Chapter I
Models and representations

1.1 Introduction.

In this mathematical chapter we provide general structures for theories in physics linking models to observations. We introduce quaternions as a four dimensional and noncommutative type of complex number. We give an instance of such a structure for the special theory of relativity, which deals with pairs of coordinate systems moving with uniform velocity with respect to each other, linking the line element in special relativity with its apparently non equivalent analogue as the norm for quaternions. We thus establish a mapping from the quaternion norm, which is in the mathematical theory, to the line element as an observable. We then achieve the same type of mapping from quaternions themselves to objects describing the equation for the relativistic theory of the election. In this model for special relativity, time is described in the observational part as a real variable, and space in the mathematical part is transformed to space in the observational part by specifying that space for the observer is always quaternion imaginary.

We include in this chapter a discussion of fields in the mathematical sense, but not in the terminology of physics, and of vector spaces. There are two types of vector space, vector spaces with base point, and vector spaces without base point, known as affine vector spaces. The latter are part of the philosophy of relativity theory. We discuss matrices too.

In section 9 we will relate the mathematical and observational structures for special relativity with our reasoning, and discover that we have introduced a new relativistic transformation so that time and space variables are independent. We will develop the comparison of this theory with experiment in the chapter which follows.

Section 11 looks at how local space-time globalises. The physics is discussed in chapter X.

1.2 Models and observations.

A mathematical model consists of functions also known as mappings from the mathematics to experimental observations, the latter of which we include two types: on the one hand by human observers and on the other using mechanical apparatus which could vary from complicated machinery down to, say, a single electron. We will precisely describe both the mathematics of the set of states and transformations of the model, which we will intuitively identify, in the words of John Bell [Be87], as the *beables* of the theory, which if we have got the theory right indicate what is actually there, and the set of mappings from the beables to *observables*, which are what can be observed in experiments. These two sets need not be the same. Together with the model and the observables, the pair of which forms the meaning of the physics, a theory will often include a philosophical interpretation and restriction of a way of thinking about these.

1.3 Special relativistic models.

Special relativity ought to be compatible with the more general theory known as general relativity, describing acceleration of objects in a gravitational field. General relativity is based on a description of a space with time and distance, the non-acceleration part of which can be described by a *line element*, similar to a four dimensional Pythagoras theorem but of the form

$$s^2 = c^2t^2 - x^2 - y^2 - z^2, \qquad (1)$$

where we will interpret c as the speed of light, and in a four dimensional coordinate system t is a measurement of time, x a coordinate of distance, and similarly y and z are distance coordinates each mutually at right angles to the other space coordinates.

For the moment, we will interpret (1) as holding within it an essential feature of special relativity. The reader may feel that if a type of mathematics can describe consistently a four dimensional Pythagoras theorem as a natural extension of a physically constructible three dimensional one, then it should be possible to develop a type of mathematics in which (1) holds, and this will be as consistent as the Pythagoras theorem, if developed properly.

We will say that equation (1) is an invariant, meaning that if we have two coordinate systems or frames, ct, x, y, z and ct′, x′ y′, z′, then they both satisfy (1), that is

$$s^2 = c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2. \qquad (2)$$

This means that if we take the square root of s, it satisfies

$$ct\sqrt{1 - \left(\frac{x}{ct}\right)^2 - \left(\frac{y}{ct}\right)^2 - \left(\frac{z}{ct}\right)^2} = \pm ct'\sqrt{1 - \left(\frac{x'}{ct'}\right)^2 - \left(\frac{y'}{ct'}\right)^2 - \left(\frac{z'}{ct'}\right)^2}. \qquad (3)$$

It would not necessarily be the case for equation (2) that the three dimensional Pythagoras theorem

$$x^2 + y^2 + z^2 = r^2 = \text{invariant} \qquad (4)$$

holds, but if we assume it in each frame, say

$$r^2 = x^2 + y^2 + z^2, \quad r'^2 = x'^2 + y'^2 + z'^2$$

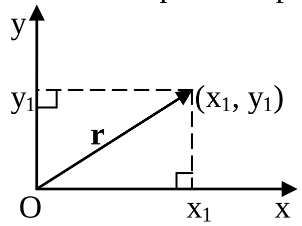
then we can write (3) as

$$ct\sqrt{1 - \left(\frac{r}{ct}\right)^2} = \pm ct'\sqrt{1 - \left(\frac{r'}{ct'}\right)^2}. \qquad (5)$$

This works perfectly well when we are considering real time variables. We will be cautious enough to delay until section 9 the case of space coordinates, where we introduce a system of interpretation in which space is measured by an extension of the idea of imaginary variables similar to $i = \sqrt{-1}$. ☐

1.4 Floating spaces and vector spaces with base point.

We have glossed over a feature of this description which is important, which we will put in the context of vector spaces. A vector is a number with *magnitude* and *direction*. Without going immediately into the axioms, or rules, for a vector space we will document two types of vector space. To take a simple example, consider a two dimensional vector space.

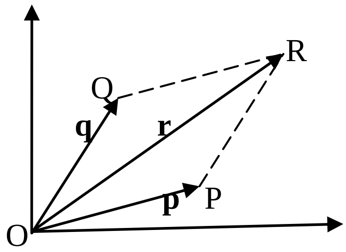


In the diagram O is the origin of this vector space. It has two axes, x and y, and a vector **r** attached to O at the base with two components, x_1 projected at right angles to the x axis and y_1 projected at right angles to the y axis. We write, with vectors written in bold

$$\mathbf{r} = (x_1, y_1).$$

We will call O the *base point* of the vector space, and the vector space is called a vector space with base point, because all vectors in the space are attached to O.

It is possible to define vector addition in this space.



In the diagram OQ is parallel to PR, QR is parallel to OP and the vector **r** satisfies

$$\mathbf{r} = (r_1, r_2)$$
$$= \mathbf{p} + \mathbf{q} = (p_1, p_2) + (q_1, q_2)$$

where

$$r_1 = p_1 + q_1$$
$$r_2 = p_2 + q_2. \qquad (1)$$

There is another type of space, called an *affine* or *floating* space used in relativity theory. In an affine space we jettison the base point O, and all vectors are allowed to 'float' parallel to themselves. Another, mathematical, way of putting this is that each vector **p**, **q** etc. in an affine space is an *equivalence class* of vectors. The vectors for **p** are then a set of vectors with the same magnitude and direction as **p**, that is, parallel to **p**. More formally, the equivalence class is defined as an equivalence relation, spoken as 'equivalent to' satisfying three rules

(1) the vector **p** is equivalent to **p**
(2) if **p** is equivalent to **q**, then **q** is equivalent to **p**
(3) if **p** is equivalent to **q** and **q** is equivalent to **r**, then **p** is equivalent to **r**.

