Polynomial equations II

Transcendental solutions

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Abstract. We demonstrate that there exist transcendental solutions to complex polynomial equations of arbitrary large degree, firstly only for the modulus, on adjoining roots to convert any complex polynomial to a real one, then by splitting the roots into real and imaginary parts, for any complex polynomial. Galois theory specifies that there are no solutions for degree $> 4$ by radicals. In [Ad16] we provide a pedagogic introduction to the convergent solution of polynomial equations using the matrix QR algorithm.

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1. Introduction.

In this work, we demonstrate that there exist transcendental solutions for the modulus $x x^*$ of a complex polynomial equation of arbitrary large degree in a variable $x$, where $x^*$ is the complex conjugate. Galois theory specifies that there are no solutions for degree $> 4$ by radicals. We are not challenging this.

We assume the fundamental theorem of algebra. We introduce Kronecker’s theorem to determine the range over which the roots span, when $x$ is real.

In theorem 2.2, without losing essential information, we adjoin roots to convert a polynomial to one with entirely real coefficients.

In the case of theorem 2.3, on considering real and imaginary parts of $x$, a complex polynomial is converted to a real one in the real variable $x x^*$.

Then, in section 2.3 we apply what is essentially Newton’s method to obtain by descent transcendental solutions of the modulus, $x x^*$, of any polynomial in complex $x$ and complex coefficients.

In section 2.4, we separate out a polynomial into its real and complex parts. These two equations enable the real part of $x = y + zi$ to be specified. However, the resulting equation for $y$ and $n = 2$ is in double the degree of the original polynomial. This means that as well as the real solutions for $y$ and $z$, which must exist, if $x$ does not contain duplicate solutions then other $y$ and $z$ must be complex. Since the real solutions do exist, we can still use Newton’s method on the real zeros, however.

In [Ad14] we have introduced ladder numbers as a replacement for the reals. However, for numbers of the form $a_1 + bi$, we will continue to use the phrase real part, for $a_1$.

2. Transcendental solutions to polynomial equations.

2.1. Kronecker’s theorem.

The following proof, due to Kronecker (Crelle 101, p. 347), is given in Netto [Ne1892].

**Theorem 2.1.** Let

$$E(x) = \sum_{n=0}^{m} (a_n x^n).$$

where $E(x)$ is not identically zero. There is a value $\xi$ such that when every value of $x$, with $x^*$ the complex conjugate of $x$, has absolute value $|x| = +\sqrt{x x^*}$ greater than $\xi$, then the function $E(x)$ is different from zero.

**Proof.** This follows immediately from (1) in the form

$$E(x) = \prod_{n=1}^{m} (x + p_n),$$

for which the maximum value of $|p_n|$ may be chosen.
However, our objective is to obtain the result as follows. Let \( a_k \) be the numerically greatest of the \( m \) coefficients \( a_0, a_1, \ldots, a_{m-1} \) in (1), and denote \( \frac{|a_k| + |a_m|}{|a_m|} \) by \( r \). We have then

\[
\frac{a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_0}{a_m} \leq \frac{a_k}{a_m} \left( |x|^{m-1} + |x|^{m-2} + \cdots + 1 \right) \\
\leq \frac{a_k}{a_m} \left( |x|^{m-1} \right) \\
\leq \frac{|r - 1| (|x|^{m-1})}{|x| - 1}.
\]

Hence, for any value of \( x \) not lying between \(-r\) and \( r\)

\[
\frac{a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_0}{a_m} \leq |x|^m
\]

\[
|a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_0| \leq |a_m x^m|,
\]

so that if \( x \) is real and choosing \( a_m = 1 \), the sign of \( E(x) \) is the same as that of \( x^m \). \( \square \)

2.2. Conversion of polynomials to real roots and coefficients.

**Theorem 2.2.** If equation 2.1.(1) holds, then when multiplied out

\[
F(x) = \left[ \sum_{n=0}^{m} (a_n x^n) \right]\left[ \sum_{n=0}^{m} (a_n^* x^n) \right] = 0
\]

has all real coefficients.

**Proof.**

\[
F(x) = \prod_{n=1}^{m} (x + p_n)(x + p_n^*),
\]

and for each \( n \)

\[
x^2 + (p_n + p_n^*)x + p_n p_n^*
\]

has real coefficients. \( \square \)

**Theorem 2.3.** Let \( m, n \in \mathbb{N}, \ m \geq n, \) and \( a_n \) and \( x \) be ladder numbers, with \( a_m = 1 \). Suppose

\[
G(x) = \sum_{n=0}^{m} (a_n x^n) = 0,
\]

then if \( a_n^* \) is the complex conjugate of \( a_n \) and \( x^* \) is the conjugate of \( x \), for some real \( K_n \)

\[
\sum_{n=0}^{m} (-1)^n (a_n a_n^* - K_n) (xx^*)^n = 0.
\]

**Proof.** Let \( x = y + zi \) and \( p_n = q_n + r_n i \). The root

\[
x + p_n = 0
\]

may be written in the equivalent form for real and imaginary parts

\[
x^* + p_n^* = 0
\]

and hence

\[
x x^* - (p_n p_n^*) = 0.
\]

Thus we derive effectively duplicate solutions under the transformations

\[
x \to -x x^* \\
p_n \to (p_n p_n^*)
\]

