# Division Algebras

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This paper is under development. It is believed to be correct. A section on non-associative division algebras will be added.

## Introduction.

I introduce a representation of 2 X 2 matrices called the *intricate* representation, which contains the complex numbers as a subalgebra, and for $2^n$ X $2^n$ matrices a corresponding representation called the *hyperintricate* representation.

A frequently asked question is "what is the relationship between intricate numbers and quaternions?" I answer this, and show the quaternions are the most general standard, but not non-standard, matrix division algebra.

## Intricate numbers.

A complex number, represented by g = a1 + bi, where i = √-1, can also be represented by matrices, where

$$1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \qquad i = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

This representation follows all the rules for a *field*, including the existence of a multiplicative inverse $g^{-1}$ of a complex number, satisfying

$$g^{-1} = (a1 - bi)/(a^2 + b^2).$$

If we wish to extend this algebra to include all possible 2 X 2 matrices with real elements, then we can introduce two more *basis elements* – the *actual* matrix

$$\alpha = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$$

and the *phantom* matrix

$$\phi = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

Just as for complex numbers where we can represent the (a, b) pair of real and imaginary components as vectors in what is called an *Argand diagram*, we can also have a 4-dimensional diagram representing what I call an *intricate number*

$$h = a1 + bi + c\alpha + d\phi.$$

The linearly independent intricate basis elements satisfy

$$1^2 = 1,\ i^2 = -1,\ \alpha^2 = 1,\ \phi^2 = 1,$$
$$1i = i = i1,\ 1\alpha = \alpha = \alpha 1,\ 1\phi = \phi = \phi 1,$$
$$i\alpha = -\phi = -\alpha i,\ i\phi = \alpha = -\phi i \text{ and } \alpha\phi = i = -\phi\alpha. \qquad (1)$$

An intricate number can represent uniquely any real 2 X 2 matrix.

A 2 X 2 real matrix

$$\begin{Vmatrix} p & q \\ r & s \end{Vmatrix}$$

does not have an inverse if its determinant ps – rq = 0, in which case it is called a *singular* matrix. For a complex number the basis elements 1 and i have determinant 1

– so all matrices except the zero matrix for a complex number have multiplicative inverses. We can see in contrast that non-zero intricate numbers may have no multiplicative inverse.

In more detail, the matrix above has the intricate representation

$$h = a1 + bi + c\alpha + d\phi$$
$$= \tfrac{1}{2}(p + s)1 + \tfrac{1}{2}(q - r)i + \tfrac{1}{2}(p - s)\alpha + \tfrac{1}{2}(q + r)\phi.$$

The intricate conjugate is $(a1 - bi - c\alpha - d\phi)$. If the multiplicative inverse exists, it is

$$h^{-1} = (a1 - bi - c\alpha - d\phi)/(a^2 + b^2 - c^2 - d^2),$$

so the denominator is non-zero.

## Hyperintricate numbers.

We can define n-*hyperintricate* numbers recursively, by building up starting from intricate ones. Consider a $2^n$ X $2^n$ matrix. Let "+" be a chosen $2^{n-1}$ X $2^{n-1}$ matrix which is a hyperintricate basis element of lower dimension, for example an intricate basis element 1, i, $\alpha$ or $\phi$. Let "–" be the corresponding matrix with all negative entries from "+". Consider the set of $2^n$ X $2^n$ hyperintricate basis elements

$$\begin{Vmatrix} + & 0 \\ 0 & + \end{Vmatrix} \qquad \begin{Vmatrix} 0 & + \\ - & 0 \end{Vmatrix} \qquad \begin{Vmatrix} + & 0 \\ 0 & - \end{Vmatrix} \qquad \begin{Vmatrix} 0 & + \\ + & 0 \end{Vmatrix}$$

Any $2^n$ X $2^n$ matrix can be represented uniquely by a linear combination of these.

We note that all hyperintricate basis elements beyond the intricate have determinant +1. This is because the "+" and "–" components both have the same determinant, say -1 as intricate numbers, but multiplied together to form a higher dimensional determinant, its value is always +1.

I now introduce some notation. I will do this by giving examples of 4 X 4 matrices. Write

$$1_1 = \begin{Vmatrix} 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} & 0 \end{Vmatrix} \qquad \alpha_i = \begin{Vmatrix} 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \end{Vmatrix}$$

$$i_1 = \begin{Vmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{Vmatrix} \qquad \phi_i = \begin{Vmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{Vmatrix}$$

So "+" corresponds with the subscript, for example in $\alpha_i$. Mnemonically 'subscripts are the little part'.

