CHAPTER XVII

Superexponentiation

17.1. Introduction.

In referring to Wilson's theorem, Gauss was led to exclaim "in our opinion truths of this kind should be drawn from notions rather than from notations". In the general case we disagree; we need both. The substitution of verbal arguments using typical numbers was only replaced by letters for numbers (algebra) in the last 500 years. Mathematics in its modern form would have been impossible without this notational development. The replacement of the Roman numeral system I, II, III, IV, V, VI, VII, VIII, IX, X, ... by the essentially Arabic numerals 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ... was thought on its introduction to be a more complicated system and more difficult to deal with. The reason for its usefulness, of course, is the final term, 10, in the displayed sequence, which allows a development of addition and multiplication which is available by repeating the method of simpler cases in a relatively easy way. We seek to introduce a notation for superexponetiation, sometimes called hyperoperations, of similar usefulness. In the phrase of Tim Gibbs, this is "a map to a new mathematical world".

We will extend the operations +, which we will write as ${}^{1}\Uparrow$ and speak as "one up", × written as ${}^{2}\Uparrow$ and pronounced "two up", exponentiation $\uparrow$ as in $a \uparrow b$ more usually written as a^b, and written with ${}^{3}\Uparrow$, and a general nth superexponentiation operation ${}^{n}\Uparrow$.

Usual notation	Superexponential notation
a + b	$a\ {}^{1}\Uparrow b$
ab = a × b	$a\ {}^{2}\Uparrow b$
$a^b = a \uparrow b$	$a\ {}^{3}\Uparrow b$
area of sphere = $4\pi r^2$	area of sphere = $4\ {}^{2}\Uparrow \pi\ {}^{2}\Uparrow (r\ {}^{3}\Uparrow 2)$

Details will follow, but we note the following points.

The nth superexponentiation operation generates an (n + 1)th operation by recursion. Then

$$a + a + a \ldots + a \text{ (m terms)} = am$$

$$(\ldots((a \uparrow a) \uparrow a) \ldots) \text{ (m terms)} = a\ {}^{4}\Uparrow m,$$

so that, for instance for +, given by ${}^{1}\Uparrow$

$$(\ldots((a\ {}^{1}\Uparrow a)\ {}^{1}\Uparrow a) \ldots) \text{ (m terms)} = a\ {}^{2}\Uparrow m,$$

a general case being

$$(\ldots((a\ {}^{n}\Uparrow a)\ {}^{n}\Uparrow a) \ldots) \text{ (m terms)} = a\ {}^{n+1}\Uparrow m.$$

Even for natural numbers, superexponentiation operations for low values quickly generate numbers greater than the number of states in the observed universe. A difficulty now exists in checking calculations, where plugging in numbers directly may not result in a feasible calculation. There is the option of using mod, called congruence or finite, arithmetic here and in other cases of looking for transformation formulas from high values in superexponential operators to low values by stepping down the dimensions of the operators, so that equations can be checked.

We have introduced in chapter XVI the $\mathbb{D}$w exponential algebras. In its form for complex numbers, this differs from the standard exponential algebra. The standard exponential algebra branches in multiple ways, so that for $n \in \mathbb{N}$

$i^i = (e^{i\pi/2 + i2\pi n})^i = e^{-\pi/2 - 2\pi n}$,

but the $\mathbb{D}$w algebra only branches due to its $e^{i\pi n} = \pm 1$ terms, because in this algebra

$(a^{ib})^{ic} = a^{iwbc}$,

so

$i^i = (e^{i\pi/2})^i$

$= (e^{iw\pi/2})$

$= 1, i, -1$ or $-i$. □

In this chapter we will extend the $\mathbb{D}$w exponential algebra to an algebra for $2^m \times 2^m$ matrices as the *JAF* algebra for hyperintricate matrices.

The complex $\mathbb{D}1$ exponential algebra was developed further in [Ad14] for ladder numbers, which contain infinities and infinitesimals, where it was extended to superexponential operations, solving the 'tetration problem' for the evaluation of i ^{4}⇈i and i⇈^{4} i. We will reiterate this development here, and indicate how we may extend these ideas to matrix algebra in the hyperintricate representation.

All these are 'up' operations. We indicate how the reverse process 'down', which includes logarithms, is implemented.

The gamma function for complex numbers generalises the factorial function n! for natural numbers. This is an example of *interpolation*, in which the values of the codomain are extended by analytic continuation. We will need to provide interpolation formulas for superexponential operations.

Our superexponential operations can be extended by interpolation to negative and zero n in ^{n}⇈. In order to develop this algebra we need to introduce the idea of a polymagma. This is because superexponential algebra in its commutative and associative form fail for n > 2, and where n = 0 the normal rules of a nonassociative algebra fail too.