This means that if we have a set of formulas satisfied by a vector space with base point, we might expect the same set of formulas to hold for its affine analogue. If we can describe 1.3.(1) as an equation in a vector space, we have two possible philosophical interpretations, the space is a vector space with base point or it is an affine space. The formulas could remain the same in both.

Indeed, if we look at the diagram above, where vectors OP and OQ combine to give vector OR, we can consider the vector OR to be a vector sum OP + PR, where PR is equivalent to OQ. In this triangle the vectors may have a distance associated with them. These distances remain invariant when the triangle is allowed to float away from the origin O. Distances can be of relativistic type as in equation 1.3.(1), which we will relate to the case of quaternions.

The question arises as to whether there exist formulas that are valid in a vector space with base point but not in an affine space. There are, and we will eventually meet them. They are formulas relating one base point to another. Since affine spaces do not have base points, we will say this type of formula does not apply to them.

1.5 The rules for mathematical fields and vector spaces.

The axioms for a *field* $\mathbb{F}$, +, ×, which we will denote simply by $\mathbb{F}$, with a, b, c ∈ $\mathbb{F}$ (the symbol ∈ means 'belongs to'), and multiplication a × b being written as ab satisfy

(1) additive closure: $a + b \in \mathbb{F}$
(2) associativity: $a + (b + c) = (a + b) + c$
(3) abelian addition: $a + b = b + a$

(4) existence of a zero: there exists a $0 \in \mathbb{F}$ satisfying

$$a + 0 = a$$

(5) existence of negative elements: there exists a $(-a) \in \mathbb{F}$ with

$$a + (-a) = 0,$$

which we write introducing subtraction as

$$a - a = 0$$

(6) multiplicative closure: $ab \in \mathbb{F}$

(7) associativity: $a(bc) = (ab)c$

(8) commutativity: $ab = ba$

(9) existence of a 1: there exists a $1 \in \mathbb{F}$ satisfying

$$a1 = a$$

(10) existence of inverse elements: there exists an $a^{-1} \in \mathbb{F}$ for $a \neq 0$ with

$$a(a^{-1}) = 1,$$

which we write introducing division as

$$a/a = 1$$

(11) distributive law: $a(b + c) = (ab) + (ac)$.

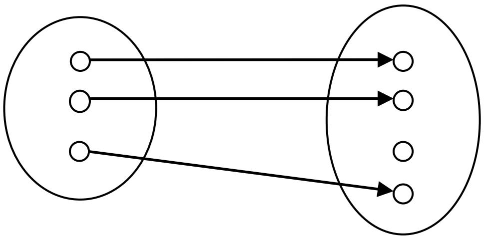
A vector space is a collection of objects, **V**, called *vectors*, denoted in bold type, which can be added together and multiplied by numbers called *scalars*, given in ordinary letters. Scalars can be real numbers, but there can also be scalar multiplication by complex numbers, rational numbers or generally any field. Vector addition and scalar multiplication satisfy the axioms

Axiom	*Meaning*
(a) Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
(b) Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(c) Identity element for addition	There exists an element **0** belonging to **V**, called the *zero vector*, so that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all **v** that belong to **V**.
(d) Inverse elements for addition	For every **v** that belongs to **V**, there exists an element **–v** that belongs to **V** called the additive inverse of **v**, with the property $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
(e) Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$.
(f) Scalar multiplication identity element	There is a scalar 1 satisfying $1\mathbf{v} = \mathbf{v}$ for all **v**.
(g) Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
(h) Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

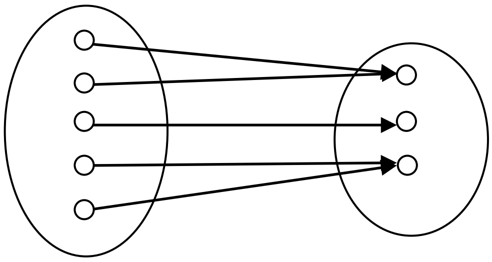
1.6 Functions and propagators.

A *function*, *mapping*, *map* or *transformation* f(x) is a set of pairs mapping from a set {x} of elements called the *domain* (or range) of the function to the set {f(x)} of elements called the *codomain* (or target, or *image*) of the function.

A function $S \to T$ is called *injective* (or *one-to-one*, or an injection) if $f(a) \neq f(b)$ for any two different elements a and b of the domain. It is called *surjective* (or *onto*) if $f(S) = T$. That is, it is surjective if for every element y in the codomain there is an x in the domain such that $f(x) = y$. The function f is called *bijective* if it is both injective and surjective.



injective function



surjective function

If t is time and three space coordinates are x, y and z mutually at right angles to one another, then as time varies we will say the set {t, x, y, z} is a *propagator*.

For an entity there will usually be a relation between t and x, y and z, denoted by R(t, x, y, z), for example a formula satisfying R(t, x, y, z) = 0. We may attach to say the variable t a function $f_t(t, x, y, z)$, and similarly for the other variables. In general we do not have to be restricted to four space-time coordinates.

1.7 The hyperintricate representation of matrices.

We now provide a description for the addition and multiplication firstly for 2 × 2 matrices called the intricate representation, and then for $2^n \times 2^n$ matrices, in a regular extension of the intricate case that we call hyperintricate numbers. The intricate representation was discovered by James Cockle, published by the London Edinburgh Dublin Philosophical Magazine in 1849, which he called coquaternions. The current terminology is often split-quaternions. A fuller account of intricate and hyperintricate numbers is given in [Ad15], chapter I and II. A useful feature of intricate and hyperintricate numbers is that they give a more detailed insight into the structure of matrices, and contain complex numbers in a natural way. We will use them to represent the quaternions in section 8.

A square n × n matrix, A, is an array of numbers with n rows and n columns, say for n = 2

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

where p, q, r and s are numbers. We will say the element at the j^{th} row and k^{th} column starting from 1 is a_{jk}, so in the matrix above $a_{11} = p$.

To add two matrices A and A′, add their corresponding elements. If

$$A' = \begin{bmatrix} p' & q' \\ r' & s' \end{bmatrix},$$

then

$$A + A' = \begin{bmatrix} p + p' & q + q' \\ r + r' & s + s' \end{bmatrix}.$$

To multiply A and A′, for the element in the j^{th} row and k^{th} column, multiply in turn each element from the j^{th} row with the corresponding element from the k^{th} column, adding the results together, so

$$AA' = \begin{bmatrix} pp' + qr' & pq' + qs' \\ rp' + sr' & rq' + ss' \end{bmatrix},$$

where for example the first row of A is $[p \quad q]$ and the first column of A′ is $\begin{bmatrix} p' \\ r' \end{bmatrix}$, so to get the first row and first column element of the matrix, this is (pp′ + qr′).

