Creative mathematics – a viewpoint

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1 Creative mathematical research and the system.

1.1 Supporting creative intentions.

Creative mathematics is not like creative accounting; mathematics is about the truth. The objective of this work is to seek ways of explaining how creative and significant writing in mathematics can be attained, and to encourage the reader to get involved in it. The remainder of the book contains some examples of my own work in which I have tried to apply my own principles.

Mathematics is like a vast ocean, but its boundaries may be traversed. To deepen the analogy, with modern technology the Atlantic can be crossed in a few hours. If challenged to produce such a mathematical journey, I would say we first have to design the airplane, and that this could be accomplished as a project.

I believe from experience that writing a research paper on mathematics takes about 50 times the effort of composing a piece of music. I also know that with application this effort can be achieved.

Creativity cannot properly be invigilated within a system. Its results often do not correspond with a set of boxes which can be ticked on a form, either because what is deemed necessary the author may not wish to attain, or just as frequently, the box is not there in which the result can be graded. In the main innovation is marked down, and successful innovation requires more work than its non-innovative counterparts to complete the same objectives.

Creativity quotas reduce people to mechanical automatons, so the solution cannot usually be found within a bureaucratic system. Although the system often wishes to monitor itself, there is 'Heisenberg's principle of teaching', which states that the more the teaching of a subject is monitored, the more the teaching process is perturbed. The best results in mathematics are obtained by clear insight, which requires calm and receptiveness, and is not obtained exclusively by strenuous exercise of routine processes. Moreover, the system fails to grade work extending beyond a deadline which it defines, even though some problems require it, and fails the creativity test by operating on answers to questions which are already known.

Nor can I put myself in sympathy with a culture of fear which seeks to grade people by their ability, and my personal intention is to step outside the system to a place where this is absent.

There is the idea that mathematics is defined by its utility and not its content, but I think that has never been so. There is a sense that the subject of mathematics is outside society, even though people promote and communicate its purpose, insight and results. In contradistinction, for the career minded the progress through academia may at first be envisaged as an upward trajectory without termination. The search for grade and social recognition are self-limiting, however.

There are cases where grading people is necessary. To pursue a musical analogy, one cannot have an orchestra where a musician is reduced to tears when confronted with a piece of music. Analogous remarks can be made for those professions where mathematics is needed for the technical requirements of, say, engineering projects. So proficient execution can and should be graded, so that the system can provide an efficient and standard service for itself.

On the other hand, the composer of a work of music does not have to be able to play every instrument, to conduct or to perform in the orchestra, and there are many ways, say in the administration of a concert and rehearsals, or in the production of its printed programme, which require other abilities which are vital, and may even be performed by the deaf.

We end this analogy by stating that we are here concerned with the creation of mathematics, similar to the way in which the composer writes a piece of music. There are other activities that the mathematician can be involved in, and the non-mathematician can be of enormous support. Some people may not wish to participate in this endeavour, either through lack of interest, which is fine, or through a fear of their own abilities. But even if the reader is not interested in the creative process, there are other works here which I believe are innovations, and from that point of view, can be studied with interest by the working mathematician for insight.

The famous mathematicians of the past, and the present, are not supermen or superwomen, they are ordinary individuals like you or me. Do not be told that this or that person is a genius; results are not sacrosanct, mathematics is about the truth, and you can find it through your own resources.

There is an enormous culture of mathematics, and much productive effort may be accomplished and much insight may be gained through its study, and the study of mathematical history, but if you cannot understand a result, even after substantial effort, never mind, you can gain insight even through your own resources as well as the community of mathematicians, and mathematics does not always require an enormous technological infrastructure, unlike physics for its particle accelerators. You can find the truth by use of your own reasoning. So have courage, not only are great results attainable, you can attain them yourself. And if at the end of this you gain prestige, remember that we are all human, and we are not numbers. So forgive mistakes, because even those who have made the most efforts make them. Indeed, getting it wrong is the first stage in getting it right!

1.2 Emancipation and control.

Innovation by its very nature disrupts the system. If we look at the history of innovation we see more than one method used by its progenitors in bringing this about. If we look at those conceptual revolutions at the beginning of the twentieth century, in art by Picasso, in music by Schönberg and in physics by Einstein, we see three different stratagems adopted by these people with reference to the system.

In the case of Picasso, he did not try to interface with the academic system. If we compare the work of the academic painter Alma Tadema – Helen of Troy looking down on the Greek fleet – and the weeping woman of Picasso, we see an enormous gulf in style, obsession of theme and subject, and intended audience.

The Picasso is immediately subversive, whereas it has only been in subsequent generations that the idea of subversion has been appropriated by the academic art system as part of its central, regulated core.

