Chapter III
Wavelets

3.1 Introduction.

Given that physics is often reduced to the study of either particles or waves, it is appropriate that we now give a mathematical description of wave phenomenons. This will provide a general context for the study of the Schrödinger equation in quantum mechanics. In the conventional approach we often come across waves in terms of the exponential function applied to complex variables. Fourier series describing infinite waves are just a summation of such sine waves. Wavelets are waves which cut off outside the central distribution. The description of waves and wavelets given here introduces extensions to hyperintricate numbers and novanions from the idea of complex numbers, and also introduces nonstandard algebras for these, which we call $\mathbb{D}$w exponential algebras. In physics the idea of waves and wavelets can be combined with descriptions of exponentially increasing or decreasing processes, and can be extended further to include superexponential operations. Currently these higher order generalisations of exponentiation are not used to describe physical phenomenons.

We provide the mathematical basis for the description of these objects, firstly by developing the intricate formalism of chapter I to incorporate the J-abelian idea. The meaning of this is that 2 × 2 matrices are noncommutative ($AB \neq BA$) but when two intricate numbers share the same J-abelian structure then they commute between themselves. This idea can be extended to hyperintricate numbers, but not all hyperintricate numbers can be expressed in this way. We introduce a change of 'coordinate system' for intricate numbers. The basis i, α and ϕ can be changed to another basis by linear transformations of the original basis. This is called a $\mathcal{JAF}$ transformation.

We then consider a standard extension of the exponential idea to intricate matrices. This exponential algebra is not the only one possible, so we introduce $\mathbb{D}$w exponential algebras which contain the standard theory as a special case, and extend these algebras to novanions. This provides the setting for a discussion of generalised Fourier series and Fourier transforms that the reader could find is developed at greater length in [SS03], and also the wavelet idea.

3.2 Ordinal infinities, infinitesimals and ladder algebra.

Definition 3.2.1. $\mathbb{N}$ is the set of positive whole numbers, called *natural numbers*. They satisfy assumptions called the Peano axioms. If we wish to emphasise that $\mathbb{N}$ does not contain 0, we write $\mathbb{N}_{\neq 0}$, and the set $\mathbb{N}$ together with 0 is denoted by $\mathbb{N}_{\cup 0}$.

Definition 3.2.2. For given t, $\mathbb{M}_t$ is the set of *transnatural num*bers satisfying the Peano axioms, where $\mathbb{M}_1 = \mathbb{N}$, there is no bijection from $\mathbb{M}_t$ to $\mathbb{M}_{t'}$ if $t \neq t'$, and there does not exist an $\mathbb{M}'$ that satisfies $\mathbb{M}_t$ is properly included in $\mathbb{M}'$ and $\mathbb{M}'$ is properly included in $\mathbb{M}_{t+1}$.

Definition 3.2.3. The *ordinal infinity* $\boldsymbol{\Omega}$ is the number $\sum_{\text{all N}} 1$.

In what follows, $\boldsymbol{\Omega}$ is irreducible; it cannot be split up, rearranged or truncated. By definition it follows the axioms of a field given in chapter I, section 5.

Definition 3.2.4. A positive *infinity* is a number $w > 0$ such that for every $n \in \mathbb{N}_{\neq 0}$ there does not exist a $w < n$. Negative infinities can be defined as numbers $w' < 0$ with correspondingly no $w' > -n$.

Definition 3.2.5. A positive noninfinite (that is, finite) number u is *Eudoxus* whenever there is a natural number $n \in \mathbb{N}_{\neq 0}$ such that for any $m \in \mathbb{N}_{\neq 0}$, un > m. Negative Eudoxus numbers can be defined, reversing the > sign.

Definition 3.2.6. A positive *infinitesimal* is a number d such that there is no natural number $n \in \mathbb{N}_{\neq 0}$ such that for any $m \in \mathbb{N}_{\neq 0}$, dn > m. Negative infinitesimals can be defined.

As an example, $1/\boldsymbol{\Omega}$ is an infinitesimal.

Definition 3.3.7. $\boldsymbol{\Omega}^n$, n > 1 is a *hyperinfinity*, $\boldsymbol{\Omega}^{-n}$ is a *hyperinfinitesimal*.

Definition 3.2.8. A number $\ldots a_n\boldsymbol{\Omega}^n + a_{n-1}\boldsymbol{\Omega}^{n-1} + \ldots + a_0 + a_{-1}\boldsymbol{\Omega}^{-1} + \ldots$ is a *ladder number*.

A ladder number satisfies the axioms of a field.

3.3 J-abelian hyperintricate numbers. [Ad15]

An n-*hyperintricate number is* J-*abelian* *if* U, V, ... W *are intricate numbers for the layers of the* n-*hyperintricate number* $\Sigma U_{V\ldots W}$, *where for each layer the value of* J *is constant* (*but* J *can vary over different layers*), J *is not real and* $J^2 = 0$ *or* ±1.

The n-hyperintricate representation has 4^n independent components, but the number of independent components in a J-abelian n-hyperintricate number $U_{V\ldots W}$ is 4n, and this is maximally incremented for n > 2 by forming the sum $\Sigma U_{V\ldots W}$, there being c = 3n of the J components plus one scalar, plus $\sum_{r=0}^{n-1} \frac{n!}{r!(n-r)!}$ independent mixed J components.

3.4 The 𝒥𝒜ℱ basis for intricate numbers.

An intricate number p1 + qi + rα + sϕ = p1 + JK satisfies

$$(qi + r\alpha + s\phi)^2 = (\pm qi \pm r\alpha \pm s\phi)^2 = -q^2 + r^2 + s^2. \ \square$$

Composite basis elements are obtained from other basis elements using operators like + and × and satisfy the same formal properties as the original basis elements.

