Polynomial equations II

Transcendental solutions

Jim H. Adams 24th February 2014 © 2014

Abstract. We demonstrate that there exist transcendental solutions to complex polynomial equations of arbitrary large degree, firstly only for the modulus, on adjoining roots to convert any complex polynomial to a real one, then by splitting the roots into real and imaginary parts, for any complex polynomial. Galois theory specifies that there are no solutions for degree > 4 *by radicals*. In [Ad16] we provide a pedagogic introduction to the convergent solution of polynomial equations using the matrix QR algorithm.

Table of Contents

1. Introduction.

In this work, we demonstrate that there exist transcendental solutions for the modulus xx* of a complex polynomial equation of arbitrary large degree in a variable x, where x* is the complex conjugate. Galois theory specifies that there are no solutions for degree > 4 *by radicals*. We are not challenging this.

We assume the fundamental theorem of algebra. We introduce Kronecker's theorem to determine the range over which the roots span, when x is real.

In theorem 2.2, without losing essential information, we adjoin roots to convert a polynomial to one with entirely real coefficients.

In the case of theorem 2.3, on considering real and imaginary parts of x, a complex polynomial is converted to a real one in the real variable xx*.

Then, in section 2.3 we apply what is essentially Newton's method to obtain by descent transcendental solutions of the modulus, xx*, of any polynomial in complex x and complex coefficients.

In section 2.4, we separate out a polynomial into its real and complex parts. These two equations enable the real part of x = y + zi to be specified. However, the resulting equation for y and n = 2 is in double the degree of the original polynomial. This means that as well as the real solutions for y and z, which must exist, if x does not contain duplicate solutions then other y and z must be complex. Since the real solutions do exist, we can still use Newton's method on the real zeros, however.

In [Ad14] we have introduced ladder numbers as a replacement for the reals. However, for numbers of the form a1 + bi, we will continue to use the phrase *real part*, for a1.

2. Transcendental solutions to polynomial equations.

2.1. Kronecker's theorem.

The following proof, due to Kronecker (Crelle 101, p. 347), is given in Netto [Ne1892].

Theorem 2.1. *Let*

$$E(x) = \sum_{n=0}^{m}(a_n x^n). \qquad (1)$$

where E(x) *is not identically zero. There is a value* ξ *such that when every value of* x, *with* x* *the complex conjugate of* x, *has absolute value* $|x| = +\sqrt{xx^*}$ *greater than* ξ, *then the function* E(x) *is different from zero.*

Proof. This follows immediately from (1) in the form

$$E(x) = \prod_{n=1}^{m}(x + p_n),$$

for which the maximum value of $|p_n|$ may be chosen.

However, our objective is to obtain the result as follows. Let a_k be the numerically greatest of the m coefficients a_0, a_1, ... a_{m-1} in (1), and denote $\frac{|a_k| + |a_m|}{|a_m|}$ by r. We have then

$$\left|\frac{a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_0}{a_m}\right| \leq \left|\frac{a_k}{a_m}\right| (|x|^{m-1} + |x|^{m-2} + \ldots + 1)$$

$$\leq \left|\frac{a_k}{a_m}\right| \frac{(|x|^m - 1)}{|x| - 1}$$

$$\leq |r - 1| \frac{(|x|^m - 1)}{|x| - 1}.$$

Hence, for any value of x not lying between –r and r

$$\left|\frac{a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_0}{a_m}\right| \leq |x^m|,$$

$$\left|a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_0\right| < |a_m x^m|,$$

so that if x is real and choosing $a_m = 1$, the sign of E(x) is the same as that of x^m. □

2.2. Conversion of polynomials to real roots and coefficients.

Theorem 2.2. *If equation* 2.1.(1) *holds, then when multiplied out*

$$F(x) = [\textstyle\sum_{n=0}^{m}(a_n x^n)][\sum_{n=0}^{m}(a_n^* x^n)] = 0 \tag{1}$$

has all real coefficients.

Proof.

$$F(x) = \textstyle\prod_{n=1}^{m}(x + p_n)(x + p_n^*),$$

and for each n

$$x^2 + (p_n + p_n^*)x + p_n p_n^*$$

has real coefficients. □

Theorem 2.3. *Let* m, n ∈ ℕ, m ≥ n, *and* a_n *and* x *be ladder numbers, with* $a_m = 1$. *Suppose*

$$G(x) = \textstyle\sum_{n=0}^{m}(a_n x^n) = 0, \tag{2}$$

then if a_n^* *is the complex conjugate of* a_n *and* x^* *is the conjugate of* x, *for some real* K_n

$$\textstyle\sum_{n=0}^{m}(-1)^n(a_n a_n^* - K_n)(xx^*)^n = 0. \tag{3}$$

Proof. Let x = y + zi and $p_n = q_n + r_n i$. The root

$$x + p_n = 0$$

may be written in the equivalent form for real and imaginary parts

$$x^* + p_n^* = 0$$

and hence

$$xx^* - (p_n p_n^*) = 0. \tag{4}$$

Thus we derive effectively duplicate solutions under the transformations

$$x \to -xx^*$$

$$p_n \to (p_n p_n^*). \tag{5}$$

The fundamental theorem of algebra states that there exists a bijection between G(x) in the form (2) and

$$G(x) = \prod_{n=1}^{m}(x + p_n), \qquad (6)$$

so that precisely the same solutions (duplicated) exist by (5) for the polynomial

$$G(x) = \prod_{n=1}^{m}\left(-xx^* + (p_n p_n^*)\right), \qquad (7)$$

and this bijection gives a solution of the form (3). For example, if

$$G(x) = (-xx^* + pp^*)(-xx^* + qq^*) = 0, \qquad (8)$$

then if A = p + q and B = pq,

$$(xx^*)^2 - (AA^* - 2\sqrt{BB^*})(xx^*) + BB^* = 0, \qquad (9)$$

where $K_1 = 2\sqrt{BB^*}$, but for degree > 4 all the real values K_n cannot be determined by radicals, even though they exist. □

2.3. Transcendental determination of root moduluses for any complex polynomial.

For a polynomial in real variables and coefficients, denote

$$H(x) = \sum_{n=0}^{m}(-1)^n(b_n b_n^*)\,(xx^*)^n = 0. \qquad (1)$$

A real polynomial is asymptotically divergent outside the region x = ±ξ, given by Kronecker's theorem.