For Schönberg, although he wrote the Theory of Harmony in 1922, intensively teaching the subject of music theory within academia came relatively late, and included the necessity to provide income for himself. His writings on composition show a reverence for what came before, and almost always ignore the revolutions inherent in his own work. The subject he taught is conservative, even whilst the production of compositions continues to be revolutionary.

For Einstein, the intention from the beginning was to become a professor, and an initial intense disappointment in furthering his interests in physics prompted him to overcome this through the publication of his early work. His employment at the Patent Office in Berne allowed him access to research and further study of physics, but it was not until his academic tenure that his revolutionary ideas contained in general relativity allowed him to *become* the system and part of its purpose.

Whereas each of these archetypical options is available, I wish to promote the idea that being outside the system is feasible today. It may be felt that being a freelance mathematician is like being a freelance brain surgeon; it is a technical subject and in the interests of society intruders should be strongly discouraged. Nevertheless, we can itemise several advantages.

A number of writers voice frustration with the process by which some articles are selected. Although submitting an article successfully to a journal enhances prestige and maintains academic tenure, it is sometimes remarked that important works by mathematical experts have subsequently sunk and not been read by the mathematical community, even when it would be highly desirable to do so.

A journal will have a particular ethos and format, and an audience it has accumulated over time, with expectations of its contents and style. You would not go into a doctor's surgery and ask for a cheese sandwich, and it may be futile to submit an article to a journal which transgresses its editorial policy. This policy can be determined most simply by looking at back issues and forming a value judgement of its aims and methods.

The researcher has to obtain

(1) income, so that conditions are available where research can take place,
(2) time, so that research tasks can be completed,
(3) access to research materials,
(4) access to media promoting the work.

It is increasingly prevalent that books are self-published; ISBN numbers may be readily purchased, critiques and observations by mathematicians may be obtained and from them consequences selected, social media can be used to promote the work, and above all the author is in control, with all its delights, privileges and dangers. This emancipation of the pen is feasible, and whatever the circumstances the potential author, sometimes weighed down by the bureaucratic system, has the option of pursuing it.

1.3 The sociology of group innovation.

In mathematics, even for erudite research, there are occasions where the development of the subject, or a series of criticisms, results in an upturning of the original thesis and its replacement by another. The question then arises, at which level may a result be regarded as safe, and at which point do the appointed gate-keepers of the system allow entry?

Mathematics is not usually regarded as a dialectical subject, and is often thought of as being about facts which are irrefutable. If a piece of research was correct in the year 1750, it is also correct today. According to this reasoning, the system, to function correctly, can only allow the introduction of research which is indubitably correct. Several things need to be said about this point of view.

Firstly, the subject is not only about facts, but about conjectures, and conjectures, partly because of their uncertainty, are often more interesting than facts, and allow an opening to further research.

Secondly, the sociology of the system allows imperfections. I introduce three considerations here. Status allows publication of a work by a high-status mathematician, where publication of the same article by a low-status mathematician would be blocked (the reader is invited to devise an experiment in which this hypothesis can be tested). There are also considerations on the system of referees, the selection process of those referees, and the formal and informal relation of the referee system to editorial control. The journal itself exists in an economic environment, so that external constraints often determine journal policy.

The language used in an article, and its intended audience, may inhibit variation of style or social acceptability, irrespective of its mathematical content.

Lastly, it is often the case that a subject is not developed by one individual, but by an extended series of authors over a long period of time. That the system sometimes fails in locating an error can be a beneficence. It allows multi-author development by successive approximation and improvements.

2 The creative mathematician.

2.1 What is mathematics?

Mathematics has specialisations in which the mathematician does her or his work. The mathematician operates somewhere between two extremes, the conceptual mathematician dealing with the global subject, and the specialist problem solver dealing with a specific and fixed axiom system and the interrelationships within it.

A question that can be raised for the creative mathematician, is whether the subject can be specified, so that the activities on which work takes place can be put into context. I think this specification can be found, and universal mathematics is about sure reasoning with respect to an arbitrary axiom system. That is, we start with an arbitrary set of assumptions, and ask how the system behaves under related rules of deduction. For this relative mathematics to exist uniquely, it must have only one type of universal generalisation. Is the state of maximal generalisation one of no assumptions at all? I don’t know, but the relative truth of such an axiom system to its stably deduced theorems is absolute, valid and specific.

If there is only one true world, there is only one true description of it. The programme of mathematics is to evince by reasoning what is inherently possible in its maximum generality. By this process the description of the physical world must correspond to some aspect of mathematical structure. There is the question, raised by the classical Greeks and long since abandoned as impractical, of whether the nature of the world can be determined by thought alone. If so, this is the domain of mathematics.

