Chapter XVIII
The classification of novanions

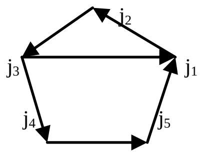
18.1 Introduction.

Having already introduced the existence of novanion algebras, in this chapter we will try to classify the types of novanion algebras available. The 10-novanions and 26-novanions will be used to develop Heim theory further to a novanionic universal physics.

18.2 The search for other novanion algebras.

Are there other novanion algebras of a type not already covered? This question has been stimulated by a first-year Sussex University student's identification of novanions with strings in physics, [Ad17], where the allocation n = 10 is the same number as the dimensionality of the heterotic striong, and for which we wish to investigate the bosonic allocation $1 + 5^2 = 26$.

In the pentagonal diagram shown next, an initial attempt depicts only one out of five subtriangles.



The pentagon can be enumerated cyclically, so that

$$j_1j_2 = j_3,\ j_2j_3 = j_4,\ j_3j_4 = j_5,\ j_4j_5 = j_1,\ j_5j_1 = j_2, \qquad (1)$$

and jumping a vertex we evaluate the closest triangle

$$j_3j_1 = j_2,\ j_4j_2 = j_3,\ j_5j_3= j_4,\ j_1j_4 = j_5,\ j_2j_5 = j_1, \qquad (2)$$

where on inverting the orientation, we get a minus sign.

This latter fact implies we have an inbuilt norm and inverse; the inverse of

$$a1 + \sum_{n=1}^{5} b_n j_n$$

is

$$a1 - \sum_{n=1}^{5} b_n j_n/(a^2 + \sum_{n=1}^{5} b_n^2), \qquad (3)$$

which is nonassociative, as is demonstrated by

$$(j_3j_1)j_4 = -j_3 \neq j_3(j_1j_4) = -j_4.$$

The question arises as to whether this constitutes a novanion algebra, which would now be extended from previous considerations to include the dimensions

$$n = 1 + 3^f5^g7^h.$$

The possibility of the existence of the division algebra violating equation

$$(a1 + bj_1 + cj_2 + dj_3 + ej_4 + fj_5) \times (p1 + qj_1 + rj_2 + tj_3 + uj_4 + vj_5) = 0 \qquad (4)$$

will now be investigated. Under the constraints (1) and (2) we obtain the set of equations

real part:

$$ap - bq - cr - dt - eu - fv = 0, \qquad (5)$$

j_1 part:

$$bp + aq - fr + 0 - fu + (c + e)v = 0, \qquad (6)$$

j_2 part:

$$cp + (f + d)q + ar - bt + 0 - bv = 0, \qquad (7)$$

j_3 part:

$$dp - cq + (b + e)r + at - cu + 0 = 0, \qquad (8)$$

j4 part:

$$ep + 0 - dr + (c + f)t + au - dv = 0, \qquad (9)$$

j5 part:

$$fp - eq + 0 - et + (d + b)u + av = 0, \qquad (10)$$

from which it follows that the E type matrix is not antisymmetric, but it may be represented as the sum of two matrices F and G, where F has all pure imaginary eigenvalues:

$$F = \begin{bmatrix} 0 & -b & -c & -d & -e & -f \\ b & 0 & -f & c & -f & e \\ c & f & 0 & -b & d & -b \\ d & -c & b & 0 & -c & e \\ e & f & -d & c & 0 & -d \\ f & -e & b & -e & d & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & c \\ 0 & d & 0 & 0 & -d & 0 \\ 0 & 0 & e & 0 & 0 & -e \\ 0 & -f & 0 & f & 0 & 0 \\ 0 & 0 & -b & 0 & b & 0 \end{bmatrix},$$

and we do not have pure imaginary eigenvalues for $F + G - \lambda I$.

Indeed a general matrix H may be represented as a sum of a symmetric part H_{sym} and an antisymmetric part H_{anti}. Then H_{sym} and H_{anti} are linearly independent over real coefficients, meaning there exist no real numbers c and d satisfying

$$cH_{sym} + dH_{anti} = 0.$$

By a demonstration analogous to that in section 11.3, and proved directly in [Uh01], the eigenvalues of a symmetric matrix are real. Further, a complex number with real coefficients h_1 and h_2 is linearly independent between h_1 and h_2i over real coefficients. It now follows that the eigenvalue equation

$$H = hI \qquad (11)$$

which has a unique set of n solutions is satisfied by

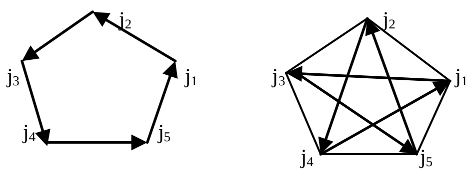
$$H_{sym} = h_1I$$

with n solutions and

$$H_{anti} = h_2I,$$

also with n solutions. These possible solutions are the only ones, since the solution set of (11) is unique. So if H_{sym} is not the zero matrix, a real h_1 exists. This implies there is no novanion algebra available.

