Introduction to intricate and hyperintricate numbers II

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7th October 2012, revised 19th November 2013

Abstract. Part I [Ad12] discussed a representation of 2 X 2 matrices called *intricate numbers* containing the complex numbers as a subalgebra, and for 2^n X 2^n matrices the corresponding *hyperintricate* representation. We develop further in Part II some of their properties. This includes a description of hyperintricate compression and expansion, a discussion of norms (determinants) and traces, which contains an extension of some aspects of the theory of group characters, and a method for computing hyperintricate determinants, and inverses when these exist. As an application of the hyperintricate methodology, we prove a theorem due to Frobenius that any matrix may be represented by the product of two symmetric ones. We also introduce the spectral $\mathbb{D}1$ exponential algebra of [Ad14e].

A nonassociative algebra, the *extricate* algebra, is also presented here, in the first place as a modification of the intricate representation. Then intricate and extricate operations are combined to form the *duplicate* algebra. The extricate non-matrix algebra is not a division algebra for either the exquaternions which are derived from the quaternions or the exoctonions from the octonions. Nor are the 16-dimensional sedenions a division algebra.

We mention an implication; the quaternion representation displays handedness, as it does for the weak interaction in physics.

Of significance in the hyperintricate methodology is the existence of an emergent phenomenon investigated here. This is the property of being or not being J-abelian. Two hyperintricate numbers which are J-abelian for the same J's are consequently relatively abelian. The J-abelian property is shared by all sums with real coefficients of powers of constant intricate numbers, but not all such constant hyperintricate ones, and has a restrictive application in the extension of Galois theory to matrix variables [Ad14a].

Keywords: matrices, hyperintricate number, division algebra.

1 Intricate and hyperintricate numbers.

Part I [Ad12] contains the basic material on intricate and hyperintricate numbers, and is assumed known. Notation carries over from that article.

2 Deriving hyperintricate basis element coefficients from a matrix.

The trace (the sum of diagonal entries) of an intricate 1 is 2, and of α is zero. If we take consecutively the diagonal entries of α, which are (1, -1), and subtract the second entry, -1, we get 2 for α, and applied to 1 the same process now gives 0. For each diagonal matrix, the component of say 1_α, represented by (1, -1, 1, -1) is not zero over its average only on multiplying, for negative values, by -1, and this procedure applied to 1_α, α_1 and α_α gives 4. This idea can be generalised to non-diagonal hyperintricate values to obtain general coefficients from the defining matrix.

For example, let $Я_2$ be the 2-hyperintricate number

$$\begin{aligned} Я_2 = {} & a_{11}1_1 + a_{1i}1_i + a_{1\alpha}1_\alpha + a_{1\phi}1_\phi \\ & + b_{i1}i_1 + b_{ii}i_i + b_{i\alpha}i_\alpha + b_{i\phi}i_\phi \\ & + c_{\alpha1}\alpha_1 + c_{\alpha i}\alpha_i + c_{\alpha\alpha}\alpha_\alpha + c_{\alpha\phi}\alpha_\phi \\ & + d_{\phi1}\phi_1 + d_{\phi i}\phi_i + d_{\phi\alpha}\phi_\alpha + d_{\phi\phi}\phi_\phi. \end{aligned}$$

For $Я_2 = r_{jk}$ as elements of a matrix, we consider r_{11}, r_{22}, r_{33} and r_{44} in sequence

$$r_{11} = [a_{11} + c_{\alpha1} + a_{1\alpha} + c_{\alpha\alpha}]/4$$
$$r_{22} = [a_{11} - c_{\alpha1} + a_{1\alpha} - c_{\alpha\alpha}]/4$$
$$r_{33} = [a_{11} + c_{\alpha1} - a_{1\alpha} - c_{\alpha\alpha}]/4$$

and

$$r_{44} = [a_{11} - c_{\alpha1} - a_{1\alpha} + c_{\alpha\alpha}]/4.$$

For a general n-hyperintricate matrix, each term is divided by 2^n.

Substituting in sequence r_{12}, r_{21}, r_{34} and r_{43} for the above expressions, we obtain equivalent results, by substituting in the second subscript above $1 \to \phi$ and $\alpha \to i$.

Likewise for r_{13}, r_{24}, r_{31} and r_{42}, equivalent results are obtained via $a \to d$, $c \to b$, and for the first subscript $1 \to \phi$ and $\alpha \to i$.

For r_{14}, r_{23}, r_{32} and r_{41} in sequence, the substitutions are $a \to d$, $c \to b$, and for both the first and second subscripts $1 \to \phi$ and $\alpha \to i$.