If in general each of the 16 real 4 X 4 matrices are represented by e.g. $\alpha_i = A_B$, then

$$(A_B)(C_D) = (AC)_{BD},$$
$$A_{-B} = -(A_B) = (-A)_B.$$

For further nesting of matrices, consider instead of stepping down a further level, introducing (possibly) a comma, thus: $A_{B,C}$, so that for instance

$$(AB)_{CD,EF} = (A_{C,E})(B_{D,F}).$$

The *indices* of a basis element $m_{n \dots p}$, are the vectors m, n, ... p. The indices may be *permuted*, and when applied uniformly the resulting algebraic relations under addition and multiplication are the same.

## Representations of quaternions by hyperintricate numbers.

The quaternions are extensions of the complex numbers with 3 'imaginary' – or quaternionic – parts. So we can represent a quaternion by

$$a1 + bi + cj + dk$$

where

$$1^2 = 1,\ i^2 = j^2 = k^2 = -1,$$
$$1i = i = i1,\ 1j = j = j1,\ 1k = k = k1,$$
$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik$$

and the inverse is

$$(a1 - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2).$$

This (1, i, j, k) basis is representable by four hyperintricate numbers – in fact the four given above – $1_1$, $\alpha_i$, $i_1$ and $\phi_i$. An alternative representation, under swapping of index levels, is $1_1$, $i_\alpha$, $1_i$ and $i_\phi$. We can reveal another representation: $1_{11}$, $i_{\alpha\phi}$, $\alpha_{\phi i}$ and $\phi_{i\alpha}$.

## Non-existence of new standard associative division algebras.

The only standard associative division algebras are the reals, complex numbers and quaternions [1]. We will represent the basis elements of these associative division algebras by hyperintricate numbers.

In category theory, a basis of a vector space is an example of a universal arrow, which shows this result is independent of basis. Nevertheless, we can also prove this is so in the case where the hyperintricate basis elements are transformable to the generalised case studied next.

These basis elements have square ±1, and any other representation can be reduced to a linear combination of these basis elements, for which the basis element squares are also ±1. In detail, any representation of the set {1, i, α, ϕ} under a change of basis which preserves squares maps each element to the set

$$\{1, \pm\sqrt{(\gamma_i^2 + \delta_i^2 + 1)}i + \gamma_i\alpha + \delta_i\phi, \pm\sqrt{(\gamma_\alpha^2 + \delta_\alpha^2 - 1)}i + \gamma_\alpha\alpha + \delta_\alpha\phi,$$
$$\pm\sqrt{(\gamma_\phi^2 + \delta_\phi^2 - 1)}i + \gamma_\phi\alpha + \delta_\phi\phi\}$$

with the coefficients $\gamma_i \neq \gamma_\alpha \neq \gamma_\phi$ etc. real. This extends to all indices.

It is not initially clear, for example, whether $1_{11}$, $i_{\alpha 1}$, $1_{i1}$, $i_{\phi 1}$, $1_{1i}$, $i_{\alpha i}$, $1_{ii}$ and $i_{\phi i}$ can form a normed division algebra in which more than one square of a basis element is 1, e.g.$1_{ii}$.

For any hyperintricate basis element, the inverse is known. For a basis element A whose square is 1, the inverse $A^{-1} = A$. These A amount to all basis elements which have an even number (including zero) of i's in their hyperintricate representation. For any basis element, B, whose square is -1, the inverse $B^{-1} = -B$. The set of all B's is those basis elements which have an odd number of i's in their hyperintricate representation.

There are only two possibilities for basis elements, they either commute, AB = BA, or they anticommute, AB = -BA.

Consider finding the inverse of $aA_1 + bA_2$, $A_1 \neq A_2$, where $A_1^2$ and $A_2^2 = 1$. This is then

$$(aA_1 - bA_2)/(a^2 - b^2) \qquad (2)$$

when $A_1$ and $A_2$ commute and

$$(aA_1 + bA_2)/(a^2 + b^2)$$

when $A_1$ and $A_2$ anticommute. If we incorporate the fact that 1 is always present amongst such A's, then for some values of a and b, (2) holds, which implies that there exist a's and b's for which (2) includes the possibility of dividing by zero. The statement that we can do division is incorporated in the definition of a division algebra (although we have to specifically exclude division by zero, as for a field), therefore there exists in such division algebras only one basis element with square 1, and this must be the real basis element. We will extend these considerations later.

