Polynomial equations I

Duplicate roots

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Abstract. We prove that the sextic containing roots $(x + a)^2$ or $(x + a)(x – a)$ is solvable. We introduce the non degree conserving differential operator Δ, by which we obtain the duplicate roots solution. We reiterate that if the composition series criterion of Galois theory acts as a model for the solution of polynomial equations, the quintic is not solvable by radicals.

We introduce a leaped Jordan-Hölder series

$$S_6 \rhd G_3 \rhd G_2 \rhd 1,$$

for the sextic, where the algebra uses multifunctions, but limit further discussion on this approach.

The situation for matrix varieties will be described in [Ad14c].

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1 Prologue

Jim thinks a lot – but he does not know when to stop! – Tim Gibbs

Galois theory contains the most famous no-go theorem in mathematics, concretely that there is no direct solution using radicals of a general polynomial of degree 5, the quintic. It is a theory of radicals, but not comprehensively about all types of solution. Historically Galois theory led to the development of the theory of groups, which describe symmetry operations, and some of its generalisations are pervasive in modern algebra, combining the theory of groups with the theory of ordered structures called lattices – the theory of Galois connections.

Asserted in Galois theory is a definition of radical solvability (the inversion problem) employing normal subgroups and describing a descending chain condition called a Jordan-Hölder series, which models a descending series of solutions, in steps from the equation of highest degree down to a set of linear solutions. In effect, the concrete manifestation of this technique is that the polynomial

$$F(x) = \sum_{m=0}^{n} a_m x^m = 0 \qquad (1)$$

is reduced to a series of solutions of cyclotomic equations

$$x^k - b_k = 0$$

for $0 < k \leq n$, in which the intervening coefficient terms have been eliminated, so that the reduction

$$x = \omega_k \sqrt{b_k}$$

where ω_k is a kth root of unity, is possible.

In modern terms [Ar59], the automorphisms of the roots are the permutations of n objects, the group being called the symmetric group, denoted by S_n. A normal subgroup N of the group S_n satisfies

$$N * g \equiv g * N$$

where $g \in S_n$, so that the cosets $N * g$ are partitioned into equivalence classes. For S_5 there are no group mappings, called homomorphisms, to S_4, so there is no Jordan-Hölder series

$$S_5 \rhd G_4 \rhd G_3 \rhd G_2 \rhd 1,$$

describing the descent of a series of solutions from the quintic to the quartic, to the cubic, to the quadratic, to the linear set of solutions. Polynomials are not groups, which have only one operation, group multiplication, but rings with + and ×, but does this alter the situation? A product of roots has the right zeros, and a general ring theory can be developed [MC64].

The common interpretations of the Galois theory of polynomial rings may be extended, specifically in two ways which we will meet in this paper.

Firstly, the group-theoretic backdrop is contingent on the assumption that there are no other transformations decrementing the degree of a general polynomial equation. However, it is possible to introduce a formal operator Δ acting on polynomials with complex values and complex coefficients, which is an analogue of differentiation in the usual sense. The existence of this operator is significant.

That such techniques have historically implicitly been employed is evident from the Newton-Raphson method which obtains convergent approximations to a general polynomial in the regions of its zeros [New1707], in which the solution space may be searched by an algorithm, where the maximum difference between roots, given in [Ne1892], and the minimum non-zero difference provide a search net in which the zeros of the polynomial may be located, uses the tangent to the curve, and this can be obtained by differentiation [Ad14a]. There is no direct indication from the existence of this method, that a solution by radicals may be obtained.

It is a remarkable fact that a sextic polynomial and polynomials of lower degree containing duplicate roots are solvable. Further, that the equation multiplying $(x^2 - a^2)$ and a quartic, for instance, and for the previous example of duplicates, may be solved not only in this multiplicative form but also as the additive polynomial of equation (1).

The criterion for duplicate roots is that F(x) and ΔF(x) are both zero at the duplicate root. The existence of these two solutions means that they may be compared and a descending chain of equations of successively lower degree may be formed, the penultimate containing the duplicate root, and in the final reduction giving a constraint on the coefficients.