To determine composite basis elements obtained from addition, first set

$$J_1 = ui + v\alpha + w\phi, \qquad J_1^2 = -1 = -u^2 + v^2 + w^2, \qquad (1)$$

$$J_2 = xi + y\alpha + z\phi, \qquad J_2^2 = -1 = -x^2 + y^2 + z^2. \qquad (2)$$

Then for 1 ≥ L, M ≥ 0, if the denominator is positive, putting

$$J = (J_1L + J_2M)/\sqrt{[(uL + xM)^2 - (vL + yM)^2 - (wL + zM)^2]},$$

since iα = -αi, etc., J satisfies the same type of constraint,

$$J^2 = [J_1^2L^2 + (J_1J_2 + J_2J_1)LM + J_2^2M^2]/[(uL + xM)^2 - (vL + yM)^2 - (wL + zM)^2],$$

so on using (1) and (2)

$$J^2 = -1.$$

A mapping, also called a function, can be represented by a set of *ordered pairs*, {x, f(x)}. The set of x might be denoted by [0, 1], which is the set of all numbers between 0 and 1, with 0 and 1 included. A mapping is continuous if it is not discontinuous, so there are no gaps of at least a finite constant in adjacent values of f(x) as x is squeezed closer between two values.

If u > 0, x < 0, a continuous mapping

$$[L, M]: [1, 0] \overset{t}{\to} [0, 1] \qquad (3)$$

carries (uL + xM) impossibly through zero, which also happens in the complex case by setting v = w = y = z = 0 in equations (1) and (2), but if u and x are of the same sign, then the positive constraint on the denominator is unnecessary, because $-u^2 + v^2 + w^2 = -1$, $-x^2 + y^2 + z^2 = -1$, so $(uL + xM)^2 > (vL + yM)^2 + (wL + zM)^2$. □

Next, multiplicatively, let $\mathcal{J}^2 = -1$, $\mathcal{A}^2 = 1$ and $\mathcal{F}^2 = 1$, where we put

$\mathcal{J} = qi + r\alpha + s\phi$,
$\mathcal{A} = bi + c\alpha + d\phi$,
$\mathcal{F} = ei + f\alpha + g\phi$,

and we allocate

$$\mathcal{A}\mathcal{F} = \mathcal{J}. \qquad (4)$$

Since $\mathcal{J}$ does not have a real part, it follows from the relations

-be + cf + dg = 0, (real part)
cg – df = q, (i part)
bg – de = r (α part)

and

-bf + ce = s (ϕ part)

that

$$\mathcal{A}\mathcal{F} = -\mathcal{F}\mathcal{A} = \mathcal{J}. \qquad (5)$$

Multiplying (4) on the left by $\mathcal{A}$

$\mathcal{F} = \mathcal{A}\mathcal{J}$,

and multiplying on the right by $\mathcal{F}$

$\mathcal{A} = \mathcal{J}\mathcal{F}$.

Correspondingly, multiplying (5) on the right by $\mathcal{A}$ and the left by $\mathcal{F}$ gives

$\mathcal{F} = -\mathcal{J}\mathcal{A}$,
$\mathcal{A} = -\mathcal{F}\mathcal{J}$,

and we have established an equivalence of algebras for

$\mathcal{J} \leftrightarrow i$,
$\mathcal{A} \leftrightarrow \alpha$

and

$\mathcal{F} \leftrightarrow \phi$. □

3.5 Standard exponential algebras.

The definition of the exponential function

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots \qquad (1)$$

means that we can set

$$\cos\theta = \frac{e^{-i\theta} + e^{i\theta}}{2}, \qquad (2)$$

$$\sin\theta = i\frac{e^{-i\theta} - e^{i\theta}}{2}, \qquad (3)$$

$$\cosh\theta = \frac{e^{-\theta} + e^{\theta}}{2}, \qquad (4)$$

$$\sinh\theta = \frac{-e^{-\theta} + e^{\theta}}{2}, \qquad (5)$$

so that

$$\cos^2\theta + \sin^2\theta = 1, \qquad (6)$$

$$\cosh^2\theta - \sinh^2\theta = 1. \qquad (7)$$

An alternative definition of $\sin\theta$ as the negative of (3) changes the handedness of the coordinate system.

Complex numbers satisfy the *Euler relation* with positive determinant

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad (8)$$

which can be obtained from formulas (2) and (3) or using a Taylor series expansion.

For intricate basis elements α and ϕ, we might think a similar argument gives

$$e^{\alpha\theta} = \cosh\theta + \alpha\sinh\theta, \qquad (9)$$

$$e^{\phi\theta} = \cosh\theta + \phi\sinh\theta. \qquad (10)$$

However, we will see for instance that if

$$e^{\alpha\theta} = \alpha(\cosh\theta + \alpha\sinh\theta), \qquad (11)$$

so that, in effect, we are defining

$$e^{\alpha\lambda} = \alpha(1 + \lambda + \lambda^2/2 + \lambda^3/3! + \ldots), \qquad (12)$$

then in the case of equation (12), equation (7) still holds.

