Ad12d]

The multiplicative inverses of the intricate basis elements are as follows.

$$1^{-1} = 1,\ i^{-1} = -i,\ \alpha^{-1} = \alpha,\ \phi^{-1} = \phi.$$

We shall see that for $\alpha^{1/2}$ and $\phi^{1/2}$, square roots of intricate basis elements can be represented by hyperintricate numbers. For square roots

$$1^{1/2} = \pm 1 \text{ or } \pm(ui + v\alpha + w\phi),$$

with $-u^2 + v^2 + w^2 = 1$, allowing us to expand the list of *possibilities* below, also

$$i^{1/2} = \pm(1 + i)/\surd 2,$$

giving the dependent relation (with the same positive or negative sign)

$$i^{-1/2} = \pm(1 - i)/\surd 2,$$

with

$$\alpha^{1/2} = \pm \begin{Vmatrix} 1 & 0 \\ 0 & \pm i \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \alpha & 0 \\ 0 & \pm i \end{Vmatrix}$$

or

$$\pm \begin{Vmatrix} \phi & 0 \\ 0 & \pm i \end{Vmatrix}$$

and

$$\phi^{1/2} = \pm(1/\surd 2) \begin{Vmatrix} i^{-1/2} & i^{1/2} \\ i^{1/2} & i^{-1/2} \end{Vmatrix}$$

For integers m and n we give the following roots.

$$1^{1/(2n+1)} = e^{i2\pi m/(2n+1)}$$
$$i^{1/(2n+1)} = e^{i\pi(4m+1)/[2(2n+1)]}$$
$$\alpha^{1/(2n+1)} = \alpha$$
$$\phi^{1/(2n+1)} = \phi.$$

$$1^{1/(2n)} = e^{i\pi m/n}$$
$$i^{1/(2n)} = e^{i\pi(4m+1)/(4n)}$$

with

$$\alpha^{1/(2n)} = \pm \begin{Vmatrix} 1 & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \alpha & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \phi & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$\phi^{1/(2n)}$ may be obtained recursively from roots with smaller n. For example, since

$$\phi^{1/2} = \pm(1/\surd 2)\, i^{1/2} \begin{Vmatrix} -i & 1 \\ 1 & -i \end{Vmatrix}$$

we have

$$\phi^{1/4} = \pm(1 \text{ or } i)[i^{1/4}/2^{1/4}] \begin{Vmatrix} a & b \\ b & a \end{Vmatrix}$$

where (see the next paragraph for an indication of how to obtain this)

$$a^2 = (-½ \pm 1/\surd 2)i$$
$$b^2 = -i - a^2.$$

More generally, for natural numbers $t = 2^n$ and $u = 2^{n-1}$, if

$$\phi^{1/t} = e^{i2\pi m/t}[i^{1/t}/2^{1/t}] \begin{Vmatrix} P & Q \\ Q & P \end{Vmatrix}$$

where we have previously determined that

$$\phi^{1/u} = e^{i2\pi m/u}[i^{1/u}/2^{1/u}] \begin{Vmatrix} p & q \\ q & p \end{Vmatrix}$$

then on squaring the matrix in P and Q

$$P^2 + Q^2 = p$$
$$2PQ = q$$

which reduces to a solvable quadratic equation, say in P^2, so $\phi^{1/t}$ is determined.

If a natural number T = ty, with y an odd number, is encountered instead of t, the determination of $\phi^{1/T}$ can be found from the relation, valid for natural numbers y and t

$$\phi^{1/T} = [\phi^{1/y}]^{1/t} = \phi^{1/t}.$$

The results of section 2, where an intricate number may be represented by e^z for a positive determinant, can be used to give a root $e^{z/n}$, although for $J^2 = 1$, cosh cannot take the value zero.

These ideas may be combined. For example, to get around the cosh restriction on representations of α, we may write using an exterior coefficient algebra

$$\alpha^{1/n} = \begin{Vmatrix} e^{2\pi mi/n} & 0 \\ 0 & e^{\pi(2m+1)i/n} \end{Vmatrix}$$
$$\equiv ½(e^{2\pi mi/n} + e^{\pi(2m+1)i/n})1_1 + ½(e^{2\pi mi/n} - e^{\pi(2m+1)i/n})\alpha_1.$$

Further details are given in reference [Ad12d].

11 Composites.

Let $J^2 = -1$, $A^2 = 1$ and $F^2 = 1$, where we put

$$J = qi + r\alpha + s\phi,$$
$$A = bi + c\alpha + d\phi,$$
$$F = ei + f\alpha + g\phi,$$

and we allocate

$$AF = J. \qquad (9)$$

Since J does not have a real part, it follows from the relations

$$-be + cf + dg = 0,$$
$$cg - df = q,$$
$$bg - de = r$$

and

$$-bf + ce = s$$

that

$$AF = -FA = J. \qquad (10)$$

Multiplying (9) on the left by A

$$F = AJ,$$

and multiplying on the right by F

$$A = JF.$$

Correspondingly, multiplying (10) on the right by A and the left by F gives

$$F = -JA,$$
$$A = -FJ,$$

and we have established an equivalence of algebras for

$J \leftrightarrow i$,

$A \leftrightarrow \alpha$

and

$F \leftrightarrow \phi$.

Notes

Except that intricate basis elements for SL(2, R) are given on page 25 of [Ge75], a partial literature search, both historical [Fr73], [Sch73] and contemporary [Se03] has failed to find reference elsewhere to this explicit idea. The matrix representation of complex numbers is mentioned by R. Remmert on page 69 of [Eb91], from where we got this notion in 2008. The representation of quaternions by matrices is given in [He68] and [No83].

There are many references to the upper triangular representation of matrices in the literature [JL70], [La80], [ST92]. A. Borel is often quoted in this context, but the idea without the technology goes back to Gauss [Ga1863].

We have derived the equivalence of algebras corresponding to the bijection between i and $J = bi + c\alpha + d\phi$, with J constant and $J^2 = -1$, as in equation (6). A consequence is that complex analysis may be extended, for which the Cauchy-Riemann equations and the Cauchy integral formula can be re-expressed in terms of J instead of i, similarly also for Laplace transforms. In like manner, Galois theory carries over for such algebras [Ad12g]. The same can be said for a theory of modular forms [Ad12b], [BN97], [Ne97]. There is also the classification of Lie groups, which can be extended to Lie J-groups [Ad12f], [Hu90], [MT00], [Ro02]. However, our representation is an extension of all these.

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