We find a similar type of situation for the pentagonal diagrams



since the diagram for j_1, j_2, j_3 is unoriented and therefore does not constitute a quaternion, or in fact a division algebra, so we have failed on modifying equations (5) to (10) to come up with other solutions, where for pentagonal diagrams we can show these pentagons consist in the general case of combined oriented and unoriented quaternion diagrams. □

However, the identification relates to the number 1 + 25, and we now wish to probe the allocation 25 = 7 + 9 + 9, where 7 is the number of non-real basis elements of the octonions, and 9 is the number for the 10-novanions. There is an analogy here. The octonion non-real basis elements of 7 may be represented as 1 + 3 + 3, where 3 is the number of such basis elements for the quaternions, and 1 for the complex numbers. We are forced for a number of reasons to decompose such an allocation into triplets, basically to retain the cyclic algebra for the quaternions.

The allocation will be as follows, where we subscript 3 and 1 to distinguish them

3_a, 3_b, 3_c (i)

3_d, 3_e, 3_f (ii)

3_g, 3_h, 1_u (iii)

where allocations (i) and (ii) are internally similar to 10-novanions, and allocation (iii) is internally an octonion. We will explain why we use the word 'similar' later.

There are a number of possible configurations.

We want an algebra linking between (i), (ii) and (iii). Vertical allocations are present. We will choose next from straight lines going from left to right, for example the diagonal going upwards from 3_g, 3_e to 3_c. This is similar to a 10-novanion algebra. The descending line from 3_a, 3_e to 1_u is an octonion algebra. We then incorporate the algebra taking for example 3_d, 3_b to 1_u, an octonion algebra, or 3_g, 3_b to 3_f, this is similar to a 10-novanion algebra.

We have used the words 'similar to a 10-novanion algebra', and we now explain why. If we look at allocation (iii), this is part of the 3_g 3_h 1_u octonion, where 3_h and 1_u are linked. Although 3_g is indeed a quaternion, we have already mentioned that 3_h is not. Therefore the vertical allocation given by 3_a 3_d 3_g is a 10-novanion, since it is made of genuine quaternions, but the vertical allocation 3_b 3_e 3_h is not. 3_b, 3_e and 3_h occur in octonion representations. If we were to state that the central triple 3_b, 3_e and 3_h algebras were quaternions, we would have an inconsistency. Therefore for these allocations as part of a 'similar to 10-novanion' structure, we decide that the octonion structure overrides the 10-novanion one. Since there is only one special 1_u part for the octonions, this part of the allocation is unique. The similar 10-novanion structure is now not a closed algebra within the 10-novanions; part of it belongs to the octonions. The corresponding situation just for 10-novanions with no octonionic overlap but with novanionic overlap is described by the octonionic allocation already discussed.

Since there is no other mixing of allocations, the result is as consistent as the 10-novanions and the octonions. This can be checked with equations like 3.5.(6) to (15), for which it is clear eigenvalues are pure imaginary. Finally a calculation like 3.5.(16) shows that this is a novanion algebra. □

The existence of 10-, 26- and 80-novanions (the latter obtained by an array cube of items like (i) to (iii) – all configurations lie in planes of the cube, and an m-cube gives rise to a $(3^m \pm 1)$-novanion) implies that results derived for division algebras have a different extension for n-novanions. □

18.3 Sedenions and 64-novanions.

The 16-dimensional sedenions are formed by the Cayley-Dickson construction [Ba01]. Since they are not alternative, they do not form a division algebra. That is, we do not have

$$x(xy) = (xx)y$$

and

$$(yx)x = y(xx)$$

for all x and y in the algebra, the proof using basis elements. Every associative algebra is alternative, but so too are some strictly non-associative algebras such as the octonions.

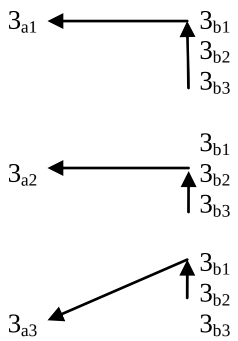
Whether there exist other 16-dimensional algebras not generated by the Cayley-Dickson construction was answered in the negative by J. F. Adams in 1960.

Irrespective of this result, the Cayley-Dickson construction generating a 64-dimensional algebra shows that this is not a division algebra, since in particular this contains the sedenions as a subalgebra. However, a 64-novanion has $63 = 3^2 \times 7$ non-scalar basis elements, and we will see that novanion algebras of this type are consistent. A 64-novanion is given by the cube with slices

$3_a, 3_b, 1_p$ $\quad 3'_a, 3'_b, 1'_p$ $\quad 3''_a, 3''_b, 1''_p$
$3_c, 3_d, 1_q$ $\quad 3'_c, 3'_d, 1'_q$ $\quad 3''_c, 3''_d, 1''_q$
$3_e, 3_f, 1_r$ $\quad 3'_e, 3'_f, 1'_r$ $\quad 3''_e, 3''_f, 1''_r$.