The determinant of an intricate number $a1 + bi + c\alpha + d\phi$ is

$$(a1 + bi + c\alpha + d\phi)(a1 - bi - c\alpha - d\phi) = a^2 + b^2 - c^2 - d^2. \qquad (13)$$

But using (9) and an intricate number $e^{\alpha\lambda}$ multiplied by its complex conjugate $e^{-\alpha\lambda}$ is $e^{\alpha\lambda - \alpha\lambda} = e^0 = 1$, which is positive, and cannot represent (13) in the case considered when $b = d = 0$, and also $a^2 < c^2$. This forces us either to admit that there exist values of intricate numbers not represented by exponentials, or to adopt equations like (11) and (12). □

3.6 $\mathbb{D}$w exponential algebras.

A natural generalisation we adopt is an analogy with multiplication of exponentiation which we represent by $\uparrow$, nested on the left, and define a type of $\uparrow$ distributivity, with the exponential constant e, as follows:

$$[e\uparrow(ti + u\alpha + v\phi)]\uparrow(xi + y\alpha + z\phi) =$$
$$e\uparrow[t(x + y + z)]i + [u(x + y + z)]\alpha + [v(x + y + z)]\phi. \qquad (1)$$

This noncommutative 'ring with unit' under + and × operations is explicit enough to generate all the relations we need. There is no division operation for $\uparrow$, because if a real value is absent, $\uparrow$ cannot generate 1. The hyperintricate exponential algebra becomes available through this structure by the means described later. □

An important question is now whether it is decidable that the above suggestion is consistent. All binomial type exponential operations are generated and defined via

$$[e^{p1 + (qi + r\alpha + s\phi)K}]\uparrow(a1 + bi + c\alpha + d\phi),$$

so we conclude the operation $\uparrow$ is as consistent as rings in general. □

The equation (1) is not $\mathcal{J}\mathcal{A}\mathcal{F}$ invariant. If we transform $i \to \mathcal{J}$, $\alpha \to \mathcal{A}$ and $\phi \to \mathcal{F}$, so that

$$ti + u\alpha + v\phi = t'\mathcal{J} + u'\mathcal{A} + v'\mathcal{F}$$

and

$$xi + y\alpha + z\phi = x'\mathcal{J} + y'\mathcal{A} + z'\mathcal{F}, \qquad (2)$$

then if the right hand side of (1) remains invariant,

$$ti + u\alpha + v\phi(x + y + z) = (t'\mathcal{J} + u'\mathcal{A} + v'\mathcal{F})(x' + y' + z') \qquad (3)$$

so we have

$$x + y + z = x' + y' + z', \qquad (4)$$

but taking the intricate conjugate of (2)

$$-x^2 + y^2 + z^2 = -x'^2 + y'^2 + z'^2,$$

for which (4) by the binomial theorem does not hold for

$$x = 1, y = z = 0,$$

$$x' = \sqrt{(1 - y'^2 - z'^2)},$$

with x′, y′ and z′ positive. □

J-abelian hyperintricate exponentiation is the natural extension of the intricate case. For example for 4 x 4 matrices we have 16 basis elements, 8 of which can be put in one combined J-abelian representation.

Since the $\mathcal{JAF}$ ring transformations map $i \to \mathcal{J}$, $\alpha \to \mathcal{A}$ and $\phi \to \mathcal{F}$, but do not provide an invariant description of the $\mathbb{D}1$ exponential and superexponential algebras that we will give, there is a spectrum of possible $\mathbb{D}1$ algebras which may be chosen, and we will present the $\mathbb{D}1$ exponential algebra spectrally, under the understanding that some specific $\mathcal{JAF}$ basis must be chosen throughout to maintain consistency.

For a hyperintricate number defined in chapter I, section 7, on iterating on k layers using a $\veebar_k$ operator

$$\sum_j [\veebar_k(a_{kj} + b_{kj}J_{kj})] = \sum_j [\veebar_k e\uparrow(\rho_{kj} + \sigma_{kj}J_{kj})],$$

where $J_{kj}^2 = 0$ or ±1, we evaluate

$$[e\uparrow(\rho + \sigma J)]\uparrow(\rho' + \sigma' J')$$

as the lower $\mathbb{D}1$ exponential algebra expression

$$e\uparrow(\rho\rho' + \sigma\rho' J + \sigma'\rho J' + \sigma\sigma' J),$$

and the Σ terms by a binomial expansion under this rule. Note that the last term for the exponentiated sum is $\sigma\sigma' J$ and not $\sigma\sigma' JJ'$ – this is the specific difference between the $\mathbb{D}1$ exponential algebra, which is minimally branched, and the standard exponential algebra.

As a $\mathcal{JAF}$ ring we select the *lower expansion*

$$e^v = [e\uparrow(r + s\mathcal{J} + t\mathcal{A} + u\mathcal{F})]\uparrow(r' + s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}'),$$

where

$$v = rr' + r(s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}') + r'(s\mathcal{J} + t\mathcal{A} + u\mathcal{F})$$
$$+ (ss' + st' + su')\mathcal{J} + (ts' + tt' + tu')\mathcal{A} + (us' + ut' + uu')\mathcal{F}. \qquad (5)$$

This is related to an *upper expansion* which is different from (5), where we choose

$$v = rr' + r(s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}') + r'(s\mathcal{J} + t\mathcal{A} + u\mathcal{F})$$
$$+ (ss' + ts' + us')\mathcal{J}' + (st' + tt' + ut')\mathcal{A}' + (su' + tu' + uu')\mathcal{F}'. \;\square \qquad (6)$$

If we denote the exponential operator by ↑, then for J-abelian 2-hyperintricate numbers with selected intricate basis element components x, y and z, we define

$$(x_y)\uparrow z = (x\uparrow z)_{(y\uparrow z)},$$

with natural extensions to the J-abelian n-hyperintricate case. This structure is the $\mathbb{D}1$ hyperintricate exponential algebra. □