There is the question of whether mathematics is invention or discovery. The working mathematician quickly becomes aware, to reawaken our musical analogy, that the creative process in music proceeds from the composer to the score, in that direction, but that for a work of mathematics, the mathematics itself tells the mathematician what to write. The reader can see this process in the penultimate work detailed in this book, where parabola diagrams were discovered and not invented in the process of investigation, and the question was initially to find out why they were there and why they have the properties they do.

So is innovation possible at all in mathematics? I sometimes give a description of it as an Eternal Room; a blind mathematician enters the room and finds items of furniture in it. The objects in the room and their configuration can be found by touching them, but the objects in this mathematical room have the characteristic of being eternal objects. They existed before the beginning of time and will never change.

2.2 Style.

The adoption of a particular mathematical style ought to be, but sometimes is not, the responsibility of the writer. A particular question here is the elevation, or oppositely the simplicity, of the level of language. It has been said that the very best footballers make the game look easy, and the average footballer makes football look difficult. To raise a question, does an analogous situation hold for writers of mathematics?

We are sometimes confronted with the hypothesis that the language of an article is designed to exclude, perhaps to limit discussion to a select group to protect its boundaries, and possibly to limit criticism of its programme that wider understanding might attract. The era of the sixties in what is sometimes known as the ‘Bourbakist terror in France’ is now past history, although the writing of the collective known as Bourbaki nowadays is not regarded as being unduly abstract. This style was characterised by abstraction without examples.

My father once mentioned that Mussolini promoted propaganda on the number of Italian warships built, and then based battle plans on the acceptance of his own propaganda. If you wish to adopt an unilluminating style, be sure you do not confuse even yourself. Language is there to describe thinking in a clear way, so find ways that help your internal thinking. There is often a gulf between what is thought or the way the result is obtained, and what is finally presented. We will broach this idea later. Be aware that you have an audience and the objective in mathematics should not be self-adulation of the mysterious author, but presentation of results in the most appealing, direct, simple and intuitive way possible.

In order to guide the line of thought further, I want to discuss meaning and syntax. The meaning of a sentence is its mapping to the world, demonstrated by pointing to examples which can be perceived. In mathematics this world sometimes consists not only of everyday objects, but also figures or diagrams on some media. In the brain, the syntax of a sentence is *parsed*, that is, scanned, by a broad range of speech processing areas, with *Wernicke’s area* implicated in lexical processing, passing over information to *Broca’s area*, which plays a significant role in speech comprehension, to understand the meaning of an utterance. So if a mathematical article is to be understood by the brain, it must contain as well as symbolic manipulations, a specification of what examples this syntax corresponds to.

I am suggesting that examples (semantics) come before generalities (abstract symbolic manipulation), if for no other reason than this reduces the burden on working memory, and some people, like myself, can only retain two items in working memory simultaneously, where the introduction of a third clears working memory and disconnects it from meaning.

Another concern nowadays is the increasing prevalence in mathematics on results which have been obtained by computer program, but for which a manual check is very difficult or practically impossible. The question for the reader of such work is then whether the results are expected also to be checked by computer program. If so, I think it is reasonable to supply the methodology by which the result was obtained, but with increasing sophistication of technology, even this may be unavailable. A related concern, is how can insight be obtained of a result which is computationally difficult and specific?

2.3 Intuition in mathematics.

These works would not have been completed as a process of research without the motivation that they consist of problems worth solving. Mathematics is not just dry results, the guiding hand is an intuitive process which directs investigation.

What is intuition? To personalise this, what has it meant to me?

Intuition in mathematics can be a vague feeling of unease, immediately grasped. It can equally be a feeling of joy at a revelation of truth, or a feeling of fear. A necessary process is to bring this out of the subconscious and onto the surface where it can be subjected to rational investigation.

On reflection, a guiding intuition I often apply on trying to understand a proof is the question: 'is information being conserved?' If information is being lost in a proof, then it can probably be strengthened. If the result contains more information than is being fed in, then the theorem is suspect. Technically, two responses to this method of thought might have been my attitude to Galois theory – it uses groups, what happens for noncommutive groups? – and the theorem obtained via the Cantor diagonal argument and elsewhere, that $\mathbb{Q}$ is dense in $\mathbb{R}$ (see the article on ladder numbers for the latter), which I deconstruct.

There are clear circumstances where intuition is wrong, even if initially plausible. In scientific theory it might be the idea that beams emit from the eye to grasp the outside world, the seat of consciousness is in the heart, or the theory of phlogiston describing heat.