To obtain a_{11}, $c_{\alpha1}$, $a_{1\alpha}$ and $c_{\alpha\alpha}$ respectively, for respective terms of r_{11}, r_{22}, r_{33} and r_{44}, there is an inverse type of relationship maintaining the signs:

$$a_{11} = [r_{11} + r_{22} + r_{33} + r_{44}]$$
$$c_{\alpha1} = [r_{11} - r_{22} + r_{33} - r_{44}]$$
$$a_{1\alpha} = [r_{11} + r_{22} - r_{33} - r_{44}]$$

and

$$c_{\alpha\alpha} = [r_{11} - r_{22} - r_{33} + r_{44}]$$

extendable to the other cases. □

3 Extricate and duplicate numbers. [Ad14b]

The linearly independent intricate basis elements of Part I, section 1 satisfy

$1^2 = 1$, $i^2 = -1$, $\alpha^2 = 1$, $\phi^2 = 1$,

$1i = i = i1$, $1\alpha = \alpha = \alpha 1$, $1\phi = \phi = \phi 1$,

$i\alpha = -\phi = -\alpha i$, $i\phi = \alpha = -\phi i$ and $\alpha\phi = i = -\phi\alpha$.

This algebra may be modified so that, for instance, all relations are maintained except for one case of the sign, which is altered to

$i\alpha = \phi = -\alpha i$.

If the resulting algebra were associative, then

$\alpha = i\phi = i(i\alpha) = (i^2)\alpha = -\alpha$

and

$-i = \phi\alpha = (i\alpha)\alpha = i\alpha^2 = i$,

so this *extricate* (in contradistinction to *intricate*) algebra is not associative, and therefore cannot be represented by a matrix.

Indeed, we may adopt the relations

$i(i\alpha) = -(i^2)\alpha$

and

$(i\alpha)\alpha = -i(\alpha^2)$.

If we apply the quaternion representation of Part I to the exquaternions, in the sense of an algebra rather than a matrix, then a representation is

$e_1 = i_1$, $e_2 = \alpha_i$, $e_3 = \phi_i$,

and the exquaternion multiplication table on elements becomes

×	***1***	***e₁***	***e₂***	***e₃***
1	1	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	-1	e_3	e_2
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1
$\boldsymbol{e_3}$	e_3	$-e_2$	$-e_1$	-1

The exquaternions do not form a division algebra, since

$(e_2 - e_3)(1 - e_1) = 0$. □

We can combine the intricate and extricate algebras to form a *duplicate* algebra. For this purpose, it is convenient under the extricate algebra to introduce basis elements written as $i_\#$ (or i# in layers) for i, $\alpha_\#$ (or α#) for α, and $\phi_\#$ or ϕ# for ϕ. The general duplicate number is now written as

$a1 + bi + c\alpha + d\phi + b'i_\# + c'\alpha_\# + d'\phi_\#$.

We need a multiplication operation for such numbers. If A and B are intricate numbers, $A_\#$ and $B_\#$ extricate numbers, define

$(A + A_\#) \times (B + B_\#) = (AB) + (A_\#B) + (A\times_\# B_\#) + (A_\#\times_\# B_\#)$,

where AB is expressed under intricate multiplication as an intricate number, $A_\#\times_\# B_\#$ under extricate multiplication $\times_\#$ as an extricate number, whereas the $A_\#B$ term is taken as an intricate multiplication with the $A_\#$ variables transformed to intricate ones, and $A\times_\# B_\#$ as extricate multiplication, with the A variables transformed to extricate ones.

We may also form the following general nonassociative algebra, for the set

$\Lambda = \{\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\}$,

where Λ_0, Λ_1, Λ_2 and Λ_3 are scalars and

$$(A + A_\#) \times_\Lambda (B + B_\#) = \Lambda_0(AB) + \Lambda_1(A_\# B) + \Lambda_2(A \times_\# B_\#) + \Lambda_3(A_\# \times_\# B_\#).$$

4 The J representation. [Ad14c]

An intricate or extricate number $p1 + qi + r\alpha + s\phi = p1 + JK$ satisfies

$$(qi + r\alpha + s\phi)^2 = (\pm qi \pm r\alpha \pm s\phi)^2 = -q^2 + r^2 + s^2.$$

When $J^2 = 0$ we obtain for J the parameterisation

$$e^\rho[\pm i \pm \cos\sigma\,\alpha \pm \sin\sigma\,\phi],$$

when $J^2 = -1$

$$\pm\cosh\rho\, i \pm \sinh\rho\cos\sigma\,\alpha \pm \sinh\rho\sin\sigma\,\phi,$$

and when $J^2 = 1$

$$\pm\sinh\rho\, i \pm \cosh\rho\cos\sigma\,\alpha \pm \cosh\rho\sin\sigma\,\phi.$$ ☐

If $J_1^2 = \pm 1$ and $J_2^2 = \pm 1$, where $J_1 = bi + c\alpha + d\phi \neq J_2 = qi + r\alpha + s\phi$, then it is possible to write

$$J_1J_2 = a + J_3f,$$

where $J_3^2 = 0$ or ± 1 and J_3 is intricate or extricate. Then $J_2J_1 = (J_1J_2)^*$, the intricate or extricate conjugate, and for intricate numbers only

$$(J_1J_2)(J_2J_1) = \pm 1 = (a + J_3f)(a - J_3f)$$
$$= a^2 \pm f^2, \text{ or } a^2 \text{ if } J_3^2 = 0.$$ ☐

5 Standard division algebras and exoctonions.

The standard complex numbers, quaternions, octonions and sedenions can be nested by inclusion and an instance is represented in the following table for the sedenions

×	***1***	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

The Cayley-Dickson construction [Ba01] can be used to form the sedenions above, which are not 'alternative' so they are not a division algebra. For the example above

$$(e_1 + e_{10})(e_{15} - e_4) = 0.$$

Representing the exquaternions of section 3 by the 4×4 block A, the exoctonion multiplication table may be represented by the blocks

A B

C D

where A or its transpose A^T may be chosen and D or its transpose D^T. An example of the complex numbers, exquaternions and exoctonions is represented in the following nested table for an instance of the exoctonions.