For lower expansion $\mathbb{D}$w exponential algebras, for w, w′ and w″ intricate numbers which could have integer coefficients, we adopt the following extended definition of these algebras

$$[e^{\mathcal{J}s}]^{\mathcal{J}} = [e^{\mathcal{J}s}]^{\mathcal{A}} = [e^{\mathcal{J}s}]^{\mathcal{F}} = e^{\mathcal{J}ws},$$
$$[e^{\mathcal{A}s}]^{\mathcal{J}} = [e^{\mathcal{A}s}]^{\mathcal{A}} = [e^{\mathcal{A}s}]^{\mathcal{F}} = e^{\mathcal{A}w's},$$
$$[e^{\mathcal{F}s}]^{\mathcal{J}} = [e^{\mathcal{F}s}]^{\mathcal{A}} = [e^{\mathcal{F}s}]^{\mathcal{F}} = e^{\mathcal{F}w''s}, \qquad (7)$$

and related extensions in the J-abelian case. ☐

3.7 Novanion exponential algebras.

We will meet novanions in detail in chapter XI, but for now we will extract a feature of these algebras and show how exponential algebras can be applied to them. The n-novanions can be represented by

$$a1 + b_1i_1 + b_2i_2 + ... + b_{n-1}i_{n-1}, \qquad (1)$$

where a and the b_k are scalars (they commute with other elements), and 1, i_1, ..., i_{n-1} are basis elements like the basis of a vector space, of which the scalars are components, but the algebra of the vectors is not the usual one.

Novanions are noncommutative, so for novanions A, B and C, in general AB ≠ BA, and they are nonassociative, so in general A(BC) ≠ (AB)C. The scalar parts commute with the i_k, and the novanion parts are anticommutative, in particular

$$i_ji_k = - i_ki_j$$

for j ≠ k, but

$$i_ki_k = -1.$$

Then there is a scalar norm obtained by multiplying

$$\left(a + \sum_{k=1}^{n-1} b_k\, i_k\right)\left(a - \sum_{k=1}^{n-1} b_k\, i_k\right) = a^2 + \sum_{k=1}^{n-1}(b_k)^2,$$

where we will put

$$\sum_{k=1}^{n-1}(b_k)^2 = K^2. \qquad (2)$$

Then if we write

$$J = \sum_{k=1}^{n-1} \frac{b_k}{K} i_k, \qquad (3)$$

expression (1) becomes JK, and we can consider by analogy with the Euler relation for complex numbers

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

the novanion exponential equation

$$e^{a+JK} = e^a(\cos K + J\sin K). \qquad (4)$$

The novanion $\mathbb{D}$w exponential algebra satisfies an analogue of 3.6.(7), namely for scalar s and novanion w

$$(e^{Js}) = (e^J)^s = (e^s)^J,$$
$$(e^{Js})^J = e^{Jws}.$$

3.8 Fourier series.

Fourier series are used to describe vibrating strings, travelling waves and heat flow. The states of a system show what is there, but for practical applications transformations of states can describe what is observed. Our approach here is the logical one of dealing with the states of the system first, even if this leads to a delay in describing practical examples as mappings.

In what follows it would be possible to replace everywhere the complex exponential equation

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

by its counterpart for novanions, the novanion exponential equation of 3.7.(4)

$$e^{JK} = \cos K + J\sin K.$$

A function is *periodic* if

$$F(\theta + u) = F(\theta)$$

for some u. So for example $F(\theta + 2u) = F(\theta + u) = F(\theta)$.

An example of a periodic function is

$$G(\theta) = e^{im\theta},$$

where

$$G(\theta + 2\pi) = G(\theta).$$

There is a mapping between periodic functions and functions on the unit circle, given by say a real function $h(\theta)$ with

$$h(\theta) = H(e^{i\vartheta}),$$

for some other suitable function H, so that $h(\theta)$ is periodic with period 2π.

Suppose we wish to describe the state of an arbitrary periodic complex function by

$$f(\theta) = \sum_{m=-\Omega}^{\Omega} a_m e^{im\theta},\ m \in \mathbb{M}_t, \qquad (1)$$

where Ω is ordinal infinity in $\mathbb{M}_t$ (the summation then means over all $\pm m \in \mathbb{M}_t$) and the a_m are complex. We see this is periodic with period 2π and this period can be changed by adjusting the values of the $e^{im\theta}$ to $e^{im\theta/L}$ for L real. We suspend discussion of the convergence of (1) until later. Using 3.5.(2) and 3.5.(3) an alternative description is

$$f(\theta) = \sum_{m=-\Omega}^{\Omega} (b_m \cos m\theta + c_m \sin m\theta) \qquad (2)$$

for complex b_m and c_m. Since an arbitrary function f on $[\pi, -\pi]$ can be represented in terms of the even parity function

$$g(\theta) = \frac{f(\theta) + f(-\theta)}{2}$$

and the odd parity function

$$h(\theta) = \frac{f(\theta) - f(-\theta)}{2},$$

and the cos function is even and sin is odd, we can write this as

$$f(\theta) = \sum_{m=0}^{\Omega} B_m \cos m\theta + \sum_{m=1}^{\Omega} C_m \sin m\theta \qquad (3)$$

for complex B_m and C_m. The expansions (1) and (3) are known as *Fourier series*.