2.4 Errors.

It is amusing but invalid reasoning to say that if people did not make mistakes, we would not need brains. The basic attitude to errors, especially amongst the novice, is to be afraid of them, and seek by training and repetition, methods by which they are at least reduced to a low minimum, and in preference are completely absent.

That this is desirable carries over to the situation of playing a musical instrument, where if an error occurs on playing a piece of music for the first time, it is necessary to unlearn the mistake, otherwise it is likely to be repeated. A good technique in music is to inspect deeply the score first, then play the piece slowly without error to the end. Doubling the execution speed is then possible, and the brain circuitry is in place to make the work at the correct tempo a good performance.

I am perhaps recommending that to learn mathematical techniques, a good strategy is first to think generally about the problem and then to do the calculations at first slowly.

A common type of error which I do not always eliminate in my own work, is a substitution error. Particularly since I have low working memory, almost all my detailed work is reasoning on paper, and not in my head. Making multiple substitutions simultaneously I have found is asking for trouble, since I only apply the first two of these, and sometimes I still forget one of them.

A particular type of substitution error is a transcription error. To reduce this, try to eliminate the following symbol pairs.

symbol	which can be confused with
S	5
c	together with C
l (lower case L)	1 (one)
O, o (alphabetic)	0 (zero)
w	ω, if used

Ambiguity increases brain processing in order to resolve it. To reduce confusion, I think it is better not to have both upper and lower case variables in the same formula. On the use of Gothic letters, I have come across German speakers who do not understand this orthography.

It is often the case that you can run out of suitable symbols. I recommend a technique which was employed frequently by Jacobi: use primed variables. The symbols p', p'', p''' and p'''' can be used, but note that ′ and ″ are not the single and double quote on the keyboard (they are present on the Symbol and Cambria Math fonts).

If a long calculation has been performed employing a long chain of reasoning, it is likely at the first attempt that an error or errors have crept in. Rather than decrying our own stupidity, it is more mature to accept that both we and the world are not perfect, and that the presence of errors is a natural occurrence.

My point of view is that calculation is not complete until it has been checked for errors. How is this to be achieved?

A check that can be made is to try an example and see if it works. If the computation is heavy, a simple example at first may reveal an error, whereas a complicated example may introduce more errors. A representative sample of conditions may be introduced, but what is representative may be more a matter of art than automatic technique. The first objective is to test the calculation in areas where it is expected to hold, and the second to explore areas by taking the calculation up to its maximum resilience.

A second technique is to perform the calculation twice or multiple times, and a preferred variant once obvious errors have been eliminated, is to do the calculation in two different ways, using different methods.

Doing the same calculation multiple times will not eliminate errors when the basic formula or assumptions are incorrect. It is possible to develop techniques that will eliminate mechanical errors, but leave conceptual errors intact. How do we detect the latter? This will be presented at the end of the next section.

2.5 Methods for creative mathematical thinking.

What is insight? Insight is an encapsulation of an entire subject or method of working in a single good idea. I need it, I seek it, and I hope in some shape or form it is always available. Possession of insight means, however complicated the problem, I have a method of dealing with it. I know where I stand, and I can allocate resources in order to achieve objectives within the area I am studying.

We continue with what is now a cliché, the method of thinking outside the box. The puzzle, originating from Sam Lloyd, is to connect the dots in *Figure* 1 by drawing four straight, continuous lines that pass through each of the nine dots, without lifting pen from paper. Some people seeing the problem for the first time find it difficult because they imagine a psychological barrier at the edge of the dot array they feel they should not cross.

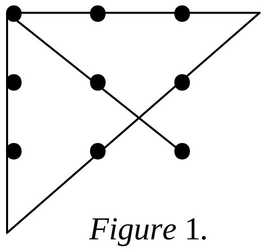

Figure 1.

Self-constructed barriers to thinking can not only be due to the nature of the problem, but also social conditioning. If we look at the penultimate article in the book, we deal with the Euclidean algorithm. It is taught in this way:

Every positive whole number n *can be written uniquely in terms of a positive whole number* w *multiplied by another number* $k \geq 0$ *with a unique remainder* $0 \leq t < w$:

$$n = kw + t.$$

What the article does, to investigate the discovered phenomenon of parabolas, is relax the condition of uniqueness so that t ranges over any value, not just < w. This is necessitated by the nature of the problem, but it contravenes the taught convention of the form of the Euclidean algorithm.

We have introduced in *Some simple proofs on general reciprocity* a convention for the introduction of new ideas. Using the ☐ symbol as usual for conclusion of a proof, a new idea, either an assumption or a new method, is begun with ☐→, and a return to standard thinking by ☐←.

