Chapter XV
Yang-Mills theory [Ze11]

15.1 Introduction.

As a student at Kunming and Chicago, Yang was impressed by the fact that gauge invariance determined all electromagnetic interactions, and he tried to generalise the concept of gauge invariance to nonabelian groups. In analogy with Maxwell's equations he tried

$$F_{ab} = \partial_a A_b - \partial_b A_a,$$

where A_a are matrices (a, b = 0, 1, 2, 3). As Yang pointed out later on "This led to a mess, and I had to give up".

15.2 The Yang-Mills equation.

In 1954, as a visiting physicist at Brookhaven National Laboratory on Long Island, New York, Yang returned once again to the idea of generalised gauge invariance. His officemate was Robert Mills. They decided to add a quadratic term

$$F_{ab} = \partial_a A_b - \partial_b A_a + A_a A_b - A_b A_a, \qquad (1)$$

which cleared up the "mess" and led to a beautiful new field theory.

15.3 Equivalence of the Levi-Civita form and the Yang-Mills form.

The fundamental equations for the components of the Riemann curvature tensor proved in chapter XIV are

$$R^d_{abc} = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac}. \qquad (1)$$

This is the generalisation of Gauss's *wonderful theorem* to higher dimensions.

The Yang-Mills field identities 15.2.(1) are identical to 15.3.(1) with the Lie product

$$[A_a, A_b] = A_a A_b - A_b A_a. \qquad (2)$$

To do this, set

$$A_a = \Gamma^d_{ac}$$

$$F_{ab} = R^d_{abc},$$

where the upper index d numbers the rows, and the lower index c numbers the columns. This implies for the Christoffel matrices, or connection matrices

$$A_a = \begin{bmatrix} \Gamma^0_{a0} & \Gamma^0_{a1} & \Gamma^0_{a2} & \Gamma^0_{a3} \\ \Gamma^1_{a0} & \Gamma^1_{a1} & \Gamma^1_{a2} & \Gamma^1_{a3} \\ \Gamma^2_{a0} & \Gamma^2_{a1} & \Gamma^2_{a2} & \Gamma^2_{a3} \\ \Gamma^3_{a0} & \Gamma^3_{a1} & \Gamma^3_{a2} & \Gamma^3_{a3} \end{bmatrix}, \qquad (3)$$

and for the Riemann curvature matrices

$$F_{ab} = \begin{bmatrix} R^0_{ab0} & R^0_{ab1} & R^0_{ab2} & R^0_{ab3} \\ R^1_{ab0} & R^1_{ab1} & R^1_{ab2} & R^1_{ab3} \\ R^2_{ab0} & R^2_{ab1} & R^2_{ab2} & R^2_{ab3} \\ R^3_{ab0} & R^3_{ab1} & R^3_{ab2} & R^3_{ab3} \end{bmatrix}, \qquad (4)$$

so that matrix multiplication gives the equivalence of 15.2.(1) and 15.3.(1).