Chapter XII
Continuity and conformality

12.1 Introduction.

The significance of conformal, or angle preserving, structures is that they define stable structures and appear as a selection principle which determines a feature of quantum mechanical systems, namely the Feynman propagator and amplituhedron propagator, which can be used for instance in quantum electrodynamic calculations. The novanion structures which we wish to apply to a universal physics also have the property of being conformal.

Holomorphic functions are the central objects of study in complex analysis. A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighbourhood of every point in its domain. The existence of a complex derivative in a neighbourhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal to its own Taylor series.

The term analytic function is often used interchangeably with holomorphic function, although the word “analytic” is also used in a broader sense to describe any function (real, complex, or of more general type) that can be written as a convergent power series in a neighbourhood of each point in its domain. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.

A holomorphic function whose domain is the whole complex plane is called an entire function. The phrase “holomorphic at a point z_0” means not just differentiable at z_0, but differentiable everywhere within some neighbourhood of z_0 in the complex plane.

In what follows we extend the meaning of these terms to matrix algebras and beyond. We will deviate from common usage by reserving the word conformal to these types of function in the case where they are angle-preserving. Indeed we show in section 4 that these analytic functions are not in general conformal.

12.2 The intricate analytic Cauchy-Riemann equations. [Ad17]

We now describe the analytic, or holomorphic, intricate Cauchy-Riemann equations.

Suppose that

$$P(g) = w(g) + x(g)i + y(g)\alpha + z(g)\phi \tag{1}$$

is a function of an intricate number, g ∊ Я. The *intricate derivative* of P at a point g_0 is defined by

$$\lim_{\substack{h \to 0 \\ h \in Я}} \frac{P(g_0 + h) - P(g_0)}{h}, \tag{2}$$

wherever this limit is path independent of h to 0. Alternatively, to obtain this derivative, we can regard h as a (hyper)infinitesimal number and discard (hyper)infinitesimal parts of the division performed in (2). Note that when an intricate numerator is divided by the intricate denominator h, the result depends on whether the reciprocal h^{-1} is multiplied first or last, so we assume two types of derivative. When h comes first we write $\lim_{h \to 0}$, otherwise the limit is from the right, denoted by $\lim_{0 \leftarrow h}$.

A standard argument indicates for this left limit to be unique, intricate polynomials satisfy the condition of being intricate left analytic, that is we have

$$P(w, x, y, z) = P(w + xi + y\alpha + z\phi, 0, 0, 0). \tag{3}$$

If the limit (2) exists, it can be computed by taking the limit h → 0 along the real, imaginary, actual or phantom axes. In all cases it should give the same result. Approaching along the real axis, we find for the limit on the left

$$\lim_{\substack{h \to 0 \\ h \in U}} \frac{P(g_0 + h) - P(g_0)}{h} = \frac{\partial P(g_0)}{\partial w}, \tag{4}$$

whereas along the imaginary axis

$$\lim_{\substack{h \to 0 \\ h \in U}} \frac{P(g_0 + hi) - P(g_0)}{hi} = \frac{1}{i}\frac{\partial P(g_0)}{\partial x}, \tag{5}$$

along the actual axis

$$\lim_{\substack{h \to 0 \\ h \in U}} \frac{P(g_0 + h\alpha) - P(g_0)}{h\alpha} = \frac{1}{\alpha}\frac{\partial P(g_0)}{\partial y}, \tag{6}$$

and for the phantom axis

$$\lim_{\substack{h \to 0 \\ h \in U}} \frac{P(g_0 + h\phi) - P(g_0)}{h\phi} = \frac{1}{\phi}\frac{\partial P(g_0)}{\partial z}. \tag{7}$$