For the quintic, on adjoining an arbitrary root (x + a), and specifying the condition ΔF(x) = 0 at (x + a), the equation cannot be solved directly by radicals as a duplicate root equation. Further, the quintic cannot be appended a root (x – a) and this be treated for the sextic as being part of the pair (x – a)(x + a), and solved by this method, and the coefficients of the sextic cannot be made to provide the coefficients of the quintic so that the solution of the quintic is expressed in terms of its own coefficients.

We note also here that we are also using an initially ascending series to a subgroup of S_6

$$S_5 \lhd G_6,$$

which by a theorem of Schreier as an end result incorporates a direct uniformly descending series. If the degree n of the maximal G_n is finite, we say the series has finite height.

Secondly, having established the nature of duplicate solutions, it is germane to ask whether there exist degree conserving Jordan-Hölder series of any type. Our investigations, which are documented later in this work, are limited for finite height to the condition that we use a multifunction, revealing for the sextic that there exists a leaped multifunction series

$$S_6 \rhd G_3 \rhd G_2 \rhd 1.$$

Further work on this topic is planned [Ad14c], including explicit calculations for matrix polynomials in chapter IX of that work, a description of Galois theory for the aspects we have covered in chapter X, and the generalisation to matrix varieties in chapter XI.

Our plans, contingent on results, were originally to be concerned there with the inversion problem, which considers the mapping from a polynomial in multiplicative form to additive form, and its reversal, comparing this with aspects of the theory of groups and rings. We plan to discuss the status of internal constraints on roots and the impact of these ideas on the current theory due to Jordan, Hölder, Schreier, Noether, Artin and Wedderburn.

2 Conceptual history

2.1. Public discourse,

J.L. Lagrange in *Réflexions sur la résolution algébrique des équations*, Oeuvres vol. 3, p 305 [La1771] says: "To apply, for example, the Tschirnhaus method to the fifth degree, we have to resolve four equations comprising four unknowns, of which the first is a first degree equation, the second of the second degree, etc., so that the first equation results in the elimination of three of these unknowns which display, in general, a degree of the form 1.2.3.4, that is, the twenty-fourth degree.

Thus, independently of the enormous work which would be necessary to obtain this equation, it is clear that when we have found it, we are hardly further forward, in that we have at least to reduce it to a degree less than the fifth, a reduction, if it is possible, which would be none other than the fruit of a new endeavour considerably more than the first ...

Also we see that the same developers of these techniques have been satisfied with applying them only to the third and fourth degree, and no-one else has displayed sufficient capacity to push forward this work any further".

It is remarkable that there appears to be no record of Gauss investigating this mathematics.

In the 1964 revision of his lecture notes at Harvard University, R. Brauer in *Galois theory* [Br64] was able to say: "Galois theory originated from the futile attempts to obtain formulas for the solution of the general equation of degree ≥ 5 in terms of radicals".

It is a guiding principle, which I interpret was realised by Grothendieck in later work at Montpellier, that if, for example

$$(x^2 + P)(y^2 + Q)(z^2 + R) = 0,$$

then if, say

$$\begin{aligned} w &= x + h \\ &= y + h' \\ &= z + h'', \end{aligned}$$

this is satisfied by

$$(w^2 + P)(w^2 + Q)(w^2 + R) = 0,$$

for two values of w with $h = 0$, two values with $h' = 0$ and two values with $h'' = 0$.

2.2. Personal reflections.

No more obdurate and intransigent stance to bypass Galois theory has been taken by any other contemporary mathematician. Failed attempts involved the development of a complete matrix theory (hyperintricate numbers) containing complex numbers as a subalgebra [Ad14c]. Dealing with permutations as matrices was tried as a resolution of the problem, and hyperintricate numbers are interesting in extending the theory.

There is the question as to whether the condition that all factor groups are abelian is relevant to the case where the variables x in a polynomial are matrices, with possible matrix coefficients. In the hyperintricate representation of matrices documented in chapters 5 and 6, a $2^n \times 2^n$ matrix, under which other finite matrices can be subsumed, may be represented faithfully by a vector with 4^n linearly independent basis elements. If we represent a_n and x as hyperintricate matrices, and equate their respective hyperintricate basis element parts, and let

$$\sum_n a_n x^n = 0,$$

if all a_n are real, then if Galois theory applies, all coefficients commute with x. But this case is a subcase of the general one, and therefore in the general case, the content of Galois theory is not modified. Moreover, if

$$x = c1_{\text{hyperintricate}} + \sum_{\text{hyperintricate}} d,$$

and $c \neq 0$ in general, then there is an equation for the real part of degree n which is non-trivial, and the non-real parts do not decrement the degree of the real part, so that Galois theory applies to matrices if it applies in the case of complex numbers.