×	***1***	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_4}$	$\boldsymbol{e_5}$	$\boldsymbol{e_6}$	$\boldsymbol{e_7}$
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e_1}$	e_1	-1	e_3	e_2	e_5	$-e_4$	$-e_7$	e_6
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
$\boldsymbol{e_3}$	e_3	$-e_2$	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
$\boldsymbol{e_4}$	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
$\boldsymbol{e_5}$	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	e_3	e_2
$\boldsymbol{e_6}$	e_6	e_7	e_4	$-e_5$	$-e_2$	$-e_3$	-1	e_1
$\boldsymbol{e_7}$	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	$-e_1$	-1

The exoctonions do not form a division algebra, since they contain the exquaternions as a subalgebra, but the inverse of $a_0 1 + a_1 e_1 + a_2 e_2 + ...$ etc. exists and is

$$(a_0 1 - a_1 e_1 - a_2 e_2 - ... \text{ etc.})/(a_0^2 + a_1^2 + a_2^2 + ... \text{ etc.}).$$

The hyperduplicate algebra is obtained by the same process that a hyperintricate algebra is obtained from an intricate one.

6 Compression and expansion.

The *compression* of a v-hyperintricate number from $2^v \times 2^v$ matrix basis elements to $2^w \times 2^w$ basis elements, where we are compressing $v - w + 1$ vectors, consists in multiplying together in order the vectors to be compressed in the v-hyperintricate algebra.

The compression operation, κ, with abelian addition and non-commutative multiplication, satisfies for basis elements, and correspondingly for composites (we may use here real numbers r and s, although we can incorporate r and s as intricate numbers via an interior or relative coefficient algebra)

$$\kappa(rA_B) = r^2AB,$$
$$\kappa(rA_B + sC_D) = (r^2AB) + (s^2CD),$$

as may be verified using basis element universal objects.

Where B or C are 1 or B = C, we connect compression with matrix multiplication via

$$\kappa(rA_BsC_D) = \kappa(rA_B)\kappa(sC_D) \qquad (1)$$

otherwise for distinct non-real B and C, by non-commutation of basis elements we obtain

$$\kappa(rA_BsC_D) = -\kappa(rA_B)\kappa(sC_D),$$

for example

$$\kappa[(\alpha_\phi)^2] = -\kappa(\alpha_\phi)\kappa(\alpha_\phi) = -i^2 = 1.$$

The zero matrix is compressed to a zero matrix, and the unit matrix to a unit matrix. However $\kappa(rA_0) = 0$, and A_0 is 0, but $\kappa(\alpha_\alpha) = 1$ and $\alpha \neq 1$. Compression is an additive epimorphism from the v-hyperintricate algebra to the w-hyperintricate algebra.

The hyperimaginary, hyperactual and hyperphantom algebras commute, so for hyperimaginary, hyperactual and hyperphantom numbers, κ is commutative, and (1) always holds.

The equation

$$\kappa(A_BC_D) = \kappa(A_C)\kappa(B_D)$$

and the definition

$$\kappa(A_B + C_D) = \kappa(A_B) + \kappa(C_D)$$

define a type of structure similar to a ring. There is a unit:

$$\kappa(1_1C_D) = 1_1\kappa(1_C)\kappa(1_D) = 1_1\kappa(C_1)\kappa(1_D)$$

and the algebra is distributive:

$$\kappa[U_V(W_X + Y_Z)] = \kappa[U_VW_X] + \kappa[U_VY_Z].$$

There is an opposite operation, κ^{op}, called *expansion*, so that for expansion

$$\kappa^{op}(r^2AB) = rA_B,$$
$$\kappa^{op}[(r^2AB) + (s^2CD)] = (rA_B) + (sC_D).$$

Let B be an intricate number of the form

$$B = b + fJ_1,$$

where $J_1^2 = 0$ or ±1, and C be of the form

$$C = c + gJ_2,$$

where

$$J_1J_2 = d - J_2J_1.$$

Then

$$BC = bc + fcJ_1 + bgJ_2 + fgJ_1J_2$$
$$CB = bc + fcJ_1 + bgJ_2 + fg(d - J_1J_2)$$

so that for some v, w and pure intricate J_3

$$\kappa(A_BC_D) = \kappa(A_CB_D) + v\kappa(A_D) + w\kappa[(AJ_3)_D].$$

We have proved that for intricate A, B, C and D there exists a v and intricate X such that

$$\kappa(A_BC_D) = \kappa(A_CB_D) + vX, \qquad (2)$$

conversely that for given A, D a selection of B and C can be made so that for arbitrary v and X, (2) holds. □

We can extend this type of notion of compression *for intricates down to reals* by taking the determinant of the matrix basis elements. First note that the number 1 we have been using is in fact a diagonal 2 X 2 matrix. To distinguish this from its real value elements, denote the latter occasionally by 1~.

