Chapter XI
Novanion rings

11.1 Introduction.

In this chapter we continue to provide general structures for theories in physics.

J. F. Adams proved in 1960 that the only possible division algebras are at maximum those of the octonions, which are nonassociative, where quaternions and complex numbers are subalgebras of these. However, there exist nonassociative algebras, the n-novanions, with nonzero division, but when just the scalar part is zero two nonzero novanions may have a zero product. I obtain here the proof for n = 10, and extend it for n greater than 10. Novanions have a dimension of $n = 1 + 3^f \prod_{i \in \mathbb{N}} (3^{g_i} - 2)_i$, for which $f \in \mathbb{N}_{\cup 0}$, $g_i \in \mathbb{N}$. Thus the n-novanions comprise a new type of quasi division algebra. We then introduce novanion rings. These do not have division, and thus can represent discrete structures.

11.2 The nonassociative octonion division algebra.

A nonassociative division algebra, D, drops the multiplicative commutative rule and drops also the multiplicative associative rule. It may introduce the following axiom:

There do not exist $a, b \neq 0 \in D$ with

$$ab = 0. \qquad (1)$$

This rule is unnecessary for associative division algebras, since if $1 \neq 0$ then (1) implies

$$ab \neq c,\ c \neq 0,$$

and therefore using the associative rule there exists no

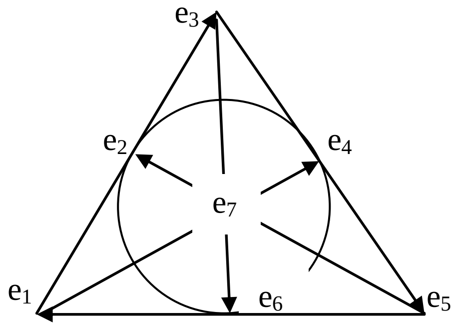
$$a(b/c) = 1,$$

but b/c can be chosen to be the multiplicative inverse of a, a contradiction. ☐

The octonions form a nonassocative division algebra, so in general

$$(AB)C \neq A(BC).$$

The algebra of the octonions, also called the Cayley numbers, was discovered by J. Graves in 1843, and is given by the Fano plane for their basis elements 1, e_1, e_2, e_3, e_4, e_5, e_6 and e_7 in the figure below.



Typical cyclic identities are

$$e_7e_5 = e_2 \qquad (2)$$
$$e_5e_6 = e_1$$

and

$$e_4e_2 = e_6.$$

Note what we have said here. The inner triple $e_2e_4e_6$ acts like a quaternion, but the outer triple $e_1e_3e_5$ does not. Nevertheless, we will need to allocate later a list ordered as right triple ($e_2e_4e_6$) + central triple ($e_1e_3e_5$) + one (e_7). Octonions form a division algebra, in particular

$$e_c^2 = -1, \quad (3)$$

$$e_ae_b = -e_be_a \ (a \neq b),$$

and the inverse of

$$a1 + \sum_{n=1}^{7} b_n e_n$$

is

$$(a1 - \sum_{n=1}^{7} b_n e_n)/(a^2 + \sum_{n=1}^{7} b_n^2). \quad (4)$$

The octonions, $\mathbb{O}$, are also generated by the Cayley-Dickson construction [Ba01]. This builds up algebras from the complex numbers, to the quaternions, to the octonions, to the sixteen dimensional sedenions, etc.

Define a T-algebra to be an algebra equipped with conjugation, a linear map T satisfying

$$a^{TT} = a, \quad (5)$$

$$(ab)^T = b^Ta^T. \quad (6)$$

Starting from any T-algebra, the Cayley-Dickson construction gives a new algebra

$$(a, b)(c, d) = (ac - db^T, ad^T + cb), \quad (7)$$

with conjugation defined by

$$(a, b)^T = (a^T, -b). \quad (8)$$

Let us look at how this works in practice. The complex numbers have multiplication

$$(a, b)(c, d) = (ac - bd, ad + bc).$$

The complex conjugate

$$(a, b)^T = (a^T, -b) = (a, -b),$$

since the scalar part a is a real number with $a^T = a$.

The complex conjugate satisfies

$$(a, b)^T(a, b) = (aa + bb, ab - ba) = (a^2 + b^2, 0).$$

We define $+\sqrt{a^2 + b^2}$ as the *norm* of the complex number. It is a non-negative real number.

For the noncommutative quaternions equations (7) and (8) hold. The order of the terms in (7) is important. We now have

$$(a, b)^T(a, b) = (a^T, -b)(a, b) = (a^Ta + b^Tb, ba^T - ba^T)$$

$$= (a^Ta + b^Tb, 0),$$

where $+\sqrt{a^Ta + b^Tb}$ is the norm.