For any point on the complex Argand plane, the types of transformation of complex numbers are limited, being a translation, described additively, a boost or rotation multiplicatively, or a mirror reflection about the real axis by conjugation, and these accommodate the most general homomorphisms $G_j \to G_k$ when any pair of the above are commutative, which happens except for boost or conjugation, with translation. Thus the group theory, which does not include translation, specifies abelian factor groups, but we need to extend it. We are open to the suggestion that five or more points in the Argand plane when subjected collectively to transformations of the above type cannot be permuted in the most general manner.

Under possibly hidden constraints Galois restrictions do not always apply, holding when the direct degree of a polynomial is 5, but not on adding a duplicate root. Independently of this, [St04] asserts the correctness of Abel-Ruffini theory and the end result of Galois theory.

Previously developed reasons for doubt were that a series of roots may be selected for which we know the solution, but that if it is claimed that there is not even a transcendental formula by which, on having selected coefficients for a polynomial, the inverse problem can be solved for degree ≥ 5, it must be asserted that there is a fundamental irreversibility in mathematics. The situation would be symmetrical if there were no *formula* describing the input of a selected set of roots, but this is not so, since the selection can conform to a polynomial, so we have a determinate input, but no *method* that is reversible.

This proposition differs from theories in classical physics in which the mixing of a system can be reversed [Bo85], and also pertains to the de Broglie-Bohm quantum theory, which is determinate.

If there were permanent obstructions to the existence of transcendental formulas, it would constitute a refutation of the philosophy in [Ad14a] that methods are subsidiary to states. Our work enables us to maintain the integrity of this principle for transcendental formulas, and the reader is invited to perform the confirmatory calculation for formulas involving duplicate solutions by radicals.

3 Results originating from the study of dependent roots

3.1. Polynomials of degree ≤ 6 with roots (x + a)(x – a).

The method we develop in the present section is the most practical for obtaining iterated roots of polynomial equations. However, historically the approach using dependent roots was first developed using the results of sections 3.2 and 3.3, when it was conjectured from the book by Netto [Ne1892] that non degree conserving techniques allowed an escape clause from Galois theory.

We will first take the case of the sextic and study the equation with roots (x + a)(x – a):

$$(x^2 - a^2)(x^4 + bx^3 + cx^2 + dx + e) = 0. \tag{1}$$

If this is put in the form

$$x^6 + Px^5 + Qx^4 + Rx^3 + Tx^2 + Ux + V = 0, \tag{2}$$

then there is a computable mapping between (2) and (1).

Indeed, we have

$$\begin{aligned} P &= b \\ Q &= -a^2 + c \\ R &= -a^2b + d \\ T &= -a^2c + e \\ U &= -a^2d \\ V &= -a^2e, \end{aligned} \tag{3}$$

and the equations (3) can be directly inverted:

$$\begin{aligned} &b = P \\ &a^4P + Ra^2 + U = 0 \end{aligned} \tag{4}$$

with the constraint

$$T = -a^2(Q + a^2) - V/a^2$$

and so

$$a^2 = \frac{-R \pm\sqrt{R^2 - 4UP}}{2P}$$

$$b = P$$

$$c = Q + \frac{-R \pm\sqrt{R^2 - 4UP}}{2P}$$

$$d = \frac{3R \pm\sqrt{R^2 - 4UP}}{2}$$

$$e = V/[\frac{R \pm\sqrt{R^2 - 4UP}}{2P}].$$

We may now directly solve (1), knowing the classical solution of the quartic. □

It is correct by Galois theory, that if we then consider the Tschirnhaus substitution

$$x = y + h, \tag{5}$$

so that in effect the polynomial is a completely general one, the equation

$$y^6 + P'y^5 + Q'y^4 + R'y^3 + T'y^2 + U'y + V' = 0, \tag{6}$$

is then unsolvable directly by the common method, that is, the mapping

$$(h, a, b, c, d, e) \rightarrow (P', Q', R', T', U', V')$$

cannot be inverted by usual Galois techniques of steady descent to equations of lower degree. In the case of duplicate roots, which follows in the next section, the duplicate roots maintain their status under Tschirnhaus substitutions. We will combine these two ideas in section 3.4 to investigate the quintic. □