Then to be intricate left analytic the equality of the results (4) to (7) gives

$$\frac{\partial P}{\partial x} = i\frac{\partial P}{\partial w} \tag{8}$$

$$\frac{\partial P}{\partial y} = \alpha\frac{\partial P}{\partial w} \tag{9}$$

$$\frac{\partial P}{\partial z} = \phi\frac{\partial P}{\partial w}, \tag{10}$$

which are the 3 left intricate Cauchy–Riemann equations at the point g_0. ☐

If we change the basis by means of continuous first derivatives to P = r + si + tα + uϕ, then

$$\frac{\partial P}{\partial w} = \frac{\partial r}{\partial w} + i\frac{\partial s}{\partial w} + \alpha\frac{\partial t}{\partial w} + \phi\frac{\partial u}{\partial w}$$

$$\frac{\partial P}{\partial x} = \frac{\partial r}{\partial x} + i\frac{\partial s}{\partial x} + \alpha\frac{\partial t}{\partial x} + \phi\frac{\partial u}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial r}{\partial y} + i\frac{\partial s}{\partial y} + \alpha\frac{\partial t}{\partial y} + \phi\frac{\partial u}{\partial y}$$

$$\frac{\partial P}{\partial z} = \frac{\partial r}{\partial z} + i\frac{\partial s}{\partial z} + \alpha\frac{\partial t}{\partial z} + \phi\frac{\partial u}{\partial z},$$

giving, using equations (8) to (10), on multiplying on the left

$$\frac{\partial P}{\partial w} = \frac{\partial r}{\partial w} + i\frac{\partial s}{\partial w} + \alpha\frac{\partial t}{\partial w} + \phi\frac{\partial u}{\partial w}$$

$$-\frac{\partial P}{\partial w} = i\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} - \phi\frac{\partial t}{\partial x} + \alpha\frac{\partial u}{\partial x}$$

$$\frac{\partial P}{\partial w} = \alpha\frac{\partial r}{\partial y} + \phi\frac{\partial s}{\partial y} + \frac{\partial t}{\partial y} + i\frac{\partial u}{\partial y}$$

$$\frac{\partial P}{\partial w} = \phi\frac{\partial r}{\partial z} - \alpha\frac{\partial s}{\partial z} - i\frac{\partial t}{\partial z} + \frac{\partial u}{\partial z},$$

so that the intricate Cauchy-Riemann equations become, on equating intricate parts

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial z} \tag{11}$$

$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x} = \frac{\partial u}{\partial y} = -\frac{\partial t}{\partial z} \tag{12}$$

$$\frac{\partial t}{\partial w} = -\frac{\partial u}{\partial x} = \frac{\partial r}{\partial y} = -\frac{\partial s}{\partial z} \tag{13}$$

$$\frac{\partial u}{\partial w} = \frac{\partial t}{\partial x} = \frac{\partial s}{\partial y} = \frac{\partial r}{\partial z}, \tag{14}$$

the number of these latter equations being 16.

We represent the Jacobian matrix of equations (11) to (14) by

$$\begin{bmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [r \quad s \quad t \quad u] = \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ c & d & a & b \\ d & -c & -b & a \end{bmatrix}, \qquad (15)$$

or as the 2-hyperintricate

$$a1_1 + b1_i + c\phi_\alpha + d\phi_\phi. \qquad (16)$$

Conversely, if P : Я → Я is a differentiable function when regarded as a function on $\mathbb{U}^4$, then P is a left intricate-valued real-differentiable function if and only if the 3 left intricate-differential Cauchy-Riemann equations hold. ☐

From equation (11)

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x},$$

then if the second differential exists

$$\frac{\partial^2 r}{\partial w^2} = \frac{\partial^2 s}{\partial w \partial x},$$

and from equation (12)

$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x},$$

so

$$-\frac{\partial^2 r}{\partial x^2} = \frac{\partial^2 s}{\partial w \partial x},$$

giving

$$\frac{\partial^2 r}{\partial w^2} + \frac{\partial^2 r}{\partial x^2} = 0. \qquad (17)$$