04.3
The fundamental theorem of algebra states that there exists a bijection between $G(x)$ in the form (2) and
\[ G(x) = \prod_{n=1}^{m} (x + p_n), \tag{6} \]
so that precisely the same solutions (duplicated) exist by (5) for the polynomial
\[ G(x) = \prod_{n=1}^{m} (-xx^* + (p_n p_n^*)), \tag{7} \]
and this bijection gives a solution of the form (3). For example, if
\[ G(x) = (-xx^* + pp^*)(-xx^* + qq^*) = 0, \tag{8} \]
then if $A = p + q$ and $B = pq$,
\[ (xx^*)^2 - (AA^* - 2\sqrt{BB^*})(xx^*) + BB^* = 0, \tag{9} \]
where $K_1 = 2\sqrt{BB^*}$, but for degree $> 4$ all the real values $K_n$ cannot be determined by radicals, even though they exist. \[\square\]

2.3. Transcendental determination of root moduluses for any complex polynomial.

For a polynomial in real variables and coefficients, denote
\[ H(x) = \sum_{n=0}^{m} (-1)^n (b_n b_n^*)(xx^*)^n = 0. \tag{1} \]
A real polynomial is asymptotically divergent outside the region $x = \pm \xi$, given by Kronecker’s theorem.

If we obtain the maximums and minimums of $H(x)$, then an approximation to the interior zeros of $H(x)$ is given, for real $x$, by the intersection of the straight line connecting adjacent maximums and minimums with the $x$ axis. For extremal zeros, the approximation to a zero can be given by the intersection connecting the straight line defined by $H(-\xi)$ and the first maximum or minimum on the left with the $x$ axis, otherwise from the straight line with a point on the last maximum or minimum on the right joining $H(+\xi)$, and its intersection with the $x$ axis.

If $x$ is real or complex, then
\[ \Delta x^m = mx^{m-1} \text{ if } m \geq 1, \text{ and if not, } \Delta x^m = 0. \]
For a curve $H(x) = 0$ of degree $m$, $\Delta H(x) = 0$, at which the maximums and minimums reside, is of degree $(m - 1)$ in $\Delta H(x)$, whereas $\Delta^2 H(x) = 0$ is of degree $(m - 2)$. Then between each maximum and minimum, there exists at most one $\Delta^2 H(x) = 0$, otherwise the total degree of $\Delta H(x)$ exceeds $(m - 1)$. So on the connecting straight line between an adjacent maximum and minimum, there can be at most one intersection with the curve $H(x) = 0$.

Thus, if the maximums and minimums can be determined, by the monotone convergence theorem the zero approximation located at $C$ between maximum $A$ and adjacent minimum $B$

![Figure 2.4.1](image-url)
can be improved by finding $H(x)$ at D in the diagram above, and the tangent at D which hits
the x axis at E. From the mid-point of C and E (at, say, F) use F as the new C. Provided this
results in an improvement at the first attempt, this method converges to the zero, otherwise
choose j attempts, where the new F is $F_j = (E + jC)/(j + 1)$. Because the polynomial is finite,
all derivatives are finitely bounded, and the value of j is finite.

Since the equation $\Delta H(x) = 0$ is of lower degree and x and its coefficients in $\Delta H(x)$ are real,
the maximums and minimums may be determined recursively by the same method. □

2.4. Transcendental solutions of complex polynomials in the general case.

Let us look at the quadratic polynomial

$$(x + 1 + i)(x + 2 + 3i) = 0,$$  

(1)

where $x = y + z$. Then equating real and complex parts to zero separately

$$y^2 - z^2 + 3y - 4z - 1 = 0,$$

(2)

$$2yz + 4y + 3z + 5 = 0$$

(3)

and on substituting

$$z = -(4y + 5)/(2y + 3),$$

(4)

from (3) into (2) we obtain

$$4y^4 + 24y^3 + 57y^2 + 63y + 26 = 0,$$

(5)

where the solutions

$$(y + 1)(y + 2) = 0,$$

(6)

clearly satisfy (5). What are the extra roots? Putting

$$(y + 1)(y + 2)(ay^2 + by + c) = 0,$$

(7)

and equating to (5) gives

$$a = 4, b = 12 \text{ and } c = 13,$$

(8)

so that additional solutions are

$$y = \frac{-3}{2} \pm i, \quad z = -2 \mp \frac{1}{2}i,$$

(9)

which also satisfy (5) and (1).

Consider again the equation

$$F(x) = \sum_{n=0}^{m} (a_n x^n) = 0,$$

(10)

in which the ladder numbers $x = y + z$ and $a_n = b_n + c_n i$, with $y$ and $b_n$ real parts and $zi$ and
$c_n i$ imaginary parts. Then by the binomial theorem

$$x^n = (y + zi)^n = y^n + my^{n-1}(zi) + ... + \frac{n!}{k!(n-k)!}y^{n-k}(zi)^k + ... + (zi)^n,$$

so the real part of $F(x)$ can be compared with the imaginary part, and by descent $z$ can be
eliminated, giving an equation for $y$, which may be solved by the method already given,
determining both $y$ and $z$ transcendentally.

Note that for $F(x) = 0$ the number of distinct real values of $y$ is $n$, and the number of real
distinct maximums and minimums given by $\Delta F(x) = 0$ is bounded, and thus the equation in $y$
is monotonically either increasing or decreasing between each adjacent pair of these for
which their connecting straight lines intersect the x axis at an x value between them. □
References