3.2. Polynomials of degree $\leq$ 6 with duplicate roots.

We will next take the case of the sextic and study the equation with duplicate roots

$$(x + a)^2(x^4 + bx^3 + cx^2 + dx + e) = 0. \qquad (1)$$

If this is put in the form

$$x^6 + Px^5 + Qx^4 + Rx^3 + Tx^2 + Ux + V = 0, \qquad (2)$$

then again there is a computable mapping between (2) and (1).

This time we have

$$P = 2a + b \qquad (3)$$
$$Q = 2ab + a^2 + c$$
$$R = 2ac + a^2b + d$$
$$T = 2ad + a^2c + e$$
$$U = 2ae + a^2d$$
$$V = a^2e.$$

The value of a in terms of P, Q, R, T, U and V can be obtained from the next section, 3.3, and so b, c, d and e can be determined.

We may now directly solve (1), knowing the classical solution of the quartic. □

In preparation for what follows, we then consider the Tschirnhaus substitution

$$x = y + h', \qquad (4)$$

when (4) retains duplicate roots in y and the modified equation (2)

$$x^6 + P'x^5 + Q'x^4 + R'x^3 + T'x^2 + U'x + V' = 0, \qquad (5)$$

is then solvable by the same method, that is, the mapping

$$(h', a, b, c, d, e) \to (P', Q', R', T', U', V')$$

may be inverted. □

3.3. A differential condition for the detection of duplicate roots.

Theorem 3.3.1. *Select a quadratic factor* $g(x) = 0$ *of a polynomial equation* $F(x) = 0$. *There exists a unique transformation* $x \to x + h$, *such that either*

g(x) *represents* $(x + a)^2 = 0$ *or* g(x) *represents* $x^2 – a^2 = 0$.

Proof. Consider the product $(x + b)(x + c) = 0$. If $b = c$, then this is the first case, and if $b \neq c$, $h = -(b + c)/2$. □

Definition 3.3.2. Let x be real or complex, then $\Delta x^m = mx^{m-1}$ if $m \geq 1$, otherwise $\Delta x^m = 0$.

Remark 3.3.3. We are not employing Cauchy-Riemann complex differentiation.

Theorem 3.3.4. *If*

$$F(x) = \sum_{m=0}^{n} a_m x^m = 0 \tag{1}$$

has duplicate roots, then at this root

$$\Delta F(x) = 0. \tag{2}$$

Proof.

$$\Delta[f(x)g(x)] = f(x)\Delta g(x) + g(x)\Delta f(x).$$

Thus if f(x) corresponds to an arbitrary polynomial and g(x) to the duplicate root $(x + a)^2$, then

$$\Delta g(x) = 2(x + a),$$

so that

$$\Delta[f(x)g(x)] = (x + a)\times(\text{a polynomial}). \ \square$$

Equation (2) is of the form

$$\Delta F(x) = \sum_{m=1}^{n} m a_m x^{m-1} = 0, \tag{3}$$

so that multiplying the above equation by x and subtracting mF(x) we get a distinct equation in the power x^{m-1}. Combining this equation with (3), we get an equation in x^{m-2}, so that we have obtained a further descent of the degree, and this process may be iterated. At each stage we retain the root (x + a), thus at the penultimate resolution we obtain a linear equation in x, which must be the root, and finally we obtain an equation in x^0 which corresponds to a constraint on the coefficients of (1).

By this technique the root (x + a) may be obtained, and we may proceed by methods already introduced. □

3.4. The quintic with the root (x + a) and an adjoined root (x + a) or (x – a).

We can take the case of the quintic with root (x + a) and study the adjoined equation with root (x – a):

$$(x - a)(x^5 + qx^3 + rx^2 + tx + u) = (x^2 - a^2)(x^4 + bx^3 + cx^2 + dx + e) = 0, \tag{1}$$

or the similar case for duplicate roots.