To find the inverse of $a1 + bB_1$, where $B_1^2 = -1$, then this is

$$(a1 - bB_1)/(a^2 + b^2),$$

which introduces no further problems.

To find the inverse of $aB_1 + bB_2$, for $B_1^2 = 1$ and $B_2^2 = -1$, then this is the permissible

$$-(aB_1 + bB_1)/(a^2 + b^2),$$

when $B_1$ and $B_2$ anticommute, which is now the only possibility.

The above argument may be generalised for more $B_r$'s, and it becomes necessary to stipulate that all $B_1, B_2, \ldots B_n$ mutually anticommute.

We know there are solutions for $B_1$, $B_2$, $B_3$ given by basis elements for the quaternions. Now assume the existence of four such basis elements, $B_1$, $B_2$, $B_3$, $B_4$, all mutually anticommuting and distinct, so that $B_rB_s = -1$. We will use associativity of these basis elements in computing from $B_1B_2B_3B_4$ its mirror reflection in two separate ways. So

$$\begin{aligned} B_1B_2B_3B_4 &= -B_1B_2B_4B_3 \\ &= B_1B_4B_2B_3 \\ &= -B_4B_1B_2B_3 \\ &= B_4B_1B_3B_2 \\ &= -B_4B_3B_1B_2 \\ &= B_4B_3B_2B_1. \end{aligned}$$

However

$$\begin{aligned} (B_1B_2)(B_3B_4) &= -(B_3B_4)(B_1B_2) \\ &= -(B_4B_3)(B_2B_1), \end{aligned}$$

a contradiction.

*Thus the maximum number of dimensions for a standard associative division algebra is* 4.

**Extension of the reasoning to possibly singular matrices.**

If we were to allow equation (2) to operate, this means that we can divide by $(a^2 - b^2)$. There exists the possibility that this is zero, but we could treat this situation on the same footing as dividing explicitly by zero, excluded as a number, but see [3], [4]. We will describe a non-standard division algebra as one in which the number of singular occurrences, divided by the total number of occurrences, is an infinitesimal, a number ε such that for any number $n \in N_{\neq 0}$ there does not exist an $m \in N$ with $\varepsilon m > n$. We will now incorporate these circumstances where $(a^2 - b^2) \neq 0$, which allows more than one timelike square, that is, we permit multiple basis element squares of 1.

However, if we generalise the example for which we began the last section, we note that $1_{p,q, \ldots 1} + 1_{p,q, \ldots \alpha}$ is a matrix with a zero bottom row and therefore corresponds to a singular matrix. Similarly $1_{p,q, \ldots 1} + 1_{p,q, \ldots \phi}$ has two equal rows and is consequently also singular.

For the remainder of the work we will be considering $\{1_{p,q, \ldots 1}, \ldots\} \cup \{1_{p,q, \ldots i}, \ldots\}$, but we will see here too that a singular matrix can be derived with the trailing index, in the example which follows by setting a = 1, g = 1 and all other coefficients zero.

To deal with the case considered next, first note that

$$P + Q1_{ii} + Ri_{\alpha i} + Si_{\phi i}$$

has inverse

$$(P - Q1_{ii} - Ri_{\alpha i} - Si_{\phi i})/(P^2 - Q^2 - R^2 - S^2). \quad (3)$$

Let us now investigate the properties of

$$\{1_{11}, 1_{i1}, i_{\alpha 1}, i_{\phi 1}, 1_{1i}, 1_{ii}, i_{\alpha i}, i_{\phi i}\}^{+,x} \quad (4)$$

under addition and multiplication. The above example is closed under multiplication. Does it form a multiplicative group?

We will write these 3-hyperintricate numbers as

$$a1_{11} + b1_{i1} + ci_{\alpha 1} + di_{\phi 1} + f1_{1i} + g1_{ii} + hi_{\alpha i} + ki_{\phi i}. \quad (5)$$

If we change (5) to an expression with inverses of basis elements substituted, we get

$$a1_{11} - b1_{i1} - ci_{\alpha 1} - di_{\phi 1} - f1_{1i} + g1_{ii} + hi_{\alpha i} + ki_{\phi i}. \quad (6)$$

Multiplying (5) by (6), a little manipulation gives the expression

$$[a^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2 + k^2] + 2[(ag - fb)1_{ii} + (ah - fc)i_{\alpha i} + (ak - fd)i_{\phi i}], \quad (7)$$

which is precisely of the form (3).