Further relations obtained in a similar way are

$$\frac{\partial^2 r}{\partial w^2} - \frac{\partial^2 r}{\partial y^2} = 0, \qquad (18)$$

$$\frac{\partial^2 r}{\partial w^2} - \frac{\partial^2 r}{\partial z^2} = 0, \qquad (19)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial x^2} = 0, \qquad (20)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial y^2} = 0, \qquad (21)$$

$$\frac{\partial^2 s}{\partial w^2} - \frac{\partial^2 s}{\partial z^2} = 0, \qquad (22)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial x^2} = 0, \qquad (23)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial y^2} = 0, \qquad (24)$$

$$\frac{\partial^2 t}{\partial w^2} - \frac{\partial^2 t}{\partial z^2} = 0, \qquad (25)$$

$$\frac{\partial^2 u}{\partial w^2} + \frac{\partial^2 u}{\partial x^2} = 0, \qquad (26)$$

$$\frac{\partial^2 u}{\partial w^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad (27)$$

and

$$\frac{\partial^2 u}{\partial w^2} - \frac{\partial^2 u}{\partial z^2} = 0. \ \square \qquad (28)$$

A function with continuous second derivative satisfying (17) or (26) is said to be *harmonic*.

The remaining relations satisfy the *wave equation*.

For the right limit to be unique, intricate polynomials P' satisfy the condition of being intricate right analytic, that is by taking the limit 0 ← h along the real, imaginary, actual or phantom axes, on approaching along the real axis we find for the limit on the right

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h) - P'(g_0)}{h} = \frac{\partial P'(g_0)}{\partial w}, \qquad (29)$$

whereas along the imaginary axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + hi) - P'(g_0)}{hi} = \frac{\partial P'(g_0)}{\partial x}\frac{1}{i}, \qquad (30)$$

along the actual axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h\alpha) - P'(g_0)}{h\alpha} = \frac{\partial P'(g_0)}{\partial y}\frac{1}{\alpha}, \qquad (31)$$

and for the phantom axis

$$\lim_{\substack{0 \leftarrow h \\ h \in U}} \frac{P'(g_0 + h\phi) - P'(g_0)}{h\phi} = \frac{\partial P'(g_0)}{\partial z}\frac{1}{\phi}. \qquad (32)$$

Then to be intricate right analytic the equality of the results (29) to (32) gives

$$\frac{\partial P'}{\partial x} = \frac{\partial P'}{\partial w}i \qquad (33)$$

$$\frac{\partial P'}{\partial y} = \frac{\partial P'}{\partial w}\alpha \qquad (34)$$

$$\frac{\partial P'}{\partial z} = \frac{\partial P'}{\partial w}\phi, \qquad (35)$$

which are the 3 right intricate Cauchy–Riemann equations at the point g_0. □

If we multiply on the right we obtain

$$\frac{\partial P}{\partial w} = \frac{\partial r}{\partial w} + i\frac{\partial s}{\partial w} + \alpha\frac{\partial t}{\partial w} + \phi\frac{\partial u}{\partial w}$$

$$-\frac{\partial P}{\partial w} = i\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} + \phi\frac{\partial t}{\partial x} - \alpha\frac{\partial u}{\partial x}$$

$$\frac{\partial P}{\partial w} = \alpha\frac{\partial r}{\partial y} - \phi\frac{\partial s}{\partial y} + \frac{\partial t}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial P}{\partial w} = \phi\frac{\partial r}{\partial z} + \alpha\frac{\partial s}{\partial z} + i\frac{\partial t}{\partial z} + \frac{\partial u}{\partial z},$$

so that the intricate Cauchy-Riemann equations now become, on equating intricate parts

$$\frac{\partial r}{\partial w} = \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial z} \qquad (36)$$

$$\frac{\partial s}{\partial w} = -\frac{\partial r}{\partial x} = -\frac{\partial u}{\partial y} = \frac{\partial t}{\partial z} \qquad (37)$$

$$\frac{\partial t}{\partial w} = \frac{\partial u}{\partial x} = \frac{\partial r}{\partial y} = \frac{\partial s}{\partial z} \qquad (38)$$

$$\frac{\partial u}{\partial w} = -\frac{\partial t}{\partial x} = -\frac{\partial s}{\partial y} = \frac{\partial r}{\partial z}, \qquad (39)$$