We can now compress intricate basis elements down to ±1~. We have the mappings

$$1 \rightarrow 1\sim,\ i \rightarrow 1\sim,\ \alpha \rightarrow -1\sim \text{ and } \phi \rightarrow -1\sim,$$

where we denote this compression mapping by λ, so that the determinant

$\lambda(r1 + s\alpha + t\phi + ui) = [(r + s)(r - s) - (t + u)(t - u)1\sim$
$= (r^2 - s^2 - t^2 + u^2)1\sim,$

and therefore we have proved

$\lambda(r1 + s\alpha + t\phi + ui) = \lambda(r1) + \lambda(s\alpha) + \lambda(t\phi) + \lambda(ui).$

However, for the real component, and except possibly for sign similarly for i, α and ϕ

$\lambda[(r_1 + r_2)1] = \lambda(r_11) + \lambda(r_21) + 2r_1r_21\sim = (r_1 + r_2)^21\sim.$

The expansion λ^{op} is defined in like manner to κ^{op}.

7 The hyperintricate trace, norm and layer algebra.

Let 1, $T = a1 + bi + c\alpha + d\phi$ and U, V, W be intricate matrices. The norm or determinant, det, which is additive for intricate numbers when every basis element for T is linearly independent of those for U: det(T + U) = det(T) + det(U), and trace operators, tr1 = 2a, tri = 2bi, $tr\alpha = 2c\alpha$ and $tr\phi = 2d\phi$, satisfy

det(a = 1, b = 0, c = 0, d = 0) = 1, det(a = 0, b = 1, c = 0, d = 0) = 1,
det(a = 0, b = 0, c = 1, d = 0) = -1 and det(a = 0, b = 0, c = 0, d = 1) = -1
tr1(a = 1, b = b, c = c, d = d) = 2, tri(a = a, b = 1, c = c, d = d) = 2i,
$tr\alpha$(a = a, b = b, c = 1, d = d) = 2α and $tr\phi$(a = a, b = b, c = c, d = 1) = 2ϕ
det(TU) = det(T)det(U)
tr1(T + U) = tr1(T) + tr1(U), etc.
det(tr1(T)) = tr1(T), det(tri(T)) = -tri(T)i,
det($tr\alpha$(T)) = -$tr\alpha$(T)α, det($tr\phi$(T)) = -$tr\phi$(T)ϕ
tr1(det(T)) = det(T), tri(det(T)) = 0
$tr\alpha$(det(T)) = 0 and $tr\phi$(det(T)) = 0.

The hyperintricate layer operator $\veebar$ behaves as a tensor product with extra structure, where the alternative notation $T_U = T\veebar U$ satisfies [JL09]

$tr1(T\veebar U) = tr1(T)tr1(U),$
$tri(T\veebar U) = -tri(T)tri(U)i$, etc.
$(T\veebar 1)(1\veebar U) = T\veebar U$
$det(T\veebar 1) = det(T)$
$det(1\veebar U) = det(U),$

so

$det(T\veebar U) = det(T)det(U),$

whereas the relation

$(T + U) \veebar (V + W) = (T\veebar V) + (T\veebar W) + (U\veebar V) + (U\veebar W)$

can be used with the determinant of the above for linearly independent basis elements between T and U, and also between V and W respectively, or to extend this further, if the basis elements 1, $\mathcal{J}$, $\mathcal{A}$, $\mathcal{F}$ of [Ad12] between T and U and similarly 1, $\mathcal{J}'$, $\mathcal{A}'$, $\mathcal{F}'$ for V and W are linearly independent. Then

$det[(T + U) \veebar (V + W)] = [det(T) + det(U)][det(V) + det(W)],$

the above generalised accordingly for k layers. But if T = V = 1 and U = W = i, then

$0 = det[T\veebar V + U\veebar W] \neq det(T)det(V) + det(U)det(W) = 2.$ □

8 J-diffeomorphisms [Ad14f] for non-fixed J.

When a diffeomorphism is applied to $J = bi + c\alpha + d\phi \rightarrow J + \delta J$ so that $J^2 = (J + \delta J)^2$, then

$$b = c(\partial c/\partial b) + d(\partial d/\partial b). \square$$

9 J-abelian intricate powers.

Let J_n for a variable n, where the J_n are distinct with $J_n^2 = 0$ or ± 1, satisfy

$$U_n = e^{x_1 + J_n K} + J_n e^{y_1 + J_n L}. \qquad (3)$$

Distinct U_n are non-commutative. This equation is of the most general form, since the determinant of U_n for $J_n^2 = -1$ is positive. Under $i \leftrightarrow J_n$ equivalence the determinant of $p1 + qi + r\alpha + s\phi$ is greater than zero with value $p^2 + q^2 - r^2 - s^2$ for $J_n^2 = -1$. □