Note that, in general (can you find an example?)

$$AA' \neq A'A,$$

and when this happens we say the matrix multiplication is noncommutative.

A complex number, represented by g = a1 + bi, where i = $\sqrt{-1}$, can also be represented by

$$1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad i = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix},$$

where we may multiply the matrix, say 1, by a to form the matrix

$$a1 = \begin{bmatrix}a & 0\\0 & a\end{bmatrix}.$$

Here a1 is the *real* part and bi the *imaginary* part of the complex number, with $i^2 = -1$. This representation follows all the rules for a *field*, which defines the rules for addition and multiplication, including the existence of a multiplicative inverse g^{-1} of a nonzero complex number, satisfying $gg^{-1} = 1$, with

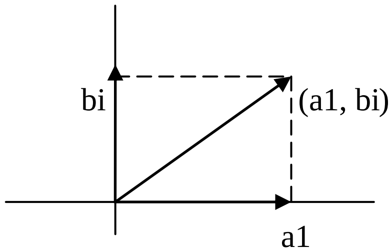
$$g^{-1} = (a1 - bi)/(a^2 + b^2).$$

If we wish to extend this algebra to include all possible 2 × 2 matrices with real elements, then we can introduce two more *basis elements* – the *actual* matrix

$$\alpha = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

and the *phantom* matrix

$$\phi = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$



Just as for complex numbers where we can represent the (a, b) pair of real and imaginary components as vectors in what is called an *Argand diagram*, we can also have a 4-dimensional diagram representing what I call an *intricate number*

$$h = a1 + bi + c\alpha + d\phi.$$

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ... $\mathbf{v}_n$ are *linearly independent* if there are no coefficients a_1, a_2, ... a_n, not all zero, satisfying

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots a_n\mathbf{v}_n = 0.$$

The linearly independent intricate basis elements satisfy

$$1^2 = 1,\ i^2 = -1,\ \alpha^2 = 1,\ \phi^2 = 1,$$
$$1i = i = i1,\ 1\alpha = \alpha = \alpha 1,\ 1\phi = \phi = \phi 1,$$
$$i\alpha = -\phi = -\alpha i,\ i\phi = \alpha = -\phi i \text{ and } \alpha\phi = i = -\phi\alpha. \qquad (1)$$

An intricate number can represent uniquely any real 2 × 2 matrix. We will show, and extend the ideas below to n × n matrices, that a 2 × 2 real matrix

$$A = \begin{bmatrix}p & q\\r & s\end{bmatrix}$$

does not have an inverse if its *determinant* det A = ps – rq = 0, in which case it is called a *singular* matrix. For a complex number the basis elements 1 and i have determinant 1. All matrices except the zero matrix for a complex number have multiplicative inverses. We will see in contrast that nonzero intricate numbers may have no multiplicative inverse.

In more detail, the matrix above has the intricate representation

$$h = a1 + bi + c\alpha + d\phi$$
$$= ½(p + s)1 + ½(q – r)i + ½(p – s)\alpha + ½(q + r)\phi. \qquad (2)$$

The *intricate conjugate* is $(a1 – bi – c\alpha – d\phi)$. If the multiplicative inverse exists, it is

$$h^{-1} = (a1 – bi – c\alpha – d\phi)/(a^2 + b^2 – c^2 – d^2),$$

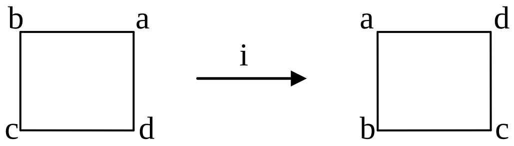
so the denominator is non-zero. This denominator is the determinant, because from (2)

$$a^2 + b^2 – c^2 – d^2 = ¼ [(p + s)^2 + (q – r)^2 – (p – s)^2 – (q + r)^2] = ps – rq. \square \qquad (3)$$

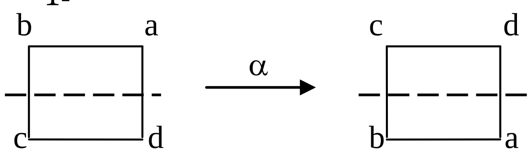
In diagrams we will call a line – – – of the following type on the left a *diagonal*

and a line of the type on the right an *antidiagonal*. If 0 is the origin of the coordinate system, the diagonal at an angle of $3\pi/4$ radians anticlockwise from the right horizontal axis, and the antidiagonal at $\pi/4$ radians pass through it.

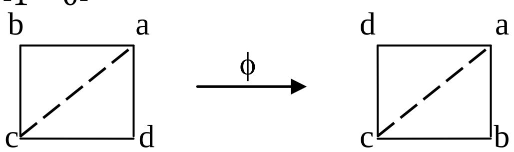
We can represent the group of the symmetries of a square by intricate basis elements. We can represent $i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as a rotation of the square anticlockwise by $\pi/2$ radians



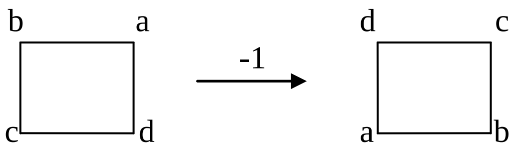
$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ as a reflection about the horizontal axis



and $\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ as a reflection about the antidiagonal



Since $i^2 = -1$, we have the two rotations of i



which is a combined diagonal and antidiagonal, or equivalently a combined horizontal and vertical reflection.

Then, as can be checked, we have the formulas of (1).

We can represent these formulas by the group multiplication table

$\times$	1	i	α	ϕ
1	1	i	α	ϕ
i	i	-1	$-\phi$	α
α	α	ϕ	1	i
ϕ	ϕ	$-\alpha$	-i	1

and extend the table for multiplication by the further elements -1, -i, $-\alpha$ and $-\phi$.☐

The sum of two m × m matrices A and B, with elements for A given by a_{ij}, where i is the ith row and j is the jth column, and for B by b_{ij}, is the matrix C where

$$C = c_{ij} = a_{ij} + b_{ij}.$$

The corresponding product D is

$$D = d_{ik} = AB = \textstyle\sum_j a_{ij}b_{jk},$$

where $\sum$ indicates summation, in this case over the variable j. This is the generalisation of a matrix product already given for 2 × 2 matrices. We seek to develop this idea within an extended framework already given for these intricate numbers.