We represent the Jacobian matrix of equations (36) to (39) as

$$\begin{bmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [r \quad s \quad t \quad u] = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}, \qquad (40)$$

where the 2-hyperintricate

$$a1_1 + b\alpha_i + c\phi_1 + di_i, \qquad (41)$$

has been obtained from equation (40). □

More generally, an intricate polynomial P″ may be allocated any combination of the following intricate mixed analytic expressions

$$\frac{\partial P''}{\partial x} = i\frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w}i, \qquad (42)$$

$$\frac{\partial P''}{\partial y} = \alpha\frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w}\alpha, \qquad (43)$$

$$\frac{\partial P''}{\partial z} = \phi\frac{\partial P''}{\partial w} \text{ or } \frac{\partial P''}{\partial w}\phi. \qquad (44)$$

Thus the structure of these transformations differs depending on whether multiplication occurs on the left or on the right.□

12.3 J-diffeomorphisms and non-fixed J.

P may be expressed for intricate numbers in J format as

$$P = r + Jv, \qquad (1)$$

It may or may not be the case that J is constant. For intricate numbers constant J implies J-abelian.

When a diffeomorphism is applied to $J = si + t\alpha + u\phi \rightarrow J + \delta J$ so that $J^2 = (J + \delta J)^2$, where $J^2 = 0$ or ± 1, then

$$s = t(\partial t/\partial s) + u(\partial u/\partial s). \text{ □} \qquad (2)$$

12.4 The intricate Cauchy-Riemann equations are nonconformal.

If we look at the Cauchy–Riemann equations 12.2.(11) to (14) written in 12.2.(15) as a 2-hyperintricate Jacobian matrix, and likewise for the Jacobian matrix 12.2.(40), a matrix of this form contains within it components of non-complex type (is not hyperimaginary), that is the matrix is not composed of layers containing only 1 and i. Geometrically, multiplying a complex number by a complex number is always the composition of a rotation with a scaling, and in particular preserves angles, but a non-complex matrix does not. Expressed more simply, for a complex number

$$e^\rho e^{i\theta}.e^{i\theta'} = e^\rho e^{i(\theta + \theta')} = e^{\rho + i(\theta + \theta')}, \qquad (1)$$

and this is the core idea of what we are saying. For a general intricate number (see *Superexponential algebra*, chapter XV, section 3), the intricate $e^{p + (bi + c\alpha + d\phi)K}$ cannot usually be expressed as $e^w e^{xi} e^{y\alpha} e^{z\phi}$ with p = w, bK = x, cK = y and dK = z.

However, if two (hyper-)intricate numbers are J-abelian with the same $J^2 = -1$, then

$$e^\rho e^{J\theta}.e^{J\theta'} = e^\rho e^{J(\theta + \theta')} = e^{\rho + J(\theta + \theta')}, \qquad (2)$$

and the transformation is now conformal.

Consequently, a function satisfying the intricate Cauchy–Riemann equations, with a nonzero derivative does not in general preserve the angle between curves in the plane, so we say the intricate Cauchy–Riemann equations contain already the property for a function not to be conformal. □

12.5 The novanion analytic Cauchy-Riemann equations.

We wish to have a symbol designed for putting commas between ordered sets of coefficients of hyperintricate basis elements.