Binomial expansions of (3) for real powers are abelian for fixed J, using the binomial theorem for powers of intricate numbers [Ad14d], under i, α or $\phi \leftrightarrow J$ equivalence if $J^2 \neq 0$. □

10 J-abelian hyperintricate determinants and inverses.

Bourbaki writes [Bo73] in the historical note in *Algebra* I, "Toeplitz ... makes the fundamental observation that the theory of determinants is not needed to prove the principal theorems of linear algebra". Positing a greater significance to determinants (they are hypervolumes spanned by vectors) we now introduce the J-layered approach to the hyperintricate representation, developing this to accommodate the structure of linear algebra in the J-abelian case, leaving a general discussion of determinants and inverses to section 11.

We say a hyperintricate number is J-*abelian if* U, V, ... W, *each in the general form* 9.(3), *represent the layers of an* n-*hyperintricate number* $\Sigma U_{V...W}$, *where for each layer the value of* J *is constant* (*but* J *can vary over different layers*).

The n-hyperintricate representation has 4^n independent components, but the number of independent components in a J-abelian n-hyperintricate number $U_{V...W}$ is 4n, and this is maximally incremented for n > 2 by forming the sum $\Sigma U_{V...W}$, there being c = 3n J components plus one scalar, plus $\sum_{r=0}^{n-1} \frac{n!}{r!(n-r)!}$ independent mixed J components.

For intricate numbers A, B, ... H, the determinant of $A = a1 + bi + c\alpha + d\phi$ satisfies

$$\det A = AA^*,$$

where A^* is the intricate conjugate

$$A^* = a1 - bi - c\alpha - d\phi.$$

Since the 2-hyperintricate

$$A_B C_D = (AC)_{(BD)},$$

the compression of these satisfies

$$\kappa(A_B C_D) = ACBD.$$

Thus the determinant of the above satisfies

$$\det[\kappa(A_B C_D)] = \det A \det C \det B \det D$$
$$= \det[\kappa(A_B)]\det[\kappa(C_D)].$$

Compression also satisfies

$$\kappa(E_F + G_H) = \kappa(E_F) + \kappa(G_H)$$
$$= EF + GH,$$

which may be thought of as a definition, so that κ defines a ring epimorphism.

Determinants are not additive and also in general

$$\det(E_F + G_H) \neq \det[\kappa(E_F) + \kappa(G_H)]. \ \square$$

Since

$$A_B = (A_1)(1_B),$$

it follows in this case

$$\det(A_B) = \det[\kappa(A_B)] = \det A \det B.$$

For such numbers

$$(A_B)(A^*{}_{B^*}) = (AA^*)(BB^*)1_1,$$

which implies

$$(A_B)^{-1} = (A^*{}_{B^*})/[(AA^*)(BB^*)]. \ \square$$

A second idea is to represent each J-abelian n-hyperintricate number as a sum of $\veebar$ composite layers, where each $\veebar$ layer of the composite, $1 \leq k \leq n$, is given by the intricate number $(a_{rk}1 + J_kL_{rk})$, where $J_k = b_ki + c_k\alpha + d_k\phi$ and $J_k^2 = 0$ or ±1.

We will represent this n-hyperintricate number by

$$Я_n = \sum_{r=1}^{\lceil c/4n \rceil} [\veebar (k = 1 \text{ to } n)(a_{rk}1 + J_kL_{rk})],$$

where we are using the ceiling function, $\lceil c/4n \rceil$, and the iterated composite layer operator $\veebar$.

For each layer we select the value of $J_k \in \{i, \alpha, \phi\}$, or its corresponding $\mathcal{JAF}$ format $J_k \in \{\mathcal{J}, \mathcal{A}, \mathcal{F}\}$, and with $J_k \neq \pm1$ for any layer, where the J_k are identical over r and independent over k.

We are now able to introduce, for an intricate number $X = (a1 + bJ_k) + (c1 + dJ_k)$ a type of conjugate, $X^{k\sim}$, so that

$$X^{k\sim} = (a1 - bJ_k) + (c1 - dJ_k),$$

which implies that $XX^{k\sim}$ is real. On expanding out $Я_n$, the kth layer is selected so that

$$Я_n{}^{k\sim} = \sum_{r=1}^{\lceil c/4n \rceil} [$$
$$\veebar (m = 1 \text{ to } n)(a_{rk}1 + J_kL_{rk})(\text{except } m = k \text{ with})(a_{rk}1 - J_kL_{rk})].$$

For each k, $Я_nЯ_n{}^{k\sim}$ is real in layer k.

Let $Y^1 = Я_nЯ_n{}^{1\sim}$ and $Y^k = Y^{k-1}(Y^{k-1})^{k\sim}$. We are able to form a real value from the product

$$Y^n = Я_nЯ_n{}^*,$$

where we have introduced the n-hyperintricate conjugate $Я_n{}^*$.