If f is an integrable function on the interval [a, b] of length L, so L = b – a, then formally we define the Fourier series of f for period L as

$$f(\theta) = \sum_{m=-\Omega}^{\Omega} a_m e^{2\pi im\theta/L},\ m \in \mathbb{M}_t. \qquad (4)$$

We now observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\theta}d\theta = 0 \text{ if } n \neq m, \text{ and } 1 \text{ if } n = m \qquad (5)$$

so we find for period $L = 2\pi$

$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-im\theta}d\theta. \qquad (6)$$

The quantity a_m is called the m^{th} *Fourier coefficient* of f.

The m^{th} Fourier coefficient of f for period L is then

$$a_m = \frac{1}{L}\int_{a}^{b} f(\theta)e^{-2\pi im\theta/L}d\theta. \qquad (7)$$

Let $N \in \mathbb{M}_t$. We define the N^{th} *partial sum* of a Fourier series

$$S(N, \theta) = \sum_{m=-N}^{N} a_m e^{2\pi i m\theta/L}, \tag{8}$$

so that $S(\Omega, \theta)$ is the Fourier series of $f(\theta)$, thought of as its limiting value.

We define the N^{th} *Dirichlet kernel* as

$$D_N(\theta) = \sum_{m=-N}^{N} e^{im\theta}, \tag{9}$$

so this is obtained by setting $a_m = 1$ in the partial sum for period 2π.

Theorem 3.8.1. *The Dirichlet kernel has value*

$$\frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)}.$$

Proof. Let $\omega = e^{i\theta}$. Equation (9) can be written as the sum of two geometric series in ω.

$$D_N(\theta) = \sum_{m=0}^{N} \omega^m + \sum_{m=-N}^{-1} \omega^m,$$

so using the formula for sums of geometric series this is

$$\frac{1-\omega^{N+1}}{1-\omega} - \frac{1-\omega^{-N}}{1-\omega} = \frac{\omega^{-N} - \omega^{N+1}}{1-\omega} = \frac{\omega^{-N-1/2} - \omega^{N+1/2}}{\omega^{-1/2} - \omega^{1/2}},$$

which by the definition of the sine function in equation 3.5.(3), is just

$$\frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)}. \square$$

3.9 Convolutions.

The notion of a convolution is an important feature of Fourier analysis, and we will use it in later developments. Roughly speaking, convolutions are 'weighted averages'.

In what follows we change the variable of integration and adopt the standard notation

$$\hat{f}(m) = a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)e^{-im\mu}d\mu. \tag{1}$$

The partial sum of equation 3.8.(8) with $L = 2\pi$ now satisfies

$$\begin{aligned} S(N, \theta) &= \sum_{m=-N}^{N} \hat{f}(m)e^{im\theta} \\ &= \sum_{m=-N}^{N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)e^{-im\mu}d\mu\right) e^{im\theta} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)\left(\sum_{m=-N}^{N} e^{im(\theta-\mu)}\right)d\mu \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)\left(D_N(\theta-\mu)\right)d\mu. \square \end{aligned} \tag{2}$$

To generalise equation (2) we introduce the convolution.

Definition 3.9.1. *Given two* 2π*-periodic continuous real functions* f *and* g, *the convolution* $f * g$ *on* $[-\pi, \pi]$ *is*

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)g(\theta - \mu)d\mu. \tag{3}$$

Thus we may think of $f(\mu)$ as belonging to a 'weighted average' of $g(\theta - \mu)$.

Equation (2) for the partial sum may now be represented using the convolution of f and the Dirichlet kernel D_N as

$$S(N, \theta) = (f * D_N)(\theta). \square \tag{4}$$

Theorem 3.9.2. *Given three* 2π*-periodic continuous real functions* f, g *and* h, *the convolution on* $[-\pi, \pi]$ *satisfies*

(i) $f * (g + h) = (f * g) + (f * h)$,

(ii) For any complex c, $(cf) * g = f * (cg)$,

(iii) $f * g = g * f$,
(iv) $f * (g * h) = (f * g) * h$,
(v) $f * g$ is continuous,
(vi) $\widehat{f * g} = \hat{f} * \hat{g}$.

Proof. Properties (i) and (ii) follow from the linearity of the convolution integral. □

Property (iii) on commutativity follows because the functions f and g are both periodic over the same interval. We can see if we interchange f and g that

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\mu)f(\theta - \mu)d\mu = (g * f)(\theta), \tag{5}$$

since if $F(\mu) = f(\mu)g(\theta - \mu)$ is continuous and periodic in this interval then

$$\int_{-\pi}^{\pi} F(\mu)d\mu = \int_{-\pi}^{\pi} F(\theta - \mu)d\mu \tag{6}$$

for any real θ. Then the result follows on changing variables $\mu \to -\mu$ which reverses the order of integration but leaves the integral fixed, and translating $\mu \to \mu - \theta$ which keeps the periodic nature of F. □

The proof of associativity in (iv) depends on swapping two integrals, together with changes of variables like (6). □

To prove (v), that $f * g$ is continuous, if f differs from $f + \delta$ by an infinitesimal and g differs from $g + \delta'$ by an infinitesimal, then from (i) and by the Eudoxus nature of $f * g$,

$$(f + \delta) * (g + \delta') = f * g + f * \delta' + \delta * g + \delta * \delta'$$

differs from it by an infinitesimal. □

Proposition (vi) is proved by evaluating

$$\begin{aligned}
\widehat{f * g}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f * g)(\theta)e^{-im\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)g(\theta - \mu)d\mu\, e^{-im\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)e^{-im\mu}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta - \mu)e^{-im(\theta - \mu)}\, d\theta\right)d\mu \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mu)e^{-im\mu}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-im\theta}\, d\theta\right)d\mu \\
&= \hat{f}(m) * \hat{g}(m). \quad \square
\end{aligned}$$

Remark 3.9.3. Theorem 3.9.2 holds not only when f, g and h are continuous, but also when they are integrable but not necessarily continuous. It uses the following lemma.