We can define n-*hyperintricate* numbers by repeatedly building them up starting from intricate ones. Consider a $2^n \times 2^n$ matrix. Let "+" be a chosen $2^{n-1} \times 2^{n-1}$ matrix which is a hyperintricate basis element of lower dimension, for example an intricate basis element 1, i, α or ϕ. Let "–" be the corresponding matrix with all negative entries from "+". Consider the set of $2^n \times 2^n$ hyperintricate basis elements, where an intricate number has "+" = 1, "–" = -1

$$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}, \begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}, \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}.$$

Any $2^n \times 2^n$ matrix can be represented uniquely by a linear combination of these.

A j × j matrix may be extended both right and below with zero entries to give a larger $2^n \times 2^n$ matrix, or main diagonal entries of 1 may be substituted here to keep determinants non-zero. By this means matrix theorems may be expressed hyperintricately.

I now introduce some notation. I will do this by giving examples of 4 × 4 matrices. Write

$$1_1 = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{bmatrix}, \qquad \alpha_i = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & 0 \\ & 0 & 0 & -1 \\ & & 1 & 0 \end{bmatrix},$$

$$i_1 = \begin{bmatrix} & 0 & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & 0 \end{bmatrix}, \qquad \phi_i = \begin{bmatrix} & 0 & 0 & 1 \\ & & -1 & 0 \\ 0 & 1 & & \\ -1 & 0 & & 0 \end{bmatrix}.$$

So "+" corresponds with the subscript, which will be described as an example of a layer, for example in α_i. A memory aid is '*subscripts are the little part*'.

If in general each of the 16 real 4 × 4 matrices are represented by e.g. $\alpha_i = A_B$, then

$$(A_B) + (A_C) = A_{(B + C)},$$
$$(A_B) + (C_B) = (A + C)_B,$$
$$(A_B)(C_D) = (AC)_{BD},$$
$$A_{-B} = -(A_B) = (-A)_B.$$

For further nesting of matrices, consider instead of stepping down a further layer, introducing (possibly) a comma, thus: $A_{B,C}$, so that matrix multiplication becomes

$$(AB)_{CD,EF} = (A_{C,E})(B_{D,F}).$$

The *layers* of a basis element $m_{n \, ... \, p}$, are the vectors m, n, ... p, and its *layer dimension* is the number of layers.

We define an n-*hyperimaginary* number to be an n-hyperintricate number with each layer restricted to the set {1, i}. We can also define *hyperactual* numbers, containing elements of {1, α} in all layers and *hyperphantom* numbers for which every layer $\in$ {1, ϕ}. Hyperactual and hyperphantom number are not members of a field. This arises because (1 + α) and (1 + ϕ) have determinant zero, and so have no inverse and $(a1_1 + bi_i)$ has inverse $(a1_1 - bi_i)/(a^2 - b^2)$, which does not exist for a = b.

Intricate and hyperintricate numbers appear in four ways – as scalars, satisfying a non-commutative algebra, as vectors (or eigenvectors described later) with a linearly independent basis, as matrices – where the first instance is intricate numbers, and in the hyperintricate case, say as the object similar to a tensor, $m_{n,p}$, where m, n and p are vectors.

1.8 Quaternions as beables (what is there) with relativistic observables.

We will now describe the algebra of quaternions sometimes using the formalism of the hyperintricate representation. These are a noncommutative (AB $\neq$ BA) four dimensional type of complex number with three imaginary quaternion dimensions. We will use quaternions, and eventually in chapter XI an extension of the idea to the n-novanions, to describe the mathematical model part of our theory. We will also introduce mathematical twisting of the quaternion spaces, which can be either an even or an odd number of twists. We will then be able to map the mathematical model onto the observables of the theory. In later developments this will allow us to describe light, or photons, which have an even number of twists, in fact two, and are called bosons, and the Dirac relativistic equation of the electron, with an odd number of twists, in fact just one, and called a fermion, and extensions of this idea.

The quaternion algebra of the Irish mathematician William Rowan Hamilton is an extension of the complex numbers with a scalar part and 3 ‘imaginary’ – or quaternionic – parts. So we can represent a quaternion by

$$a1 + bi + cj + dk$$

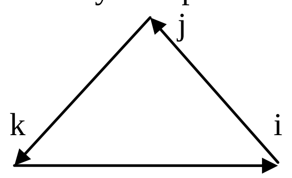
where a is the size of the scalar part and b, c and d are the sizes of the quaternion imaginary parts, for which

$$1^2 = 1,\ i^2 = j^2 = k^2 = -1,$$
$$1i = i = i1,\ 1j = j = j1,\ 1k = k = k1,$$
$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik \qquad (1)$$

and where the multiplicative inverse is

$$(a1 - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2). \qquad (2)$$

The quaternions may be represented by a triangle diagram



where the nodes i, j and k satisfy

$$ij = k,\ jk = i,\ ki = j, \tag{3}$$

in other words, for a positive sign in the above relations, we are following the arrows. When we are going in a direction opposite to the arrows, we have a negative sign:

$$ji = -k,\ kj = -i,\ ik = -j. \tag{4}$$

We have here that 1 commutes with all elements, and also

$$1^2 = 1,\ i^2 = j^2 = k^2 = -1. \tag{5}$$

This (1, i, j, k) basis is representable by four hyperintricate numbers – in fact the previously given 1_1, α_i, i_1 and ϕ_i. An alternative representation, under swapping of layer levels, is 1_1, i_α, 1_i and i_ϕ. Some other representations are 1_{11}, $i_{\alpha\phi}$, $\alpha_{\phi i}$ and $\phi_{i\alpha}$ or 1_{1111}, $i_{11\alpha\phi}$, $\alpha_{ii\phi i}$ and $\phi_{iii\alpha}$. ☐

We will use quaternions as a 'toy model' for our mathematics describing physical states. The operations on these states will be at least addition, subtraction and multiplication.

We now introduce a scalar which can be derived from any quaternion – the *quaternion norm*. We have noticed that the inverse of the quaternion

$$a1 + bi + cj + dk$$

is

$$(a1 - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2).$$

The denominator of this expression, which given nonzero real a, b, c or d can never become zero, is known as the quaternion norm,

$$a^2 + b^2 + c^2 + d^2,$$

and is a type of distance function for quaternions. It is the analogue of the norm for complex numbers, which are obtained under the restriction of quaternions to c = 0, d = 0.

Moreover square roots of norms satisfy the rules for the distance function in a *metric space*. The distance function d(x, y) between two points x and y in a metric space satisfies

(1) $d(x, y) > 0$ unless $d(x, y) = 0$, when $x = y$

(2) $d(x, z) \leq d(x, y) + d(y, z)$,

where for complex numbers we have taken the origin, or base point, to be x = 0.