This is equivalent to the inversion of the coefficients

$$(x^5 + qx^3 + rx^2 + tx + u) = (x + a)(x^4 + bx^3 + cx^2 + dx + e) = 0, \tag{2}$$

which cannot be obtained using Galois methods. □

However, we are not using Galois theory for duplicate roots, since we are applying a second technique. The question now arises, if the conditions of section 3.1 and 3.2 hold simultaneously, which is similar to the combined conditions

$$(x + a)(x - a) = 0 \text{ and } (x + a)^2 = 0,$$

does there exist a situation where the quintic is invertible?

We now have simultaneously

$$-a = P = b \tag{3}$$
$$q = Q = -a^2 + c$$
$$(r – aq) = R = -a^2b + d$$
$$(t – ar) = T = -a^2c + e$$
$$(u – at) = U = -a^2d$$
$$-au = V = -a^2e,$$

and

$$a = P'' = 2a + b \tag{4}$$
$$q = Q'' = 2ab + a^2 + c$$
$$(r + aq) = R'' = 2ac + a^2b + d$$
$$(t + ar) = T'' = 2ad + a^2c + e$$
$$(u + at) = U'' = 2ae + a^2d$$
$$au = V'' = a^2e,$$

together with the degree decrementing constraints obtained on (4) for duplicate roots. It is seen that equations (3) encapsulate identical information to equations (4). The dependent root constraint for (4) maps in the case of the quintic to the constraint for (3)

$$T = -a^2(Q + a^2) – V/a^2 ,$$

becoming the identity

$$-a^2c + e = -a^2(-a^2 + c + a^2) + e,$$

so that no further information is available to invert (3), hence

$$u = ae,$$
$$t = ad + e,$$
$$r = d + c,$$
$$c – aq = a^3,$$

so that, as expected by standard Galois theory, this reduces to

$$a^5 + qa^3 – ra^2 – ta + u = 0. \square$$

4 Results on leaped Jordan-Hölder series using multifunctions

4.1. The cubic via $(x + A + B + C)(x + A + \omega B + \omega^2C)(x + A + \omega^2B + \omega C) = 0$.

We will solve the cubic where each occurrence of the variable x is represented by three linearly independent variables expressed as

$$(x + A + B + C)(x + A + \omega B + \omega^2C)(x + A + \omega^2B + \omega C) = 0, \tag{1}$$

where ω is a cube root of unity:

$$\omega = e^{2\pi i/3} = \tfrac{1}{2}(-1 + \sqrt{3}).$$

On multiplying out, this equation reduces to

$$x^3 + 3Ax^2 + 3(A^2 – BC)x + A^3 + B^3 + C^3 – 3ABC = 0. \tag{2}$$

If this equation is of the form

$$x^3 + px + q = 0, \tag{3}$$

then on putting A = 0, we obtain

$$p = -3BC, \tag{4}$$

$$q = B^3 + C^3, \tag{5}$$

so that we obtain the classical quadratic in B^3:

$$27B^6 – 27qB^3 – p^3 = 0,$$

with solution

$$B^3 = \frac{q}{2}\left[1 \pm \sqrt{1 + \frac{4p^3}{27q^2}}\right]. \square \tag{6}$$

Alternatively, we can apply the symmetry-breaking Tschirnhaus substitution

$$x = y + h, \tag{7}$$

so that

$$y^3 + 3(h + A)y^2 +3[h^2 +2Ah + (A^2 – BC)]y$$
$$+ h^3 + 3Ah^2 + 3(A^2 – BC)h + A^3 + B^3 + C^3 – 3ABC) = 0, \tag{8}$$

and with A + h = 0

$$y^3 – 3BCy + B^3 + C^3 = 0, \tag{9}$$

which gives the same as (4), (5) and (6). □

4.2. The sextic as a leaped series $S_6 \rhd G_3 \rhd G_2 \rhd 1$.

The approach developed next seems to be the method devised by A. Grothendieck. It is complementary to dependent root techniques.

Consider the sextic in multiplicative form with linearly independent roots

$$(w + A + B + C + D + E + F)(w + A + B + C – D – E – F)\times$$
$$(w + A + \omega B + \omega^2C + D + \omega E + \omega^2F)(w + A + \omega B + \omega^2C – D – \omega E – \omega^2F)\times$$
$$(w + A + \omega^2B + \omega C + D + \omega^2E + \omega F)(w + A + \omega^2B + \omega C – D – \omega^2E – \omega F) = 0. \tag{1}$$

In effect, we are dealing with sixth roots of unity, since these are

$$1, -\omega^2, \omega, -1, \omega^2, -\omega.$$

To prove linear independence, we note that the matrix corresponding to these linearly independent roots,

$$\begin{bmatrix} Y & Y \\ Y & -Y \end{bmatrix},$$

with Y = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$, by the algorithm of Boltz-Banachiewicz demonstrated in [Ad14b] has an inverse and is therefore non-singular.