Thus

$$(5)x(6)x(3) = 1$$

provided

$$P = [a^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2 + k^2],$$

$$Q = 2(ag - fb),$$

$$R = 2(ah - fi)$$

and

$$S = 2(ak - fd),$$

so that (5) does indeed have a multiplicative inverse.

This is not a *normed* division algebra in the usual sense, since the denominator contains terms of degree 4.

Further, we can continue such a process recursively, considering n-hyperintricate examples with trailing index 1 or i, for instance derived from the above example. Let us look at this next.

We will consider both the next stage up, and indicate how we can generalise in an induction procedure, and describe these in parallel. It is possible to be more formal, but then we can lose the thread of the idea.

The case corresponding to (4) is

$$\{1_{111}, 1_{i11}, i_{\alpha 11}, i_{\phi 11}, 1_{1i1}, 1_{ii1}, i_{\alpha i1}, i_{\phi i1}, 1_{11i}, 1_{i1i}, i_{\alpha 1i}, i_{\phi 1i}, 1_{1ii}, 1_{iii}, i_{\alpha ii}, i_{\phi ii}\}^{+,x}. \quad (8)$$

In an induction procedure, we assume a set of basis elements, and append as a trailing index both 1 and i to those elements, thereby doubling the number of basis elements from its previous instance. By the induction procedure, there are $2^{n-1}$ elements to begin with, doubled to $2^n$ in the next stage.

Corresponding to (5), in the specific example we have chosen we consider the hyperintricate number

$$a1_{111} + b1_{i11} + ci_{\alpha 11} + di_{\phi 11} + f1_{1i1} + g1_{ii1} + hi_{\alpha i1} + ki_{\phi i1}$$
$$+ a'1_{11i} + b'1_{i1i} + c'i_{\alpha 1i} + d'i_{\phi 1i} + f'1_{1ii} + g'1_{iii} + h'i_{\alpha ii} + k'i_{\phi ii}, \quad (9)$$

whereas corresponding to (6), we generate the hyperintricate

$$a1_{111} - b1_{i11} - ci_{\alpha 11} - di_{\phi 11} - f1_{1i1} + g1_{ii1} + hi_{\alpha i1} + ki_{\phi i1}$$
$$- a'1_{11i} + b'1_{i1i} + c'i_{\alpha 1i} + d'i_{\phi 1i} + f'1_{1ii} - g'1_{iii} - h'i_{\alpha ii} - k'i_{\phi ii}. \quad (10)$$

In the general situation we will have coefficients in lower case of hyperintricate numbers with a trailing 1 index, minus, in the format corresponding to (10), hyperintricates in primed lower case coefficients each with basis element with a trailing i index.

Multiplying (9) and (10) together gives

$$[a^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2 + k^2]$$
$$+ [a'^2 + b'^2 + c'^2 + d'^2 + f'^2 + g'^2 + h'^2 + k'^2]$$
$$+ 2[(ag - fb - a'g' + f'b')1_{ii1}$$
$$+ (ah - fc - a'h' + f'c')i_{\alpha i1} + (ak - fd - a'k' + f'd')i_{\phi i1}]$$
$$+ 2[(ab' - a'b - fg' + gf')1_{i1i}$$
$$+ (ac' - a'c - hf' + fh')i_{\alpha i1} + (ad' - a'd - kf' + f'k)i_{\phi 1i}]. \quad (11)$$

In general there is a set of positive squares of coefficients both primed and unprimed, followed by twice a number of coefficients times basis elements with an even number of i's.

The inverse of

$$P + Q1_{ii1} + Ri_{\alpha i1} + Si_{\phi i1} + T1_{i1i} + Ui_{\alpha 1i} + Vi_{\phi 1i}$$

is

$$(P - Q1_{ii1} - Ri_{\alpha i1} - Si_{\phi i1} - T1_{i1i} - Ui_{\alpha 1i} - Vi_{\phi 1i})$$
$$/(P^2 - Q^2 - R^2 - S^2 - T^2 - U^2 - V^2), \quad (12)$$

this being a generalisation of (3), involving in its typical characteristic basis elements an even number of i's, so that once again, in its general form, the inverse of (11) can be obtained.

## References

1. J.F. Adams, *On the non-existence of elements of Hopf invariant one,* Ann. Math., **72**, No. 1, 1960.
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