If we obtain the maximums and minimums of H(x), then an approximation to the interior zeros of H(x) is given, for real x, by the intersection of the straight line connecting adjacent maximums and minimums with the x axis. For extremal zeros, the approximation to a zero can be given by the intersection connecting the straight line defined by H(-ξ) and the first maximum or minimum on the left with the x axis, otherwise from the straight line with a point on the last maximum or minimum on the right joining H(+ξ), and its intersection with the x axis.

If x is real or complex, then

$$\Delta x^m = mx^{m-1} \text{ if } m \geq 1, \text{ and if not, } \Delta x^m = 0.$$

For a curve H(x) = 0 of degree m, ΔH(x) = 0, at which the maximums and minimums reside, is of degree (m – 1) in ΔH(x), whereas $\Delta^2H(x) = 0$ is of degree (m – 2). Then between each maximum and minimum, there exists at most one $\Delta^2H(x) = 0$, otherwise the total degree of ΔH(x) exceeds (m – 1). So on the connecting straight line between an adjacent maximum and minimum, there can be at most one intersection with the curve H(x) = 0.

Thus, if the maximums and minimums can be determined, by the monotone convergence theorem the zero approximation located at C between maximum A and adjacent minimum B

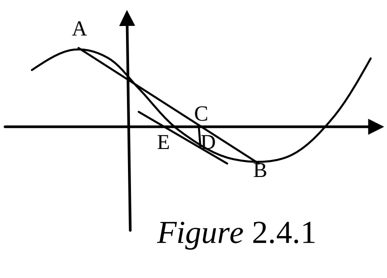

Figure 2.4.1

can be improved by finding H(x) at D in the diagram above, and the tangent at D which hits the x axis at E. From the mid-point of C and E (at, say, F) use F as the new C. Provided this results in an improvement at the first attempt, this method converges to the zero, otherwise choose j attempts, where the new F is $F_j = (E + jC)/(j + 1)$. Because the polynomial is finite, all derivatives are finitely bounded, and the value of j is finite.

Since the equation $\Delta H(x) = 0$ is of lower degree and x and its coefficients in $\Delta H(x)$ are real, the maximums and minimums may be determined recursively by the same method. □

2.4. Transcendental solutions of complex polynomials in the general case.

Let us look at the quadratic polynomial

$$(x + 1 + i)(x + 2 + 3i) = 0, \quad (1)$$

where x = y + zi. Then equating real and complex parts to zero separately

$$y^2 - z^2 + 3y - 4z - 1 = 0, \quad (2)$$

$$2yz + 4y + 3z + 5 = 0 \quad (3)$$

and on substituting

$$z = -(4y + 5)/(2y + 3), \quad (4)$$

from (3) into (2) we obtain

$$4y^4 + 24y^3 + 57y^2 + 63y + 26 = 0, \quad (5)$$

where the solutions

$$(y + 1)(y + 2) = 0 \quad (6)$$

clearly satisfy (5). What are the extra roots? Putting

$$(y + 1)(y + 2)(ay^2 + by + c) = 0 \quad (7)$$

and equating to (5) gives

$$a = 4,\ b = 12 \text{ and } c = 13, \quad (8)$$

so that additional solutions are

$$y = \frac{-3}{2} \pm i, \qquad z = -2 \mp \frac{1}{2}i, \quad (9)$$

which also satisfy (5) and (1). □

Consider again the equation

$$F(x) = \sum_{n=0}^{m}(a_n x^n) = 0, \quad (10)$$

in which the ladder numbers x = y + zi and $a_n = b_n + c_n i$, with y and b_n real parts and zi and $c_n i$ imaginary parts. Then by the binomial theorem

$$x^n = (y + zi)^n = y^n + my^{n-1}(zi) + \ldots + \frac{n!}{k!(n-k)!}y^{n-k}(zi)^k + \ldots + (zi)^n,$$

so the real part of F(x) can be compared with the imaginary part, and by descent z can be eliminated, giving an equation for y, which may be solved by the method already given, determining both y and z transcendentally.

Note that for F(x) = 0 the number of distinct real values of y is n, and the number of real distinct maximums and minimums given by $\Delta F(x) = 0$ is bounded, and thus the equation in y is monotonically either increasing or decreasing between each adjacent pair of these for which their connecting straight lines intersect the x axis at an x value between them. □

References

Ad14 J.H. Adams, *Innovation in mathematics*, chapter II, to appear.

Ad16 J.H. Adams, *Superexponential algebra*, chapter XI, 2015.

Ne1892 E. Netto, *Theory of substitutions*, Chelsea reprint.

New1707 I. Newton, *Arithmetica Universalis*, 1707, available in *The mathematical papers of Isaac Newton*, vol 5, 1683-84, particularly from p 351, ed. D.T. Whiteside, Cambridge U. P., 1967-82.

Uh01 F. Uhlig, *Transform linear algebra*, Pearson, 2001.