Lemma 3.9.4. *Let* f *be an either Eudoxus or infinitesimal integrable function on the circle. Then there exists a sequence of continuous functions* f_k *on the circle so that* $|f_k|$ *is Eudoxus or infinitesimal for all* k *and*

$$\int_{-\pi}^{\pi}|f(\mu) - f_{\Omega}(\mu)|\, d\mu$$

is infinitesimal or zero.

For a proof, see the appendix of [SS03]. This condition is important because we will deal with discrete structures and we need to know that the mathematics we will be applying is appropriate for them.

Indeed, there is an interpretation of discrete integrable structures in which the quantum system for one periodic state is distributed discretely about a number of continuous states, so that the one quantum state can be considered as a probability distribution of many separate continuous states.

3.10 Good kernels, the Cesàro mean and Fejér's theorem.

Good kernels are a type of infinitely peaked weight distribution on a circle. We define the three properties of good kernels and give their meaning. Using convolutions we then show how these kernels can be employed to recover a specified function. Dirichlet kernels are not good kernels, but we will see that the Fegér kernels derived from them are good kernels.

Definition 3.10.1. If it exists, the *limit* L, with no infinitesimal part, of a function f is found from the value of f, say as a series, whenever (L – f) is infinitesimal or zero.

Definition 3.10.2. A family of kernels $\{K_m(\mu)\}_{m=1}^{\Omega}$ on the circle are *good kernels* whenever they satisfy the properties:

(i) For every m > 0,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_m(\mu)d\mu = 1.$$

This property says the kernel has unit mass around the circle.

(ii) For every m > 0 and absolute value $|K_m(\mu)|$ there exists an N > 0 satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|K_m(\mu)|d\mu \leq N,$$

so when $K_m(\mu) > 0$, property (ii) reduces to (i).

(iii) The *Dirac* δ *function* property holds. For every Eudoxus δ > 0

$$\int_{\delta\leq|\mu|}^{\pi}|K_\Omega(\mu)|d\mu \text{ is infinitesimal.}$$

Good kernels are related to convolutions as follows.

Theorem 3.10.3. (**good kernels as identity approximations**). *Let* $\{K_m(\mu)\}_{m=1}^{\Omega}$ *be a family of good kernels. Let* f *be an integrable function continuous at* μ. *Then* $(f * K_\Omega)(\mu)$ *differs from* $f(\mu)$ *by at most an infinitesimal*.

What we are implying above is that the convolution

$$(f * K_m)(\mu) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - \mu)K_m(\mu)\, d\mu$$

has an average of $f(\theta - \mu)$ weighted by $K_m(\mu)$, and in the limit m → ***Ω*** its mass is infinitely concentrated at $\mu = 0$.

Proof. Let δ > 0 be Eudoxus. Then by property (i) of good kernels

$$(f * K_\Omega)(\mu) - f(\mu) = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - \mu)K_\Omega(\mu)\, d\mu\right] - f(\mu)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta - \mu) - f(\mu)]K_\Omega(\mu)\, d\mu,$$

so that

$$|(f * K_\Omega)(\mu) - f(\mu)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta - \mu) - f(\mu)||K_\Omega(\mu)|\, d\mu,$$
$$= \frac{1}{2\pi}\int_{|\mu|<\delta}^{\pi}|f(\theta - \mu) - f(\mu)||K_\Omega(\mu)|\, d\mu$$
$$+ \frac{1}{2\pi}\int_{\delta\leq|\mu|\leq\pi}^{\pi}|f(\theta - \mu) - f(\mu)||K_\Omega(\mu)|\, d\mu. \quad (1)$$

By lemma 3.9.4 and properties (ii) and (iii) of good kernels these two terms on the right of (1) are at most infinitesimals. □

Under arbitrary bracketing an infinite series of complex numbers $s = \sum_{k=0}^{\Omega-1} c_k$, where $c_k \in \mathbb{C}$, may have ambiguous values. We will assume it is evaluated unambiguously as

$s = (...((c_0 + c_1) + c_2) + ...$

We define the N^{th} partial sum as the finite series

$$s_N = \sum_{k=0}^{N-1} c_k.$$

If we now look at the infinite series

$$s = 1 - 1 + 1 - 1 + ..., \tag{2}$$

its partial sums are the ordered set, or sequence, {1, 0, 1, 0, ...}, so that we can form its average

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

In general this average value is known as the N^{th} *Cesàro mean* of s. The limit as N → $\boldsymbol{\Omega}$ is the *Cesàro sum*, σ.

As can be seen from example (2) where $\sigma = \frac{1}{2}$, Cesàro summability can be applied to a wider class of cases than convergent series.