Under a rotation without boost to a new coordinate system to the quaternion given by a′, b′, c′ and d′, the norm is retained, in other words it is an invariant for rotations. This aspect is very useful in using quaternions for the computation of the motion of mechanical gyroscopes.

More mathematically, in the intricate representation of quaternions given above,

$$a1_1 + bi_1 + c\alpha_i + d\phi_i,$$

under the transformation changing these matrices to the matrix transpose, no other intricate basis element than i changes sign, so $i \to -i$ maps quaternion matrices $A_{uv} \to A_{vu}$, and we obtain

$$a1_1 - bi_1 - c\alpha_i - d\phi_i,$$

which in the i, j, k notation we are also using is

$$a1 - bi - cj - dk,$$

and we have

$$(a1 + bi + cj + dk)(a1 - bi - cj - dk) = a^2 + b^2 + c^2 + d^2.$$ ☐

We now devise some principles which map the quaternion model to special relativistic observables.

Principle 1: *Time in the quaternion model and its observable transformation to special relativity is real.*

Principle 2: *Space in the quaternion model refers to a real coefficient of a quaternion imaginary variable and its observable transformation to special relativity transforms this coefficient to be part of a quaternion imaginary variable.*

Interpretation: *The observer is embedded in the model. Since space is quaternionic she must, being embedded in the model which describes her, always use quaternion imaginary variables for space measurements, and real variables for time measurements*.

Consider the real coefficients a, b, c and d. Then the scalar part or norm is a time invariant in the model given by

$$a^2 + b^2 + c^2 + d^2 = (a1 + bi + cj + dk)(a1 – bi – cj – dk).$$

It transforms the time variable a, and space variables ib, jc and kd, with $i^2 = j^2 = k^2 = -1$, to

$$a^2 – (ib)^2 – (jc)^2 – (kd)^2 \qquad (6)$$

as the time invariant mapping to the observable special relativistic case. This is the same as equation 1.3.(1). □

In section 9 we will link the quaternion observables of equation (6) to the modified or direct Lorentz and Poincaré transformations of special relativistic observables as demonstrated in experiments.

It is interesting that if, under the interpretation, the observer is measuring along (ix), then the quaternion (jy) = jx(y/x) is transformed as an observable on multiplying, say on the left, to i(jy) = ky, whereas i(kz) = -jz. Thus the coordinates in the model transform to observables in which the y and z coordinates are swapped round, and the observable coordinate z becomes negative. This transformation can be obtained from a rotation of the model coordinates to the observable coordinates, that is, it retains the handedness of the coordinate system.

In this system we can choose to embed this local quaternion structure in a global manifold which is oriented, that is say, an ix vector moving through 2π radians in a jy and kz circle returns to itself with the ix vector pointing in the same direction. When this happens, we say the observable derived from the quaternion is *globally bosonic*. An example is the particle of light called a photon. We can also implement nonoriented global manifolds in which an ix vector moving through 2π radians in a jy and kz circle returns to itself with the ix vector pointing in the opposite direction. We then say the quaternion is *globally fermionic,* for example an electron can be interpreted in this way. □

1.9 Comparison of relativistic frames of reference.

We will consider idealisations of two classic experiments, the Michelson-Morley experiment, which deals with changes to the length of measuring rods between reference frames, which is a 'space' experiment, and the special relativistic slowing down of atomic clocks which are taken round the Earth by a rocket. This is the corresponding 'time' experiment.

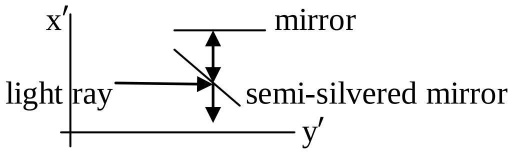
The reasoning we apply does not conform to the outcome for the Lorentz transformation for the space component, although we give reasons for the near null result of the Michelson-Morley experiment within this reasoning, as a localisation of the physics to its frame of reference. However, without this localisation the outcome of the Lorentz transformation for the time component between two separating and returning reference frames does conform to theoretical expectations.

For the remainder of the chapter we will assume that the quaternion norm, which we have seen is derived from just one quaternion, is a significant invariant in describing particles by a propagator incrementing in time and space multiplied by its back propagator reversing space coordinates. But in this picture the distance line element is not ultimately the basic entity.

We will consider two frames of reference, a ‘stationary’ frame ct, x, y, z, where t is the time measured in the stationary system, c is the speed of light, and x, y and z are space coordinates in the system, as in section 3. The moving frame is described by coordinates ct′, x′, y′ and z′. Let us arrange the coordinate systems so that the frames moving with uniform velocity with respect to each other pass at a particular (ct, x, y, z) and (ct′, x′, y′, z′) through a common origin O = O′. We will normalise these coordinates so where the origins coincide ct = x = y = z = ct′ = x′ = y′ = z′ = 0. On denoting increments of coordinates by Δct, Δx, Δy, Δz, etc., we will also assume that the Pythagoras theorem holds in each frame, so

$$(\Delta r)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$
$$(\Delta r')^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2.$$

We can now rotate coordinates so that their mutual velocity is along x and x′ alone, with no y, z, y′ or z′ motion external to these frames of reference. This action is permissible and natural in the quaternion formalism, since there exist rotational transformations of i, j and k to i′, j ′ and k′ which satisfy the same quaternion algebra.



Michelson Morley interferometer experiment
The x′, y′ coordinates are rotated to see if the interference fringes change.

Let us consider that the z and z axes are vertical, and the movement within the laboratory is in the Δx′Δy′ plane. If we now have a measuring rod along the y′ axis, it has no motion with respect to the stationary frame. To simplify, say this happens when t = t′ = 0 when the origins of the frames coincide. In the moving laboratory, let us now rotate its measuring rod to coincide with the x′ axis, parallel to the x coordinate with respect to which it is moving with uniform velocity.