The mth power Δ^m of the determinant Δ of $Я_n$, is a multiplicative function. We will allocate

$$Я_n Я_n^* = \Delta g,$$

where g is a factor.

When $Я_n$ is a real number times a hyperintricate basis element, or a *JAF* format extension of this, the value of this factor is 1. If this factor is a constant it is always 1, as can be shown since Δg is a multiplicative function. But if g is not a constant, then the degree of the powers of compressed hyperintricate components in $Я_n Я_n^*$, 2^n, is not equal to the degree of Δ in its hyperintricate matrix representation, which is not the case. Thus the inverse of $Я_n$ when it exists is $Я_n^*/\Delta$. ☐

11 The general hyperintricate inverse.

If a block diagonal of a 2-hyperintricate number is $1_{Я1} + \alpha_{Я2}$, where $Я_1$ and $Я_2$ are intricate numbers, then we have represented by the matrix

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

that $A = Я_1 + Я_2$ and $D = Я_1 - Я_2$, similarly an antidiagonal $\phi_{Я3} + i_{Я4}$ for the matrix

$$\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$$

gives $B = Я_3 + Я_4$ and $C = Я_3 - Я_4$.

The determinant of a matrix A satisfies

$$\det A = \sum_\sigma (-1)^{N\sigma} \prod_{i=1}^{n} A(i, \sigma(i)),$$

where the sum is taken over all N! permutations σ, $(\sigma(1), \ldots \sigma(n))$ of the column indices 1, ... n, and where $N\sigma$ is the minimal number of pairwise transpositions needed to transform $\sigma(1), \ldots \sigma(n)$ to 1, ... n

The determinant is multiplicative:

$$(\det P)(\det Q) = \det (PQ).$$

Let A, B, C, D, X, Y and Z be square matrix sub-blocks of the same arbitrary size, and 1 be the unit diagonal matrix. Then since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B \\ C & D \end{bmatrix}, \tag{4}$$

where we can also write

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & A^{-1}BD^{-1} \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}, \tag{5}$$

we obtain from the definition of the column expansion of a determinant

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det (1 - CA^{-1}BD^{-1}))(\det D).$$

$D - CA^{-1}B$ is known as the Schur complement of A.

Equation (5) implies when D and A can be inverted

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 1 & A^{-1}BD^{-1} \\ C & 1 \end{bmatrix}^{-1} \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

We can obtain the block inverse

$$\begin{bmatrix}1 & X\\ Y & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & -X\\ -Y & 1\end{bmatrix}\begin{bmatrix}(1-XY)^{-1} & 0\\ 0 & (1-YX)^{-1}\end{bmatrix},$$

which does not exist when X is the inverse of Y, so putting $X = A^{-1}BD^{-1}$ and Y = C, by this algorithm of Boltz-Banachiewicz [Be09] the inverse of the matrix

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix},$$

given by

$$\begin{bmatrix}E & F\\ G & H\end{bmatrix},$$

when invertible in this way satisfies in terms of A^{-1} and the inverse Schur complement

$$E = (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1},$$

where we have used with $Z = BD^{-1}C$,

$$(A - Z)^{-1} = A^{-1}(1 + Z(A - Z)^{-1}),$$

the remaining entries being

$$F = -(DB^{-1}A - C)^{-1} = -A^{-1}B(D - CA^{-1}B)^{-1},$$
$$G = -(AC^{-1}D - B)^{-1} = -(D - CA^{-1}B)^{-1}CA^{-1},$$
$$H = (D - CA^{-1}B)^{-1}.$$

There exist other solutions by similar methods, not directly obtainable by the previous formulas, for instance when det A = 0, and det B and det C $\neq$ 0.

Thus for n-hyperintricate numbers this operation can be defined recursively. ☐

We can define the hyperintricate conjugate X* of a hyperintricate number X by the formula

$$XX^* = \det X,$$

and this works for an equivalence class of X* when X is singular, otherwise

$$X^* = X^{-1} \det X.$$ ☐

12 The matrix symmetriser classification.

In 1910 Frobenius [Fr68] proved that any matrix is the product of two symmetric ones. We seek to understand the classification of these symmetric matrices.

Proposition. *Any hyperintricate number may be represented by a sum of separate* J-*abelian components, the conjuction of which need not be* J-*abelian.*

Proof. Every basis element is separately J-abelian. ☐

Lemma. (a) *If a matrix is symmetric the layers are composed each of basis element of type* i *only, multiplied by a real factor, for an even number of layers, or they are of the form* $a1 + c\alpha + d\phi$.

Proof. If an i layer was of the form $a1 + bi + c\alpha + d\phi$, then for each J-abelian part in its minimal summation, this could be expanded out to contain terms with an odd number of i's, which is antisymmetric. ☐

(b) *If a matrix is antisymmetric, the layers are basis elements of type* i *multiplied by a real factor for an odd number of layers. The remainder are of the form* a1 + cα + dϕ.