The T-algebra generates the basis element multiplication table $\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$ of the octonions

×	***1***	***e₁***	***e₂***	***e₃***	***e₄***	***e₅***	***e₆***	***e₇***
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e_1}$	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
$\boldsymbol{e_4}$	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
$\boldsymbol{e_5}$	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
$\boldsymbol{e_6}$	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
$\boldsymbol{e_7}$	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

We can generate for each

$\mathbb{O} \times \mathbb{O} \to \mathbb{O}$: $e_i \times e_j \to e_k$

a Cayley-Dickson construction of

$e_i \times (-e_j) \to -e_k$,

so that each of the 7 non-scalar basis elements in a row of the table can be multiplied by -1 to provide $2^7 = 128$ possible Cayley-Dickson constructions.

The Fano plane has 7 non-scalar basis elements. The number of non-scalar quaternionic triplets is 7, each of which, since exquaternions are excluded, operates under a forward or a reversed orientation – again 2^7 possibilities. The following Fano triplets map bijectively to the standard Cayley-Dickson construction for the octonions

(e_1, e_2, e_3), (e_3, e_4, e_5), (e_1, e_4, e_6), (e_4, e_6, e_2), (e_1, e_7, e_6), (e_4, e_7, e_3), (e_5, e_7, e_2).

Thus distinct possibilities for the Fano plane match distinct instances of the Cayley-Dickson construction. □

11.3 Eigenvalues.

Our intention now is to introduce a generalisation of quaternions and octonions called n-novanions, which will be involved in the physics we wish to describe. In order to prove the consistency of the n-novanion algebra, we will need to introduce some further ideas on matrices. Like the octonions, the n-novanions cannot be described directly by matrices, since matrices are multiplicatively associative, meaning (AB)C = A(BC), but novanions are not. Nevertheless we can characterise some of the properties of n-novanions by a set of linear equations. These can be described by a matrix equation. An important technique in analysing the matrix equations is to map the equation to a scalar equation involving complex numbers. These scalars are called eigenvalues, and satisfy a matrix equation like (1) below.

Within the field of complex numbers $\mathbb{C}$, the complex conjugate of c = a + ib is c* = a – ib. For the corresponding matrix C with entries $c_{jk} = a_{jk} + ib_{jk}$, the conjugate $C^* = a_{jk} - ib_{jk}$. The transpose of a matrix C is denoted by C^T and has entries c_{kj}. The transpose is a contravariant (order reversing) operation:

$(CD)^T = D^TC^T$.

A matrix is defined as antisymmetric if $C^T = -C$.

The following proof is derived from chapter 11 of [Uh01].

Theorem 11.3.1. *All eigenvalues of a real antisymmetric matrix* $E = -E^T$ *are pure imaginary*.

Proof. Consider the case of the eigenvalue λ and possibly complex column vector called an eigenvector $\mathbf{x} \neq \mathbf{0}$. According to the Fundamental Theorem of Algebra $\lambda \in \mathbb{C}$. Hence

$$E\mathbf{x} = \lambda\mathbf{x}. \qquad (1)$$

If we take the complex conjugate of both sides of the eigenvalue-eigenvector equation (1), we obtain

$(E\mathbf{x})^* = (\lambda\mathbf{x})^* = \lambda^*\mathbf{x}^*$.

Transposing yields

$(E\mathbf{x})^{*T} = x^{*T}E^{*T} = \lambda^*\mathbf{x}^{*T}$.

Define the norm

$$||E\mathbf{x}||^2 = (\lambda^*)\lambda(\mathbf{x}^{*T})\mathbf{x}, \qquad (2)$$

where for the set of real numbers $\mathbb{R}$

$$||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}} \in \mathbb{R}.$$

Since $E^T = -E$ and $E^{*T} = E^T$ for a real antisymmetric matrix E, we can write (2) as

$$\begin{aligned} ||E\mathbf{x}||^2 &= \mathbf{x}^{*T}E^{*T}\,E^T\mathbf{x}, \\ &= \mathbf{x}^TE^TE\mathbf{x} \\ &= -\mathbf{x}^TE^2\mathbf{x} \\ &= -\mathbf{x}^T\lambda^2\mathbf{x}, \end{aligned}$$

because $E^2\mathbf{x} = E(E\mathbf{x}) = E(\lambda\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, so

$$||E\mathbf{x}||^2 = -\lambda^2\mathbf{x}^T\mathbf{x}. \qquad (3)$$

Now $\mathbf{x}^T\mathbf{x} \neq 0$, and thus $\lambda^*\lambda = -\lambda^2$, by comparing (2) and (3). Thus a real antisymmetric matrix $E = -E^T$ can only have imaginary eigenvalues λ. □