We will now apply Cesàro summability to Fourier series, forming the N^{th} *Fejér kernel* $F_N(\theta)$ as a Cesàro mean of Dirichlet kernels.

$$F_N(\theta) = \frac{D_0(\theta) + D_1(\theta) + \cdots + D_{N-1}(\theta)}{N}.$$

Theorem 3.10.4. *The Fejér kernel has value*

$$F_N(\theta) = \frac{1}{N}\frac{\sin^2(N\theta/2)}{\sin^2(\theta/2)}.$$

Proof. We have seen in theorem 3.8.1 that putting $\omega = e^{i\theta}$ the Dirichlet kernel has value

$$D_N(\theta) = \frac{\omega^{-N} - \omega^{N+1}}{1 - \omega}.$$

Thus

$$D_0(\theta) + D_1(\theta) + \cdots + D_{N-1}(\theta) = \frac{\omega^0 - \omega^1 + \omega^{-1} - \omega^2 + \ldots + \omega^{-N} - \omega^{N+1}}{1 - \omega}, \tag{3}$$

so using

$$\frac{(1-\omega)^2}{\omega} = \omega^{-1} + \omega - 2,$$

and on multiplying the numerator and denominator of (3) by $(1 - \omega)$, from the definition of the sine function in 3.5.(3) we get

$$F_N(\theta) = \frac{1}{N}\frac{\omega^{-N-1} + \omega^{N+1} - 2}{\omega^{-1} + \omega - 2} = \frac{1}{N}\frac{\cos^2(N\theta/2) - 1}{\cos^2(\theta/2) - 1} = \frac{1}{N}\frac{\sin^2(N\theta/2)}{\sin^2(\theta/2)}. \ \square \tag{4}$$

Theorem 3.10.5. *The Fejér kernel is a good kernel*.

Proof. Properties (i), (ii) and (iii) of definition 3.10.2 are obtained from the properties of the Dirichlet kernel, also (ii) from the fact that (4) is positive and (iii) that $1/N\sin^2(\theta/2)$ is an infinitesimal for N = $\boldsymbol{\Omega}$ and $0 < \theta \leq \pi$. □

3.11 The Fourier transform.

We now develop a theory analogous to Fourier series except it is defined on the entire real line and not the circle. The functions are now generally nonperiodic.

The Fourier transform is a continuous version of the Fourier coefficients. We will replace discrete integers and sums of section 3.10 by their continuous analogues.

Let us generalise 3.9.(1) by replacing $m \in \mathbb{Z}$ by the real number $r \in \mathbb{R}$, so that the integral extends over the whole of $\mathbb{R}$

$$\hat{f}(r) = \int_{-\Omega}^{\Omega} f(\mu)\, e^{-2\pi i\mu r} d\mu, \quad (1)$$

where 3.8.(4) ceases to be a series and becomes an integral, giving the Fourier inversion formula

$$f(\mu) = \int_{-\Omega}^{\Omega} \hat{f}(r)\, e^{2\pi i\mu r} dr. \quad (2)$$

In order that formula (1) makes sense, we will need its value to be Eudoxus or infinitesimal. For instance, (2) is convergent if $f(\mu) = 1/(1 + |\mu|)$. To generalise slightly we introduce the following idea.

Definition 3.11.1. A function $f(\mu) \in \mathbb{R}$ is of *moderate decrease* if it is continuous and there is a constant $K \in \mathbb{R}$ with $K > 0$ such that for every $\mu \in \mathbb{R}$

$$|f(\mu)| \le K/(1 + \mu^2). \quad (3)$$

The set of functions of moderate decrease is denoted by $M(\mathbb{R})$.

Definition 3.11.2. If $f \in M(\mathbb{R})$ then (1) is called the *Fourier transform* of f.

Then equation (1) is well-defined because $\left|e^{-2\pi i\mu r}\right| = 1$, but there is no guarantee that $\hat{f}(r)$ is of moderate decrease, or therefore that (2) is well-defined. To remedy this, as Schwartz has pointed out, we have to introduce in some equivalent way the idea of a 'Schwartz space'.

Definition 3.11.3. A *Schwartz space* $S(\mathbb{R})$ consists of infinitely differentiable functions f, f', f'', ... $f'^{(n)}$, ... which are *rapidly decreasing*, meaning

$$|r|^m \left|f'^{(n)}(r)\right|$$

is finite for every m, $n \geq 0$.

Theorem 3.11.4. *The Schwartz space is closed under differentiation and multiplication by polynomials.*

Proof. The functions $f'(r) \in S(\mathbb{R})$ and $rf(r) \in S(\mathbb{R})$. □

Non example 3.11.5. Although $e^{-|r|}$ decreases rapidly at infinity it does not belong to $S(\mathbb{R})$, since its derivative is not continuous or differentiable at zero.

Example 3.11.6. The Gaussian

$$g(r) = e^{-r^2}$$

belongs to a Schwartz space.

Definition 3.11.7. The Fourier transform of a function f on the Schwartz space $S(\mathbb{R})$ is

$$\hat{f}(r) = \int_{-\Omega}^{\Omega} f(\mu)\, e^{-2\pi i\mu r} d\mu.$$

We will see that, except for a factor of $2\pi i$, the Fourier transform of a function f has the important property that it swaps differentiation with multiplication of $f(\mu)$ by μ.

Theorem 3.11.8. *If* $f \in S(\mathbb{R})$ *then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0 \in \mathbb{R}$

(i) $\hat{f}(r)e^{2\pi i r\tau}$ *is the Fourier transform of* $f(\mu + \tau)$,

(ii) $\hat{f}(r + \tau)$ *is the Fourier transform of* $f(\mu) = e^{-2\pi i\mu\tau}$,

(iii) $\varepsilon^{-1}\hat{f}(\varepsilon^{-1}r)$ *is the Fourier transform of* $f(\varepsilon\mu)$,

(iv) $2\pi i r\hat{f}(r)$ *is the Fourier transform of* $f'(\mu)$,

(v) $\frac{d\hat{f}(r)}{dr}$ *is the Fourier transform of* $-2\pi i\mu f(\mu)$.

