Chapter XIV
Geometry

14.1 Introduction.

14.2 The Whitney embedding theorem.

An n-dimensional Pythagoras theorem is given by

$$x_0^2 + x_1^2 + ... + x_{n-1}^2 = s^2,$$

where under coordinate transformations, s^2 is an invariant. A generalisation of this is to consider

$$g_{00}x_0^2 + g_{11}x_1^2 + ... + g_{n-1,n-1}x_{n-1}^2 = s^2.$$

a further generalisation involving more terms is the manifold

$$g_{00}x_0^2 + g_{01}x_0x_1 + g_{10}x_1x_0 + g_{11}x_1^2 + ... + g_{n-1,n-1}x_{n-1}^2 = s^2. \quad (1)$$

In what follows we assume $g_{ij} = g_{ji}$. The g_{ij} are known as gauges, so we are considering here commutative, or symmetric, gauges.

A Riemannian manifold looks like (1) in small pieces. Instead of writing the differential dx_k we will write dx^k. Then this manifold looks locally like

$$\sum_{i \geq j}^{n} g_{ij}dx^i dx^j = s^2. \quad (2)$$

A notation adopted in physics is the Einstein summation convention, which drops the summation sign in (2), so it becomes

$$g_{ij}dx^i dx^j = s^2.$$

If i = j we take this convention to mean specially

$$g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + ... + g_{n-1,n-1}(dx^{n-1})^2 = s^2. \quad (3)$$

For n = 4, there are four g_{ii} and g_{01}, g_{02}, g_{03}, g_{12}, g_{13} and g_{23}, which is six other gauges, making 10 in all. For a general n, there are n g_{ii} and

$$(n – 1) + (n – 2) + ... + 1 = ½n(n – 1)$$

other gauges, the sum being for an arithmetic series, making

$$½n(2 + (n – 1)) = n(n + 1)/2$$

gauges in all.

The Riemannian manifold may be considered as an n-dimensional sheet. If the n(n + 1)/2 gauges are linearly independent variables then this n-dimensional curved manifold may be embedded in a flat manifold where the gauges are represented in a space along n(n + 1)/2 flat coordinates. We can put a distance metric on this flat manifold, say a Pythagoras theorem. What we have constructed here is a Whitney embedding of a Riemannian manifold in a flat manifold. If the Riemannian manifold is differentiable, then so is its Whitney embedding. ☐

The Whitney embedding assumes that the local structure defined by (2) can be embedded uniquely in its global continuation, but this may not be so. A circle in a given global manifold reaches its same point after a path with a rotation of 2π radians, but from a local to a global continuation this may not match the given global manifold and there may be no reconnection of the path.

For instance, if a circle is modelled as a rotation of a vector by an angle θ in two coordinates x and y, then this may be mapped to x, y, θz for a third coordinate z where z is ignored in computing local distances in the x, y plane. In general a circle may reconnect after a number n of traversals, or not at all. ☐

14.3 Gauss's *wonderful theorem*.

When we consider a two dimensional Riemann surface with symmetric gauges, the curvature of this manifold can be measured at a point P on the surface by a circle touching the tangent at P, where the radius of the circle measures the magnitude of the curvature. This radius is called the radius of curvature. It is a remarkable theorem due to Gauss that the maximum and minimum values of this curvature correspond to circles at right angles to one another.

14.4 The Levi-Civita connection.