11.4 The 10-novanions.

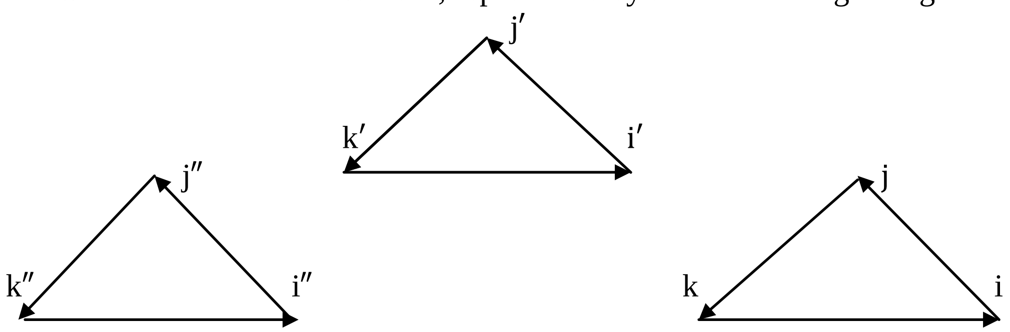
A novanion algebra B drops axiom 11.2.(1). It may substitute the following rule:

If a^2, $b^2 > 0$ for a, b ∈ B and a ∈ c, b ∈ d ∈ B, there do not exist any c, d with

$$cd = 0. \qquad (1)$$

We can provide existence proofs for these structures. The complex numbers constitute a field. The quaternions form an associative division algebra. The octonions form a nonassociative division algebra. As we will prove in this section the 10-novanions, for example, form a novanion algebra.

We now introduce the 10-novanions, represented by the set of triangle diagrams



where in general each triangle is a quaternion without 1.

The primed variables ( ), (′) and (″) act as holders of information concerning an algebra for them. When the variables all contain a common instance, for example (k), (k′) and (k″), then the algebra is that of the quaternions, in which we have a cyclic algebra

$$kk' = k'' = -\,k'k. \qquad (2)$$

When the variables contain different instances, such as k and i′, then the product contains the primed variable that does not belong to the first two elements, but the primed part commutes.

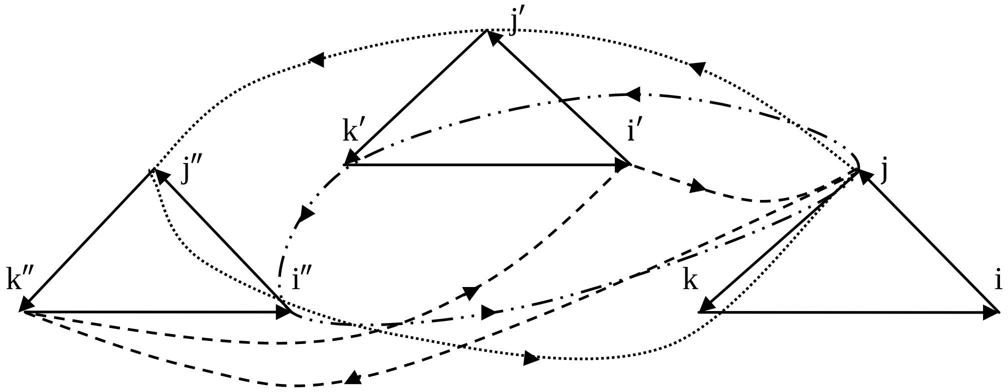
On top of this is the fact that the variables, say k and i, satisfy a quaternion algebra, so say

$$ki = j = -ik \qquad (3)$$

and consequently

$$ki' = j'' = -i'k. \qquad (4)$$

In order to picture the 10-novanions more closely, we will show the connections from node j



Our claim is that the inverse of

$$a1 + \sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} b_n^m e_n$$

is

$$(a1 - (\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} b_n^m e_n))/(a^2 + (\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} (b_n^m)^2)), \qquad (5)$$

and this constitutes a type of division algebra with no divisors of zero provided a1 ≠ 0 – the 10 dimensional 10-novanions.

We see that the n-novanions are nonassociative, since they have more than 4 basis elements; more explicitly

$$(j''k')j' = ij' = k'' \neq j''(k'j') = -j''i' = k.$$

We wish to enquire under what conditions there exist two 10-novanion numbers multiplied together giving zero:

$$(a1 + bi + cj + dk + b'i' + c'j' + d'k' + b''i'' + c''j'' + d''k'') \times$$
$$(p1 + qi + rj + tk + q'i' + r'j' + t'k' + q''i'' + r''j'' + t''k'') = 0. \qquad (6)$$