The mathematician Euler used the term induction to describe the confirmation of a proof by many examples. This was later formalised as a particular method of proof, in which the starting value of an equation is confirmed, and then assuming the formula for n, it is proved for (n + 1). For a while we will investigate methods in this older sense.

The standard presentation of a proof goes back to Euclid. It may be shown in the diagram:

$$\begin{array}{c} \text{assumptions} \\ \text{(axioms)} \end{array} \xrightarrow[\text{rules of deduction}]{\text{application of}} \text{theorem}$$

Figure 2.

Nowadays it is normal to prove existence of a solution, its uniqueness, and that the axioms employed in a proof are equivalent to a set of independent axioms. Whereas with training this presentation can correspond with the actual assembly of a theorem, not all proofs are constructed in this way, and it is our intention to describe more fully than is usual, alternative methods.

The mathematician ought to be receptive to ideas. Consider a theorem propounded by former US Secretary of Defense D. Rumsfeld, that there are two sorts of unknown: the known unknown, and the unknown unknown. Application of the method of *Figure* 2 may reveal a known unknown, but it is less likely to uncover an unknown unknown. These are sometimes stumbled upon by accident, rather than directed thought.

Within the sciences there exist different cultures according to which a method of reasoning is thought valid. There is the story repeated of an engineer, a physicist and a mathematician travelling by train over the border to Scotland. Looking out of the window, the engineer sees a solitary black sheep in a field and exclaims: ‘look, sheep in Scotland are black!’, the physicists remarks: ‘you mean *some* sheep in Scotland are black’ and the mathematician concludes ‘all we can deduce is that in Scotland there exists at least one sheep, at least one side of which is black’!

So mathematicians are trained not to jump to conclusions. We seek to reintroduce it. The ability to jump to conclusions comes naturally, and it is part of pattern recognition to identify regularities quickly. In mathematics this aspect is referred to as a conjecture or hypothesis. A method of reasoning by conjecture is itemised in the flow diagram

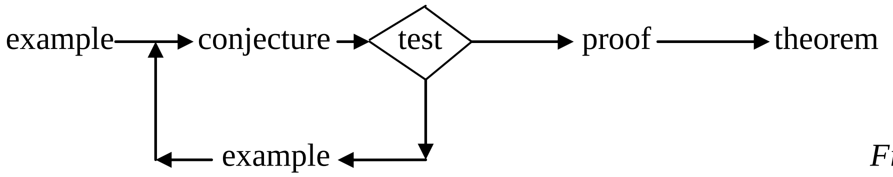

Figure 3.

If the example refutes the conjecture, the conjecture may be modified. If all examples work – including those designed to test the conjecture to failure, the next stage is to ask whether there is a logical path from assumption to conclusion.

Here is another process. Having posited a theorem, we determine the conditions under which it holds.

theorem → determination of a set of examples → establishment of conditions under which the theorem holds

Figure 4.

We have so far dealt with only half the methods for obtaining results. The remainder consists in making deliberately incorrect assumptions, as in the method of proof by contradiction: assume the opposite of a hypothesis, prove it is a contradiction, and the result is a proof of the theorem. This technique can be extended much further. For example, Galois theory states that no one equation exists for obtaining a solution of the quintic directly by radicals. But we can make a false assumption that a root of a polynomial is zero, and get closer and closer to the result, ending up an infinitesimal away from the right answer. The subject of statistics is about assuming generally inconsistently that a distribution of data follows a given rule. Yet this is not invalid mathematics; we measure the correlations between data sets precisely.

The standard presentation of a proof only gives some examples, followed by a reasoning sequence of *Figure* 2 type. This has the advantage of saving paper (the reason given by Gauss in his *Disquisitiones arithmeticae*), but the writer has the option of giving more information, if this enhances the understanding of what is going on.

We might not, using intuitive methods, be able always to articulate the assumptions we have made, but it is the standard valid process of mathematics, perhaps obtained through the work of many authors, to obtain conclusions by rationalising our intuition.

Having established a theorem by the above processes, we can determine whether a proof is correct since the standard method is to state the rules and methods of deduction, and from that by purely symbolic manipulation, the theorem, if proved correctly, is obtained.

A proof may not give insight as to the way the theorem is constructed, and it may not give a desirable and approachable meaning to some of its symbolic manipulations. Bear in mind that it is easier to encode than decode, so that we are confronted with the possibility that a proof may be easier to construct, by developing the principles behind it, than it is to be presented with the proof and understand it. For that reason, as in other aspects of life, the deepest understanding may not consist of what a subject (or theorem) is, but why it is there.