Proof. Similar.

Lemma. *Let* W *be a symmetric intricate number. Then* W^{-1} *is symmetric if it exists.*
Proof. Let W = w1 + w″α + w‴ϕ, so

$$W^{-1} = (w1 - w''\alpha - w'''\phi)/(w^2 - w''^2 - w'''^2),$$

is symmetric. □

Theorem. *Let* W *be a symmetric matrix,* W′ *be an antisymmetric matrix and* X *an arbitrary matrix. Then if* X^T *is the transpose of* X, X^TWX *is symmetric and* $X^TW'X$ *is antisymmetric.*

Proof. A symmetric matrix satisfies $A = A^T$, an antisymmetric matrix $A' = -A'^T$ and the transpose is contravariant (inverts the order of composition), so that

$$(AB)^T = B^TA^T,$$

where we can show, with a similar sort of demonstration for an antisymmetric matrix

$$X^TWX = (X^TWX)^T = X^TW(X^T)^T. \ \square$$

Corollary. *Let* A *and* D *be symmetric intricate numbers and* $C = B^T$. *Then what is known from section* 11 *as the inverse of the Schur complement*

$$S_c = (D - CA^{-1}B)^{-1}$$

is symmetric. □

Remark. The theorem implies that the cube of a symmetric matrix is symmetric and the cube of an antisymmetric matrix is antisymmetric. For squares, on considering the layers of the hyperintricate representation, in each layer the square of a basis element is real, and the product of non-real basis elements anticommutes. □

Theorem. (Frobenius, 1910). *Let* M, M′ *be* m × m *symmetric matrices to be determined. Then any* m × m *matrix* P *may be represented by* MM′.

We will first establish this theorem in the case $m = 2^n = 2$.

Proof. To demonstrate the case for n = 1, let

$$P = q1 + ri + t\alpha + u\phi,$$

and

$$L = a1 + c\alpha + d\phi.$$

Then

$$L^{-1} = (a1 - c\alpha - d\phi)/(a^2 - c^2 - d^2),$$

and

$$P = (PL^{-1})L,$$

where PL^{-1} is symmetric, that is

$$PL^{-1} = [(qa - tc - ud)1 - (td - uc)i - rd\alpha + rc\phi]/(a^2 - c^2 - d^2)$$

is symmetric, so td – uc = 0 and $(a^2 - c^2 - d^2) \neq 0$. □

For the case of general n, in the expansion of separate J-abelian terms, a layer term in the summation may be represented by the product of two separate J-abelian numbers L and L′, where

$L = a1 + c\alpha + d\phi,$
$L' = a'1 + c'\alpha + d'\phi,$
so that $LL' = (aa' + cc' + dd')1 + (cd' - c'd)i + (ac' + a'c)\alpha + (ad' + a'd)\phi.$

If each of the terms in L are fixed, multiplied by a scalar factor f for each layer, so that f, a, c and d are fixed within the term, then fLL′ represents an arbitrary intricate number. Then if f, a′, c′, d′ vary over each term, this represents a general hyperintricate number. ☐

We now consider the example n = 2.

Example. *Say* $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *is symmetric, where* A, B, C *and* D *are intricate matrices, then* A *and* D *are symmetric*, *and* B is *the transpose of* C:

$B = C^T,$

with L^{-1} *represented in the case* n = 2 *as follows.*

From section 11, $L^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$, where

$E = A^{-1} + A^{-1}BS_cB^TA^{-1},$
$H = S_c,$

and since A^{-1} and S_c are symmetric

$A^{-1} = (A^{-1})^T,$
$S_c = S_c^T$

so that by the lemma E and H are symmetric, whereas

$F = -A^{-1}BS_c,$
$G = -S_cB^TA^{-1}.$

Indeed

$(NP)^T = (n_{ij}p_{jk})^T = (p_{kj})(n_{ji}) = P^TN^T,$

so a transpose being contravariant, satisfies

$(NN^{-1})^T = (N^{-1})^TN^T = 1,$

giving

$(N^{-1})^T = (N^T)^{-1},$

which implies

$F^T = G.$ ☐

We will represent a general 4 × 4 matrix by

$V = \begin{bmatrix} Q & R \\ T & U \end{bmatrix},$

so that we wish to prove that there exists a

$V = (VL^{-1})L,$

$VL^{-1} = \begin{bmatrix} QE + RG & QF + RH \\ TE + UG & TF + UH \end{bmatrix},$

where VL^{-1} is symmetric under a suitable configuration of L. What are the constraints on E, F, H and G, so this holds for arbitrary Q, R, T and U?