From the solution of the cubic, we know this can be expressed as the two cubics

$$\{w^3 + 3(A + D)w^2 + 3[(A + D)^2 – (B + E)(C + F)]w$$
$$+ (A + D)^3 + (B + E)^3 + (C + F)^3 – 3(A + D)(B + E)(C + F)\}\times$$

$$\times\{w^3 + 3(A - D)w^2 + 3[(A - D)^2 - (B - E)(C - F)]w$$
$$+ (A - D)^3 + (B - E)^3 + (C - F)^3 - 3(A - D)(B - E)(C - F)\} = 0, \qquad (2)$$

alternatively as three quadratics

$$\{w^2 + 2(A + B + C)w + (A + B + C)^2 - (D + E + F)^2\}\times$$
$$\{w^2 + 2(A + \omega B + \omega^2 C)w + (A + \omega B + \omega^2 C)^2 - (D + \omega E + \omega^2 F)^2\}\times$$
$$\{w^2 + 2(A + \omega^2 B + \omega C)w + (A + \omega^2 B + \omega C)^2 - (D + \omega^2 E + \omega F)^2\} = 0. \qquad (3)$$

If from (3) we set $w = x + h$, giving

$$(x + h)^2 + 2(A + B + C)(x + h) + (A + B + C)^2 - (D + E + F)^2 = 0,$$
$$(x + h)^2 + 2(A + \omega B + \omega^2 C)(x + h) + (A + \omega B + \omega^2 C)^2 - (D + \omega E + \omega^2 F)^2 = 0,$$
$$(x + h)^2 + 2(A + \omega^2 B + \omega C)(x + h) + (A + \omega^2 B + \omega C)^2 - (D + \omega^2 E + \omega F)^2 = 0, \qquad (4)$$

and we choose 3 different values of h, so that

$$x^2 - (D + E + F)^2 = 0,$$
$$x^2 - (D + \omega E + \omega^2 F)^2 = 0$$

and

$$x^2 - (D + \omega^2 E + \omega F)^2 = 0, \qquad (5)$$

so that in fact

$$h^3 + 3Ah^2 + (3A^2 - 3BC)h + (A^3 + B^3 + C^3) = 0, \qquad (6)$$

then we have a multifunction solution in h for three quadratics, in other words a cubic in x^2.

If the three choices for the variable are h, h′ and h″, then the Grothendieck substitutions for say h^3 are h^3, h'^3, h''^3, hh'^2, hh''^2, $h'h''^2$, $h'h^2$, $h''h^2$ and $hh'h''$ for these multivalued functions.

However equation (1) as a product is the general form from which we have derived equations of type (5). Thus we propound, without touching on questions of invertibility

Theorem 4.2.1 *Any polynomial of the form*

$$x^6 + Px^4 + Qx^2 + R = 0 \qquad (7)$$

can be written using Grothendieck multifunctions as a general sextic polynomial. ☐

In manipulating such an equation, to maintain all solutions simultaneously, it is necessary to maintain memory of h – see the similar situation for the zero algebras of [Ad14a].

To describe a further instance of this type of situation, the equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0 \qquad (8)$$

may be solved as a multifunction by the additively solvable

$$(x - \omega\sqrt[3]{c})\left(x - \omega'\sqrt[3]{\frac{d}{a}}\right)\left(x - \omega''\sqrt[3]{\frac{e}{b}}\right) = 0, \qquad (9)$$

where ω, ω′ and ω″ are cube roots of unity. ☐

5 Simple problems

(i) Obtain explicitly the value of the duplicate root for the sextic.

(ii) By an extension of the same methods, is the octic with two distinct sets of duplicate roots solvable?

(iii) Do non degree conserving techniques disallow the solution of the decic with five pairs of duplicate roots, by radicals?