Proof. (i) holds because

$$\int_{-\Omega}^{\Omega} f(\mu + \tau)\, e^{-2\pi i\mu r} d\mu = \int_{-\Omega}^{\Omega} f(\mu)\, e^{-2\pi i\mu r} d\mu\,.$$

(ii) is a restatement of the definition of a Fourier transform.

Property (iii) on scaling under dilations follows from property (ii) and that

$$\varepsilon \int_{-\Omega}^{\Omega} f(\varepsilon\mu)\, d\mu = \int_{-\varepsilon\Omega}^{\varepsilon\Omega} f(\mu)\, d\mu = \int_{-\Omega}^{\Omega} f(\mu)\, d\mu.$$

The interchange of differentiation and multiplication properties (iv) and (v) follow for integer Ω on integrating by parts

$$\int_{-\Omega}^{\Omega} f'(\mu) e^{-2\pi i\mu r}\, d\mu = \left[\int_{-\Omega}^{\Omega} f(\mu) e^{-2\pi i\mu r}\right]_{-\Omega}^{\Omega} + \int_{-\Omega}^{\Omega} f(\mu) e^{-2\pi i\mu r}\, d\mu$$

for (iv), and for (v) on determining the convergence of

$$\frac{\hat{f}(r+\tau) - \hat{f}(r)}{\tau} - \left(\widehat{-2\pi i\mu f}\right)(r) = \int_{-\Omega}^{\Omega} f(\mu) e^{-2\pi i\mu r} \left[\frac{e^{-2\pi i\mu\tau} - 1}{\tau} + 2\pi i\mu\right] d\mu. \ \square$$

Theorem 3.11.9. *If* $f \in S(\mathbb{R})$ *then* $\hat{f} \in S(\mathbb{R})$.

Proof. As we have seen, this follows since except for a factor of $2\pi i$, the Fourier transform of a function f has the property that it swaps differentiation with multiplication of $f(\mu)$ by μ, and this implies that if f is bounded, so is $\hat{f}$. To prove that every nth differential of $\hat{f}$ is bounded in the same way that f is, note for natural numbers g and h that the expression

$$r^g \left(\frac{d}{dr}\right)^h \hat{f}(r)$$

is bounded, because by property 3.11.8 (v) this is the Fourier transform of

$$\frac{1}{(2\pi i)^g} \left(\frac{d}{d\mu}\right)^g (-2\pi i\mu)^h f(r),$$

and thus $\hat{f}$ is in a Schwartz space. □

3.12 Gaussians as good kernels.

The function

$$g(x) = e^{-ax^2}$$

for values a > 0 we have seen is the Gaussian at a = 1. These functions belong to a Schwartz space. We will now consider the case a = π, which we will also call a Gaussian.

Lemma 3.12.1. $\int_{-\Omega}^{\Omega} e^{-\pi x^2}\, dx = 1$.

Proof. Using polar coordinates r, θ

$$\begin{aligned}\left(\int_{-\Omega}^{\Omega} e^{-\pi x^2}\, dx\right)^2 &= \int_{-\Omega}^{\Omega}\int_{-\Omega}^{\Omega} e^{-\pi(x^2+y^2)} dxdy \\ &= \int_0^{2\pi}\int_0^{\Omega} e^{-\pi r^2} r dr d\theta \\ &= \int_0^{\Omega} 2\pi r e^{-\pi r^2} dr = \left[-e^{-\pi r^2}\right]_0^{\Omega} = 1. \ \square\end{aligned}$$

Theorem 3.12.2. *The Gaussian* $f(x) = e^{-\pi x^2}$ *is its own Fourier transform*.

Proof. We need to show that if $f(x) = e^{-\pi x^2}$, then $f(\mu) = \hat{f}(\mu)$. But

$$\hat{f}(\mu) = \int_{-\Omega}^{\Omega} e^{-\pi x^2} e^{-2\pi i x\mu} dx. \tag{1}$$

By property (v) of theorem 3.11.8, $\frac{d\hat{f}(x)}{dx}$ is the Fourier transform of $-2\pi i\mu f(\mu) = i f'(\mu)$, so

$$\hat{f}'(\mu) = \int_{-\Omega}^{\Omega}(-2\pi i x) f(x)\, e^{-2\pi i x\mu} dx = i\int_{-\Omega}^{\Omega} f'(x)\, e^{-2\pi i x\mu} dx. \tag{2}$$

By property (iv) of the same theorem and equation (2), the Fourier transform

$$\hat{f}'(\mu) = i(2\pi i\mu)\hat{f}(\mu) = -2\pi\mu\hat{f}(\mu). \tag{3}$$

If we form $\hat{f}(\mu)e^{\pi\mu^2}$, its derivative by (3) is $\hat{f}'(\mu)e^{\pi\mu^2} + 2\pi\mu\hat{f}(\mu)e^{\pi\mu^2} = 0$. Thus $\hat{f}(\mu)e^{\pi\mu^2}$ is a constant we will show is 1. By equation (1) and lemma 3.12.1, for $\mu = 0$, $\hat{f}(0) = 1$, which now implies the result we need

$$\hat{f}(\mu) = e^{-\pi\mu^2}. \square$$

3.13 Fourier inversion.

3.14 The Plancherel formula.

3.15 The Poisson summation formula.

3.16 Wavelets.

Arising from the periodic nature of the trigonometric functions, the Fourier transform has no localisation property, because it is defined over an infinite interval. Wavelets introduce a new transform for the frequency analysis of signals, which are time-dependent functions. They are locally defined ‘little waves’, with a zero mean value.