Their product is

real part:

$$ap - bq - cr - dt - b'q' - c'r' - d't' - b''q'' - c''r'' - d''t'' = 0, \qquad (7)$$

i part:

$$bp + aq - dr + ct - b''q' - d''r' + c''t' + b'q'' - d'r'' + c't'' = 0, \qquad (8)$$

j part:

$$cp + dq + ar - bt + d''q' - c''r' - b''t' + d'q'' + c'r'' - b't'' = 0, \qquad (9)$$

k part:

$$dp - cq + br + at - c''q' + b''r' - d''t' - c'q'' + b'r'' + d't'' = 0, \qquad (10)$$

i′ part:

$$b'p + b''q - d''r + c''t + aq' - d'r' + c't' - bq'' - dr'' + ct'' = 0, \qquad (11)$$

j′ part:

$$c'p + d''q + c''r - b''t + d'q' + ar' - b't' + dq'' - cr'' - bt'' = 0, \qquad (12)$$

k′ part:

$$d'p - c''q + b''r + d''t - c'q' + b'r' + at' - cq'' + br'' - dt'' = 0, \qquad (13)$$

i″ part:

$$b''p - b'q - d'r + c't + bq' - dr' + ct' + aq'' - d''r'' + c''t'' = 0, \qquad (14)$$

j″ part:

$$c''p + d'q - c'r - b't + dq' + cr' - bt' + d''q'' + ar'' - b''t'' = 0, \qquad (15)$$

k″ part:

$$d''p - c'q + b'r - d't - cq' + br' + dt' - c''q'' + b''r'' + at'' = 0. \qquad (16)$$

Alternative definition 11.4.1. D is a (possibly nonassociative) division algebra whenever for any element a in D and any nonzero element b in D there exists just one element x in D with a = bx and only one element y in D with a = yb.

If a = 0, the 10-novanions contain possibilities for two nonzero 10-novanions giving a product which is zero. We give an example due to Doly García, showing that the 10-novanions satisfying a = 0 do not form a division algebra of standard type

$$(i + i' + i'')(j + j' - 2j'') = 0. \qquad (17)$$

Thus the 10-novanions are not a division algebra given by the condition for equation 11.2.(1), nor do they satisfy definition 11.4.1 since for an arbitrary real number g

$$(i + i' + i'')[(j + j' - 2j'')g] = 0. \qquad (18)$$

From now on we will assume $a \neq 0$. By a symmetrical argument applied also to the following reasoning, we need to assume with this that $p \neq 0$.

Equations (7) to (16) form a matrix E + aI, where E is an antisymmetric matrix and I is the unit diagonal, multiplied on the right by the eigenvector (p, q, r, t, q′, r′, t′, q″, r″, t″). We have already given a proof that the eigenvalues of a real antisymmetric matrix are entirely imaginary, so these correspond to -a, which is real, whereas we are now excluding the only possibility for this, a = 0.

Thus 10-novanions form a novanion algebra satisfying the conditions of equation (1). ☐

11.5 n-novanions.

The octonions given by a Fano plane under suitable orientations may be given three copies in primed variables ( ) , (′) and (″), and by an analogous procedure this constitutes a 1 + (3 × 7) = 22 dimensional division algebra.

Extending these ideas further to multiple occurrences of the three or seven primed variables, say (′), (′)′ and (′)″, we obtain in general an $n = 1 + 3^f7^g$ dimensional novanion algebra, the n-novanions, in which if a common variable, k, is employed, then the lowest value within brackets of say (k), (k′)′ and (k″)″ is evaluated. ☐

11.6 Discreteness and novanion rings.

Since the n-novanions contain the quaternions or the octonions, and the octonions also contain quaternions, it follows that the arguments of chapter VI on Wedderburn's little theorem apply also to such subalgebras, and therefore

Theorem 11.9.1. *Any finite novanion division ring is commutative.*

Thus novanion rings cannot in general be finite, although they may be discrete. Discrete rings without division can be bounded from below by finite elements ignoring the sign, for example by zero, and otherwise be infinite. If a ring is bounded from above and below by the mapping of, say, plus infinity to the integer novanion part 1 and minus infinity to -1, then an infinite example can be given over real components.

If novanion rings are given a lower and an upper bound in each of their n coordinates, x_0, x_1, ... , x_{n-1} as $\pm a_0$, $\pm a_1$, ... , $\pm a_{n-1}$, then the pair $(+a_i, +a_j)$ and the pair $(-a_i, -a_j)$ can be glued in two ways as a_i to $-a_i$ and a_j to $-a_j$, or as a_i to $-a_j$ and a_j to $-a_i$. The first defines a global bosonic structure and the second a global fermionic structure.

We will label bosonic structures by b and fermionic structures by f. If we look at just the novanionic imaginary components of the novanion, then for say a quaternion, there are configurations {b, b, b}, {f, b, b}, {f, f, b} and {f, f, f}.