(iv) Suppose

$$x^6 + Px^5 + Qx^4 + Rx^3 + Tx^2 + Ux + V = K$$

has roots $(x + a)(x – a)$, and

$$x^6 + P'x^5 + Q'x^4 + R'x^3 + T'x^2 + U'x + V' = -K$$

has roots $(x + a')^2$. Can the general solution of the sextic be obtained by combining these two equations?

(v) For a polynomial of degree $m + 2n$, where $0 < m < 5$ and $i = 1, ... n$, for what degree is this equation solvable if the degree m component has no roots of the form $(x^2 – a^2)$ or $(x + a)^2$ and for each i there are roots $(x^2 – b_i^2)$ or $(x + b_i)^2$?

(vi) From section 3.1 can the decic with three sets of roots $(x^2 – b_i^2)$ be solved, and can an extension of equations (4) beyond the quartic in x^2 be introduced (which is a type of octic), to extend these procedures indefinitely by further adjoining multiple pairs of roots?

References

Nearly, the title "Galois theory" is dense in the titles of books on Galois theory. I think the book by Netto [Ne1892] is enlightening, and the third edition (not always the first or second) of Ian Stewart's book [St04] is the most persuasive and accessible under its assumptions. Hermann Weyl [We40] I find the most interesting.

Ad14a	J.H. Adams, *Discussion on ladder numbers and zero algebras*, to appear in *Innovation in mathematics*, 2014.
Ad14b	J.H. Adams, *Introduction to intricate and hyperintricate numbers* II, to appear in *Innovation in mathematics*, 2014.
Ad14c	J.H. Adams, *Superexponential algebra*, to appear, 2014.
Ar59	E. Artin, *Galois theory*, Notre Dame, Indiana, 1959.
As86	M. Aschbacher, *Finite group theory*, Cambridge U. P., 1986.
Bo85	D. Bohm, *Wholeness and the implicate order*, Ark, 1985.
Br64	R. Brauer, *Galois theory*, Harvard, 1964.
Bu1911	W. Burnside, *Theory of groups of finite order*, Dover, 1955.
Co12	D. Cox, *Galois theory*, (2nd edn), Wiley, 2012.
De60	E. Dehn, *Algebraic equations*, Dover, 1960.
Di1900	L.E. Dickson, *Linear groups with an exposition of the Galois field theory*, Dover, 1958.
Ed84	H. Edwards, *Galois theory*, Springer, 1984.
Fr26	R. Fricke, *Lehrbuch der Algebra* II, 1926.
Ga62	É. Galois, *Écrits et mémoires mathématiques*, Gauthier-Villars, 1962.
Gar86	D. Garling, *A course in Galois theory*, Cambridge U. P., 1986.
He1858	C. Hermite, *Sur la résolution de l'équation du cinquiéme degré,* Comptes Rendus de l'Acad. des Sciences XLVI, 508-513, 1858.
Jo1870	M.C. Jordan, *Traité des substitutions*, Gautier-Villars, 1957.
Ka69	I. Kaplansky, *Fields and rings*, University of Chicago Press, 1969.
Kl56	F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*, Dover, 1956.
La1771	J.L. Lagrange, *Réflexions sur la résolution algébrique des équations*, Oeuvres, vol. III.
MC64	N. McCoy, *The theory of rings*, Macmillan, 1964.
Mi14	J.S. Milne, *Fields and Galois theory*, online, 2014.
Ne1892	E. Netto, *Theory of substitutions*, Chelsea reprint.
New1707	I. Newton, *Arithmetica Universalis*, 1707, available in *The mathematical papers of Isaac Newton,* vol 5, 1683-84, particularly from p 351, ed. D.T. Whiteside, Cambridge U. P., 1967-82.
No83	E. Noether, *Gesammelte Abhandlungen*, Springer, 1983.
Ro90	J. Rotman, *Galois theory*, Springer, 1990.
Se03	J-P. Serre, *Extensions icosaédriques*, Oeuvres, Vol III, **123**, Springer, 2003.
St04	I. Stewart, *Galois theory*, 3rd edn., Chapman & Hall/CRC, 2004.
Ti88	J-P. Tignol, *Galois theory of algebraic equations*, Longman, New York, 1988.
We40	H. Weyl, *Algebraic theory of numbers*, Princeton U.P., 1940.