For $0 < k \leq m \in \mathbb{N}$ define

$$(,)_{k=1}^{m} w_k$$

to be (w_1) when $m = 1$, (w_1, w_2) when $m = 2$ and to have the properties for $m' \leq m$

$$(,)_{k'=1}^{m'} w_{k'} = (,)_{k=1}^{m'-k'+1} w_{k'}, \qquad (1)$$

$$(,)_{k=1}^{m} w_k = [(,)_{k=1}^{m'} w_k][(,)_{m'+1}^{m} w_k]. \qquad (2)$$

Let an n-hyperintricate number be denoted by $Я_n$, with coefficients of basis elements w_k. We define the hyperintricate analytic condition for a function P corresponding to 12.2.(3) to be

$$P[(,)_{k=1}^{4^n} w_k] = P[Я_n, (,)_{k=1}^{4^n-1} 0]. \qquad (3)$$

The derivative of hyperintricate numbers satisfies the same equation as 12.2.(2). Then if γ_j is a basis element with coefficient $w_{\gamma j}$, the equations corresponding to 12.2.(8)-(10) are

$$\frac{\partial P}{\partial w} = \gamma_j^{-1} \frac{\partial P}{\partial w_{\gamma j}}. \qquad (4)$$

Set

$$P = \sum_{k=1}^{4^n} \gamma_k r_k, \qquad (5)$$

so that if we change basis then

$$\frac{\partial P}{\partial w_{\gamma j}} = \sum_{k=1}^{4^n} \gamma_k \frac{\partial r_k}{\partial w_{\gamma j}}, \qquad (6)$$

and multiplying this on the left by γ_j^{-1}, using (4)

$$\frac{\partial P}{\partial w} = \gamma_j^{-1} \sum_{k=1}^{4^n} \gamma_k \frac{\partial r_k}{\partial w_{\gamma j}}. \qquad (7)$$

For multiplying on the left, let $<\gamma_j^{-1}, \gamma_k>_L$ equal the sign of $\gamma_j^{-1}\gamma_k$ in that order when $\gamma_j^{-1} = \pm\gamma_k$, and otherwise be 0, so that the hyperintricate left derivative Cauchy-Riemann equations become, on equating hyperintricate parts

$$<\gamma_{j'}^{-1}, \gamma_k>_L[<\gamma_j^{-1}, \gamma_k>_L \frac{\partial r_k}{\partial w_{\gamma j}}] = <\gamma_{j'}^{-1}, \gamma_k>_L[<\gamma_{j'}^{-1}, \gamma_{k'}>_L \frac{\partial r_{k'}}{\partial w_{\gamma j'}}]. \qquad (8)$$

For multiplying on the right, a similar situation obtains, on defining $<\gamma_j^{-1}, \gamma_k>_R$ equal to the sign of $\gamma_k\gamma_j^{-1}$ (in reversed order from previously) when $\gamma_j^{-1} = \pm\gamma_k$, otherwise 0. The right derivative hyperintricate Cauchy-Riemann equations are obtained by substituting $< , >_L$ in (8) with $< , >_R$. □

As we have seen, the quaternions are an instance of hyperintricate numbers, to which these considerations apply. The proof that quaternion analytic Cauchy-Riemann equations are conformal and the extension of this idea to the octonions and n-novanions is now discussed.

12.6 The nonconformal split representation.

For an intricate number

$$H = a1 + bi + c\alpha + d\phi$$

this can be represented as

$$H = (a1 + bi) + \alpha(c1 + di).$$

For hyperimaginary numbers an example might be

$$H' = (a1_1 + bi_i) + (c1_i + di_1),$$

where we have collected together with terms in the first brackets with an even number of i's in their hyperintricate representation, and the second brackets containing an odd number of i's. We might now expand this example to contain a general hyperintricate number, for example

$$H'' = [a1_1 + b\alpha_\alpha](c1_1 + di_i) + [e1_1 + f\alpha_\alpha](g1_i + hi_1) + [j\alpha_1 + k1_\alpha](m1_1 + ni_i) + [p\alpha_1 + q1_\alpha](r1_i + si_1). \quad (1)$$

We will use the representations of type (1) to describe conformal components of Cauchy-Riemann type and corresponding nonconformal components.

12.7 Nonconformal algebras and self-annihilation.

12.8 Novanion algebra is conformal.

12.9 The complex Cauchy integral formula.