From an equation of type (6) of section 7, on choosing a positive sign for beables

$$i\Delta x\sqrt{\left(\frac{c\Delta t}{i\Delta x}\right)^2 - 1} = i\Delta x'\sqrt{\left(\frac{c\Delta t'}{i\Delta x'}\right)^2 - 1}, \qquad (1)$$

where the observer interprets her distance derived from iΔx so that her velocity is derived from v = iΔx/Δt, but the observer is attributing real values to imaginary space observable variables. For observables

$$\Delta x\sqrt{\left(\frac{c}{v}\right)^2 - 1} = \Delta x'\sqrt{\left(\frac{c}{v'}\right)^2 - 1}. \qquad (2)$$

In terms of this interpretation of Δx, if the maximum velocity v′/c = Δx′/cΔt′ is 1, Δx′ has expanded near this high velocity, but when the reciprocal of the velocity squared

$$\left(\frac{c\Delta t'}{\Delta x'}\right)^2 > 2,$$

in other words

$$(v'/c)^2 < ½,$$

it contracts. ☐

The observers in the moving coordinate frame will not think that when they rotate from a direction with no velocity component to one with motion that they will change size (this is the real content of the Michelson-Morley experiment). They will think that they maintain invariant distances. If we assume the light used in the experiment originates in this reference frame, they will attribute $\Delta x' = \Delta y'$. The reader may be aware that this is not the derivation arrived at in textbooks on special relativity.

A possible explanation is that in the presence of a large amount of matter the moving frame locks velocities as if it were stationary, relative to this coordinate system at the stationary semi-silvered interferometer mirror. Theoretically and experimentally these and other effects are discussed in chapter II. ☐

The Poincaré transformations are a more general form of the Lorentz transformations. For instance, on putting t for time, choosing one variable x for distance, c for the velocity of light and v for the velocity of its frame, whereas in the moving frame we have variables t'', x'', c'' and $v'' = -v$, a standard text on relativity [In92] gives these transformations as

$$t'' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right) \qquad (3)$$

$$x'' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt) \qquad (4)$$

so that

$$c^2t''^2 - x''^2 = \frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left[c^2t^2 - 2tvx + \frac{v^2x^2}{c^2} - x^2 + 2tvx - v^2t^2\right],$$

and

$$c^2t^2 - x^2 = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)}[c^2t^2 - x^2] = \frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left[c^2t^2 + \frac{v^2x^2}{c^2} - x^2 - v^2t^2\right], \qquad (5)$$

which are indeed the same. Equation (5) is a hyperbola. For fixed v, (3) and (4) are linear. ☐

We now have two possible types of transformation which hold the line element invariant. In the Poincaré transformations (3) and (4), t'' and x'' are represented by using t and x together, but in equations 1.3.(5) and equation (2) in this section, although there are velocity terms, the t' variable is represented solely in terms of t and the x' variable only in terms of x. Thus these two approaches are not using the same coordinate systems. If the line element s is the same invariant in both cases however, these should represent the same physics, but if not, we can globally change the magnitude of one set of line elements to match the other set. ☐

There are issues on the badly behaved nature of the Poincaré transformations at **v** = **c**, and the badly behaved nature of our transformation (2) at **v** = **0** and **v** = **c**. The Michelson-Morley experiment looks at a stationary reference frame (velocity **0**) and another frame moving with velocity **v** with respect to the first frame. In the moving frame there is an interferometer with light at velocity **c** at right angles to **v** interfering with light at velocity **c**′ in the same direction as **v**. The Lorentz and Poincaré transformations do not relate four velocities, **0**, **v**, **c** and **c**′. There is a factor v/c in the equations, so this should relate the time and space in the stationary frame to the time and space in the second frame. The theoretical fudge used in the explanation of the null result of the Michelson-Morley experiment assumes that **c** and **c**′ are equal to **v**, whereas there are four frames, one with velocity **0**, one with velocity **v** and two with velocity **c**. The Lorentz and Poincaré transformations do not refer to four frames, but two, and indeed cannot describe the **c** frames which lead to infinities.

A resolution of this difficulty is to assume that photons, particles of light, have very small rest mass, which means that, as discussed in chapter II, for special relativity at **v** = **c** this mass becomes infinite and is unphysical. Then we can allocate under the Poincaré transformations a momentum to photons so that under available energy transactions photons are travelling very close to the velocity of light. This is an experimentally determinable assumption, and we will look into it. ☐

The situation for relativistic slowing down of clocks looks at first more reassuring. Consider an atomic clock K in a coordinate system ct, x, y, z at its origin O on the Earth's surface, and a rocket from O synchronised there with a clock K′ put in orbit to O′ at ct′, x′, y′, z′. We will assume, as if this were permissible, that both the Earth and the rocket, now travelling with uniform velocity, v′, around the Earth, can be treated as if they were inertial reference systems. Under this approximation, after a time interval $c\Delta t$ from O and $c\Delta t'$ from O′, the rocket decelerates and returns to Earth. We will additionally assume that the acceleration and deceleration times are small and have no effect on the calculation we are about to make.

Under these huge assumptions, which in explicit calculations of *global positioning system* (GPS) satellites we find are sometimes violated, equation (5) of section 3 is claimed to hold. Then the velocity of the Earth is assumed zero, and choosing a positive sign

$$c\Delta t = c\Delta t'\sqrt{1 - \frac{v'^2}{c^2}},$$

so that the interval $c\Delta t'$ is larger than $c\Delta t$. We will say that the atomic clock K′ measuring comparable events to K does this over a larger period of time in its coordinate system, so that effectively its clocks have slowed down in K. This is known as time dilation, and is the same formula as is obtained from the Lorentz transformation of special relativity. ☐

We now raise, but do not yet resolve, the issue of whether the time component of the Lorentz transformations is also locked locally in the presence of large amounts of matter. Given the near null result of the Michelson-Morley experiment, we would expect if distances are locally locked, so is time.

It is not our aim in this section to investigate the validity or invalidity of the results in these thought experiments by analysing the data. In chapter II we will look at the experimental data on this effect obtained from GPS satellites. This discussion and that of section 11 will be the background to chapter X on absolute space, and forms some of the objectives of chapter XXI on experimental tests of nonstandard physics.

1.10 Transforming from beables to observables.

The essence of the transformations we are adopting are maps from *orthogonal* spaces, also called *Euclidean* spaces, where an n-dimensional Pythagoras theorem holds, to n-dimensional *hyperbolic* spaces with one time component. These are the mappings from the beables of the theory – what is there – to the observables of special relativity theory.

Let us consider a two dimensional space with ct as a measure of time and x a measure of distance. Later in this work we will be looking at propagators where it is more natural to consider the entity

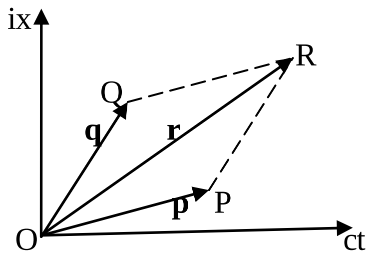
$$(ct + ix)(ct - ix) = c^2t^2 + x^2 = r^2, \qquad (1)$$

as a representation of the Pythagoras theorem.