A and D, being diagonal, have no i component, whereas if B does not contain i times a factor then nor does C, and if B contains i times a factor, then C has the same factor times –i (the transpose i^T of i is –i), so that

$A^{-1} = (a1 - a'\alpha - a''\phi)/(a^2 - a'^2 - a''^2),$
$S_c = (D - CA^{-1}B)^{-1},$

where if $K = k1 + k'i + k''\alpha + k'''\phi$,

$$K^{-1} = (k1 - k'i - k''\alpha - k'''\phi)/(k^2 + k'^2 - k''^2 - k'''),$$

when this exists. The transpose

$$K^T = k1 - k'i + k''\alpha + k'''\phi.$$

Now

$$QE + RG = (q1 + q'i + q''\alpha + q'''\phi)(e1 + e''\alpha + e'''\phi) + (r1 + r'i + r''\alpha + r'''\phi)(g1 + g'i + g''\alpha + g'''\phi),$$

is symmetric, so there is no i term, that is

$$q'e + q''e''' - q'''e'' + rg' + r'g + r''g''' - r'''g'' = 0,$$

and also

$$TF + UH = (t1 + t'i + t''\alpha + t'''\phi)(g1 - g'i + g''\alpha + g'''\phi) + (u1 + u'i + u''\alpha + u'''\phi)(h1 + h''\alpha + h'''\phi),$$

is also symmetric, consequently

$$-tg' + t'g + g''g''' - g'''g'' + u'h + u''h''' - u'''h'' = 0,$$

whereas

$$QF + RH = (q1 + q'i + q''\alpha + q'''\phi)(g1 - g'i + g''\alpha + g'''\phi) + (r1 + r'i + r''\alpha + r'''\phi)(h1 + h''\alpha + h'''\phi),$$

and

$$TE + UG = (t1 + t'i + t''\alpha + t'''\phi)(e1 + e''\alpha + e'''\phi) + (u1 + u'i + u''\alpha + u'''\phi)(g1 + g'i + g''\alpha + g'''\phi),$$

where these two equations are mutually transpose, so on equating intricate parts

$$qg + q'g' + q''g'' - q'''g''' + r''h'' - r'''h''' = te + t''e'' - t'''e''' + ug - u'g' + u''g'' - u'''g''',$$

for the i terms we obtain

$$qg' + q'g + q''g''' - q'''g'' + r'h + r''h''' - r'''h'' = -[t'e + t''e''' - t'''e'' + ug' - u'g + u''g''' - u'''g''],$$

for the α terms

$$qg'' + q''g + q'g''' + q'''g' + rh'' + r''h + r'h''' = te'' + t''e - t'e''' + ug'' - u''g + u'g'' - u'''g',$$

and for the ϕ terms

$$qg''' + q'''g - q'g'' - q''g' + rh''' + r'''h - r'h''' + r'''h'' = te''' + t'''e - t'e'' + ug''' - u'''g - u'g'' - u''g'.$$

We now have six independent bilinear equations in the 10 variables in total for E, H and G, and the constraints on VL^{-1} can be satisfied. □

13 The 𝒥𝒜ℱ spectral 𝔻1 exponential algebra.

The historical process by which exponential algebra 𝔻1 was selected is delineated in [Ad14e]. The 𝒥𝒜ℱ ring transformations map i → 𝒥, α → 𝒜 and ϕ → ℱ, but as explained in [Ad14e] do not provide an invariant description of 𝔻1 exponential and superexponential algebras. There is a spectrum of possible 𝔻1 algebras which may be chosen, and we will present the 𝔻1 exponential algebra spectrally, under the understanding that some specific lower 𝔻1 𝒥𝒜ℱ basis must be chosen throughout to maintain consistency.

For a hyperintricate number

$$\sum_j[\veebar_k(a_{kj} + b_{kj}J_{kj})] = \sum_j[\veebar_k e\uparrow(\rho_{kj} + \sigma_{kj}J_{kj})],$$

where $J_{kj}^2 = 0$ or ±1, we evaluate

$$[e\uparrow(\rho + \sigma J)]\uparrow(\rho' + \sigma' J')$$

as the lower $\mathbb{D}1$ exponential algebra expression

$$e\uparrow(\rho\rho' + \sigma\rho' J + \sigma'\rho J' + \sigma\sigma' J),$$

and the Σ terms by a binomial expansion under this rule. Note that the last term for the exponentiated sum is $\sigma\sigma' J$ and not $\sigma\sigma' JJ'$ – this is the specific difference between the $\mathbb{D}1$ exponential algebra, which is minimally branched, and the standard exponential algebra.

As a $\mathcal{JAF}$ ring we select the lower expansion

$$e^v = [e\uparrow(r + s\mathcal{J} + t\mathcal{A} + u\mathcal{F})]\uparrow(r' + s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}'),$$

where

$$v = rr' + r(s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}') + r'(s\mathcal{J} + t\mathcal{A} + u\mathcal{F})$$
$$+ (ss' + ts' + us')\mathcal{J} + (ts' + tt' + tu')\mathcal{A} + (us' + ut' + uu')\mathcal{F}. \ \square$$

Reference [Ad14e] presents the $\mathbb{D}1$ algebra as a proposal. The configurations of possible exponential algebras need to be classified completely. $\mathbb{D}n$, n = 1, 2, 3 or 4 satisfies $i^i = i^n$. Independence results indicate $\mathbb{D}1$ is consistent, and I leave to the reader the task of trying to prove this.

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