To evaluate a typical slice algebra, if we take the leftmost array above, we know that 3_b is not a quaternion, so we will build an override structure for the composition of two elements in 3_b. Within this slice 3_b belongs to three octonionic arrangements, those given by 3_a, 3_b, 1_p, or 3_c, 3_b, 1_r, or 3_e, 3_b, 1_q, so we need to select an override on the nonquaternion 3_b, so that when two elements are multiplied within it, just one allocation to an octonionic structure is selected.

We will need to look at this typical example in detail, so denote the three elements of 3_b by 3_{b1}, 3_{b2} and 3_{b3}. We will display the 3 elements of 3_b combining in pairs to form arrows with the following typical structures. We will choose at first, arbitrarily, a link to the 3_a, 3_b, 1_p octonionic structure. Of course, two arrows shown below combine to give an oriented quaternion triple, for which reversal of arrows leads to a minus value.



The central triples 3_d and 3_f have similar structures, mapping to separate values in 3_c and 3_e respectively. We have stated the 1_p, 1_q and 1_r elements combined with 3_b give on composition with one element of 3_b the octonion structures (3_a, 3_b, 1_p), (3_c, 3_b, 1_r) and (3_e, 3_b, 1_q). Because 3_{b1} links to 3_{a1}, we have to ensure that the link 1_p to 3_{b1} does not also link to 3_{a1}, but this can be arranged.

Alternative structures can be considered. For example, if 3_{b3}, 3_{b1} links to 3_{a1} as before, we could also have 3_{b3}, 3_{b2} linking to 3_{c1} and 3_{b2}, 3_{b1} linking to 3_{e1}. ☐

18.4 Further investigations.

The exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 are related to the existence of division algebras limited in number to those embedded within the octonions [Wi09], [CSM95].

We have seen in chapter IV that for matrices A, B and C the Lie bracket

$$[AB] = AB - BA$$

satisfies the Jacobi identity

$$[[AB]C] + [[BC]A] + [[CA]B] = 0. \qquad (1)$$

For octonions, we do not have a matrix algebra, but we might wish to form Lie brackets from them satisfying (1). From section 3, for the octonions the only nonquaternionic triple is $e_1e_3e_5$ and

$$[[e_1\ e_3]\ e_5] + [[e_3\ e_5]\ e_1] + [[e_5\ e_1]\ e_3] = 2[e_2\ e_5] + 2[e_4\ e_1] + 2[e_6\ e_3]$$
$$= -12e_7, \qquad (2)$$

so this does not satisfy (1).

In order to create a viable Lie bracket we note that we have constructed the octonions from the quaternions by the Cayley-Dickson construction in chapter XI, section 2. Thus, if we apply an inverse Cayley-Dickson construction to retrieve a pair of quaternions from the octonions, since the quaternions are representable by matrices, on each item of this pair we can create a Lie bracket satisfying (1). The information we might wish to keep in these Lie brackets could also be formed from the sum, difference, matrix product, and the product of a matrix by an inverse matrix of the pair, or any combination of these.

A better solution is to take the Lie brackets of equation (2) (mod 12). ☐

The 10-novanions we have considered contain the quaternions as a subalgebra. Even when an override structure is imposed, its three elements constitute a quaternion. For the octonion structures we have considered, these also contain a quaternion, and this still applies if a nonstandard override is applied. Thus the n-novanions all contain quaternion subalgebras.

The implications of the existence of algebras of novanion type for the classification of Lie algebras and of simple groups is addressed in [Ad18]. ☐

Finally in this section we mention that, just as quaternions may be given globally bosonic or globally fermionic structures, as in chapter I, section 8, so too can octonions and n-novanions. The analogous situation in the novanion algebra is that combined bosonic and fermionic structures may reside in the same n-novanion. ☐

18.5 The García classification problem.

Doly García has given a factorisation of 10-novnions so that

$$(i + i' + i'')(j + j' - 2j'') = 0.$$

It is clear from symmetry properties of 10-novanions that we also have

$$(k + k' + k'')(j + j' - 2j'') = 0,$$

and therefore

$$(i + i' + i'' + k + k' + k'')(j + j' - 2j'') = 0,$$

and for similar reasons

$$(i + i' + i'')(j + j' - 2j'' + k + k' - 2k'') = 0.$$

For n-novanions a classification problem is to find all factorisations of novanions such that for real a_k, b_k, the nonscalar basis elements i_k give

$$(a_1i_1 + \dots a_ni_n)(b_1i_1 + \dots b_ni_n) = 0.$$

In novanion physics these may be the generators of the universe at time t = 0. ☐