The variable (ct + ix) is the *forward propagator* and (ct – ix) is the *backward propagator*.

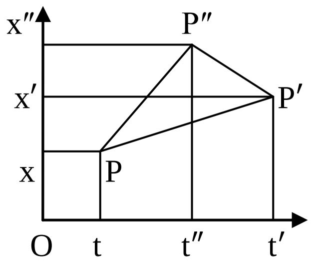
Note that these propagators separately are linear. It is a novel approach to treat propagators as fundamental, and space as a secondary consequence of these. Allowable linear increments and subtractions on propagators then describe physics. Different propagators not necessarily for the same wavelet state may then interact also by multiplication. This viewpoint and its generalisation to both space and energy will be developed from Part II onwards.

We know that in an orthogonal space that the Pythagoras theorem is a useful and consistent idea. Let us consider the diagram of section 4 repeated again here.



We have added the axes ix and ct to the diagram. Then the origin O is now the number 0 in a complex number. In complex number representations all numbers are attached to the origin. But we can consider these complex numbers also as a complex affine space where vectors of the same size are allowed to float parallel to themselves. Then the same algebra applies.

There are well-known transformation equations between distances in Euclidean space. Under a linear transformation of the coordinates, these distances remain invariant. Consider three points P, P′ and P″.



Then if c(t′ – t) is the t component of distance PP′, and (x′ – x) is the x component, etc. then in Euclidean space the distances PP′, P′P″ and P″P can be obtained. For instance

distance PP′ = $\sqrt{c^2(t'-t)^2 + (x'-x)^2}$.

These lines form a triangle, and the distances remain invariant under a translation of the origin by cΔt, Δx, since for instance the t component of PP′ is then (ct′ + cΔt – ct – cΔt), which is still (ct′ – ct). ☐

Note that the transformations of t and x are linear, but these mutual distances are nonlinear.

For the hyperbolic space of special relativity the same argument applies, but the ‘distance’, which we have called the line element, s, is now

$\sqrt{c^2(t'-t)^2 - (x'-x)^2}$.

Incidentally, if we adopt the Lorentz transformations 1.9.(3) and 1.9.(4), then as the velocity $v \rightarrow c$ the hyperboloid for s^2 in this limit becomes a cone, called the light-cone. We have now shown a well-defined mapping between an orthogonal and a hyperbolic space. ☐

1.11 Local and global space-time.

In this section we will deal with quadratic forms where the matrices and coefficients are reduced to complex variables. We have seen that quaternions can be represented by matrices. In the case of a matrix variety, a generalised example of which this is a part could be

$$aXY + bY^2 + cX^2YZ^5 + dZ = 0,$$

where X, Y and Z are matrices and a, b, c and d are coefficients.

Chapter XI, section 2 of [Ad15], proved Sylvester's law of inertia, which states that the signature, or difference between positive and negative terms, derived from the quadratic form $\sum x_j b_{jk} x_k$, where b_{jk} is symmetric, is invariant under any similarity transformation to

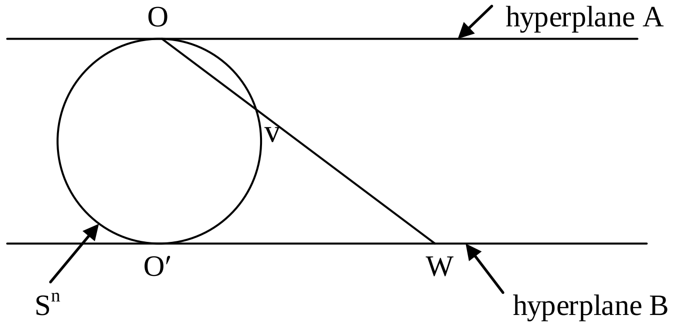
$$x_1^2 + ... + x_r^2 - x_{r+1}^2 - ... - x_n^2. \qquad (1)$$

A quaternion with time and space coordinates, together with other values associated with these coordinates form a quaternion propagator. We view these propagators as the prototype for the fundamental entities in the physical world, and we know a quaternion multiplied by its conjugate is a quadratic form. We have seen such forms in the norm of an intricate number, as an everywhere positive norm for a quaternion, and as we will also show later these exist for an octonion, or otherwise n-novanion. We have also mapped from a quaternion to the observable hyperbolic space of special relativity given by (1) with r = 1 and n = 4.

A classical case we wish to consider is stereographic projection from a sphere to a complex plane, which we will generalise to an n-sphere mapped to the special relativistic n-hyperplane (1). A sphere has n = r in equation (1) above, and we wish to generalise this to stereographic projection when n > r, so this corresponds to hyperbolic geometry.

Let S^n be the n-sphere. The n-component of numbers with quadratic norms represented by some points on S^n, may be mapped to the tangent hyperplanes A and B below. Denote the stereographic map from B, at W in the diagram, to S^n by p^{-1}.

We extend the stereographic projection map p to the mapping including the punctured point O in S^n to the codomain given by hyperplane A, tangent at O.



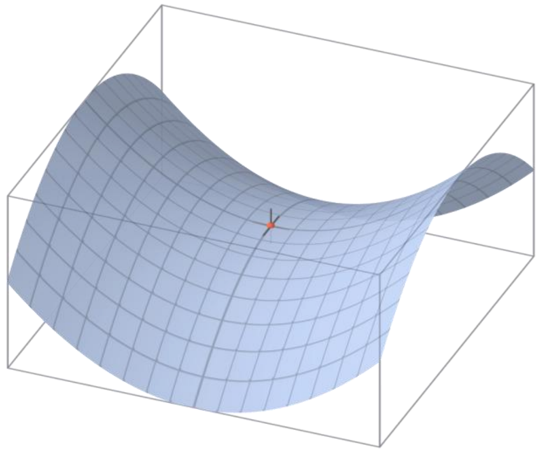
Given the time component x_1, for each space dimension r + 1 we can map the beable sphere to where the flat hyperplane B is replaced by an observable hyperboloid with cross sections

$$x_1^2 - x_{r+1}^2 = \pm R$$

giving a more general mapping. ☐

[Wikipedia] *Hyperbolic space* is a homogeneous space that has a constant negative curvature. It is hyperbolic geometry in more than 2 dimensions, and is distinguished from Euclidean spaces with zero curvature that defines the Euclidean geometry, and elliptic geometry with a constant positive curvature.

Hyperbolic n-space, denoted $\mathbb{H}^n$, is a maximally symmetric, simply connected, n-dimensional curved Riemannian manifold with a constant negative curvature. Hyperbolic space exhibits hyperbolic geometry. It is the negative-curvature analogue of the n-sphere. Hyperbolic 2-space, $\mathbb{H}^2$, is also called the hyperbolic plane. Although hyperbolic space $\mathbb{H}^n$ is differentially equivalent to $\mathbb{U}^n$, its negative-curvature metric gives it very different geometric properties.



When embedded inside a Euclidean space of a higher dimension, every point of a hyperbolic space is a saddle point. Another distinctive property is the amount of space covered by the n-ball in hyperbolic n-space: it increases exponentially with respect to the radius of the ball for large radii, rather than as a polynomial.

We will now discuss models of hyperbolic space.

Hyperbolic space, developed independently by Nikolai Lobachevsky and János Bolyai, is a geometrical space similar to Euclidean space, but such that Euclid's parallel postulate is no longer assumed to hold. Instead, the parallel postulate is replaced by the following alternative in two dimensions:

- Given any line L and point P not on L, there are at least two distinct lines passing through P which do not intersect L.

It is a theorem that there are infinitely many such lines through P. This axiom still does not uniquely characterise the hyperbolic plane. There is an extra constant, the curvature $K < 0$, which must be specified. However, it does uniquely characterise it up to only changing the notion of distance by a constant. By choosing an appropriate length scale, we can assume, without loss of generality, that $K = -1$.

Models of hyperbolic spaces that can be embedded in a flat, for example Euclidean, space may be constructed. In particular, the existence of model spaces implies that the parallel postulate is logically independent of the other axioms of Euclidean geometry.

An *isometry* is a transformation which maps elements to the same or another metric space such that the distance between the image elements in the new metric space is equal to the distance between the elements in the original metric space.

There are several important models of hyperbolic space: the *Klein model*, the *hyperboloid model*, the *Poincaré ball model* and the *Poincaré half space model* which is closely related to the Poincaré ball model. These all model the same geometry in the sense that any two of them can be related by a transformation that preserves all the geometrical properties of the space, including isometry (though not with respect to the metric of a Euclidean embedding).

(i) The hyperboloid model.

In the hyperboloid model, hyperbolic space in (n + 1) real numbers denoted by $\mathbb{U}^{n+1}$ is given by $\{x_0, ..., x_n\}$ where $x_i \in \mathbb{U}$, i = 0, 1, ..., n. The hyperboloid is the set of those points in $\mathbb{H}^n$ whose coordinates satisfy

$$x_0^2 - x_1^2 - ... - x_n^2 = 1,\ x_0 > 0.$$

In the hyperboloid model the equivalent of a straight line or *geodesic* is the curve formed by the intersection of $\mathbb{H}^n$ with a plane through the origin in $\mathbb{U}^{n+1}$.

The hyperboloid model is closely related to the geometry of special relativity which is also called Minkowski space. The quadratic form

$$Q(x) = x_0^2 - x_1^2 - ... - x_n^2,$$

which defines the hyperboloid, gives a bilinear form under the transformation

$$B(x, y) = [Q(x + y) - Q(x) - Q(y)]/2$$
$$= x_0y_0 - x_1y_1 - ... - x_ny_n.$$

The space $\mathbb{U}^{n+1}$ with the bilinear form B is an (n + 1)-dimensional Minkowski space $\mathbb{U}^{n,1}$.

We can associate a *distance* on the hyperboloid model by defining the distance between two points x and y on $\mathbb{H}$ to be

$$d(x, y) = \operatorname{arcosh} B(x, y).$$

This function satisfies the axioms of a metric space. ☐

(ii) The Klein model.

An alternative model of hyperbolic geometry is on a certain domain in projective space. The Minkowski quadratic form Q defines a subset $V^n \subset \mathbb{U}\mathbf{P}^n$ given as the locus of points for which Q(x) > 0 in the coordinates x. The domain V^n is the *Klein model* of hyperbolic space.

The lines of this model are the open line segments of the ambient projective space which lie in V^n. The distance between two points x and y in V^n is defined by

$$d(x, y) = \operatorname{arcosh}\left(\frac{B(x,y)}{\sqrt{Q(x)Q(y)}}\right).$$

This is well-defined on projective space.

This model is related to the hyperboloid model as follows. Each point x in V^n corresponds to a line L_x through the origin in $\mathbb{U}^{n+1}$, by the definition of projective space. This line intersects the hyperboloid $\mathbb{H}^n$ in a unique point. Conversely, through any point on $\mathbb{H}^n$, there passes a unique line through the origin (which is a point in the projective space). This correspondence defines a bijection between V^n and $\mathbb{H}^n$. It is an isometry, since evaluating d(x,y) along Q(x) = Q(y) = 1 reproduces the definition of the distance given for the hyperboloid model. ☐

(iii) The Poincaré ball model.

The ball model comes from a stereographic projection of the hyperboloid in $\mathbb{U}^{n+1}$ onto the hyperplane $\{x_0 = 0\}$. In detail, let S be the point in $\mathbb{U}^{n,1}$ with coordinates (−1, 0, 0, ..., 0): the *south pole* for the stereographic projection. For each point P on the hyperboloid $\mathbb{H}^n$, let P_* be the unique point of intersection of the line SP with the plane $\{x_0 = 0\}$.

This establishes a bijective mapping of $\mathbb{H}^n$ into the unit ball

$$B^n = \{(x_1, ..., x_n): x_1^2 + ... + x_n^2 < 1\}$$

in the plane $\{x_0 = 0\}$.

The geodesics in this model are semicircles that are perpendicular to the boundary sphere of B^n. Isometries of the ball are generated by spherical inversion in hyperspheres perpendicular to the boundary. ☐

(iv) The Poincaré half-space model.

The half-space model results from applying inversion in a circle with centre a boundary point of the Poincaré ball model B^n above and a radius of twice the radius. This sends circles to circles and lines, and is a conformal transformation, which we meet in chapter XII. So the geodesics of the half-space model, which are the equivalents of straight lines in curved space, are lines and circles perpendicular to the boundary hyperplane. ☐

Considering hyperbolic manifolds, every complete, connected, simply connected manifold of constant negative curvature -1 is isometric to the real hyperbolic space $\mathbb{H}^n$. As a result, the universal cover of any closed manifold M of constant negative curvature -1, a hyperbolic manifold, is $\mathbb{H}^n$.

For curved Riemann surfaces, two-dimensional hyperbolic surfaces can also be understood in the language of Riemann surfaces. According to the uniformisation theorem, every Riemann surface is elliptic, parabolic or hyperbolic. The Poincaré half plane is also hyperbolic, but is simply connected and unbounded. It is the universal cover of the other hyperbolic surfaces.