For higher order superexponentiation, there is an equivalent of differentiation. We need to see that the definition of this is meaningful. Tim Gibbs raises the question of whether there is a superexponential analogue of e^x, which satisfies the differential equation

$$\frac{d(e^x)}{dx} = e^x.$$

Tim Gibbs says that we might introduce fractal dimensions. This would mean introducing noninteger n in ^{n}⇈, including when n itself is a polymagma. He mentions a superexponential binomial theorem. The binomial theorem links +, × and ↑. In general for superexponentiation we need to link in a 'binomial' formula an arbitrary number of adjacent superexponential operations.

17.2. Polymagmas.

A *polymagma* maps from p copies of M

$$(M \times M \times ... \times M) \xrightarrow{m} M \quad (1)$$

where the mapping is enclosed within M. We will not necessarily allocate M as a set. If M is a finite set, its n elements $m_1, m_2, ... , m_n$ connect the p-fold product of (1) in a mapping from $n^p \rightarrow n$ states. We will assume the elements $m_1, m_2, ... , m_n$ can be provided with an ordering.

We do not assume

$m(M \times M \times M) = m(m(M \times M) \times M)$

or

$$m(M \times M \times M) = m(M \times m(M \times M)),$$

since the products on the right combine

$$(M \times M) \xrightarrow{m} M \qquad (2)$$

and then we compose from the codomain of (2)

$$(M \times M) \xrightarrow{m} M$$

so that the number of possible states from which the mapping m is derived is given by n^2, which maps $n^2 \rightarrow n$, whereas

$$(M \times M \times M) \xrightarrow{m} M \qquad (3)$$

maps $n^3 \rightarrow n$, which in general is a mapping of more states.

The polymagma may be represented by a hypercube of length n and dimension p describing bijectively the states of the codomain of the polymagma.

A polymagma where the codomain satisfies

$$m(m(M \times M) \times M) = m(M \times m(M \times M))$$

will be called *associative*, and when for $(M \times M \times M)$ the codomain is given by one of

$$m(m(M \times M) \times M) \neq m(M \times m(M \times M)),$$

it will be called *nonassociative*. When for example

$$m(M \times M \times M) \neq m(m(M \times M) \times M)$$

and

$$m(M \times M \times M) \neq m(M \times m(M \times M)),$$

we will refer to a *crude* (3-dimensional) polymagma.

17.3. The $\mathbb{D}$w exponential algebras.

[Ad14] proves the inconsistency of the real number system using transfinite induction, amongst other techniques. We will adopt the term ‘Eudoxus number’ to replace the words ‘real number’, and denote the set of Eudoxus numbers by $\mathbb{U}$. Let a, b, c, d $\in \mathbb{U}$. $\uparrow$ denotes exponentiation.

The exponential algebra satisfies

$$(a + b)^c = a^c + ca^{c-1}b + \frac{c(c-1)}{2}a^{c-2}b^2 + ... + b^c \qquad (1)$$

$$(a^b)^c = a^{bc}, \qquad (2)$$

$$a^{b+c} = a^b a^c. \qquad (3)$$

So

$$a\uparrow[b^c - b^c] = a^0 = 1,$$

$$a\uparrow[(b^c)(b^c)^{-1}] = a^1 = a,$$

$$a\uparrow[(b + c)^d] = a\uparrow[b^d + db^{d-1}c + \frac{d(d-1)}{2}b^{d-2}c^2 + ... + c^d]$$

$$= [a\uparrow(b^d)].[a\uparrow(db^{d-1}c)].[a\uparrow(\frac{d(d-1)}{2}b^{d-2}c^2)]. ... [a\uparrow(c^d)]. \square$$

For w an integer and to the base a, the complex $\mathbb{D}$w exponential algebra satisfies

$$(a^b)^c = a^{(bc)}, \qquad (4)$$

$$(a^{ib})^c = a^{(ibc)}, \qquad (5)$$

$$(a^b)^{ic} = a^{(ibc)}, \qquad (6)$$

$$(a^{ib})^{ic} = a^{(iwbc)}. \square \qquad (7)$$

17.4. Superexponential algebras for n $\geq$ 1.

[Ad14], but for ladder numbers, describes the following situation.

The specification of the algebra of some complex superexponential operators is currently considered an unsolved problem. We investigate this in section 6.

In order to develop notions (semantics) utilising superexponentiation, it is necessary to develop a suitable notation (syntax), so that meanings can be expressed by symbolic manipulations – 'meaning' being specific instances of mappings from symbolic or other representational manifestations to the objective world.

To this end we introduce the superexponentiation $\Uparrow$ symbol. We adjoin a number, say n, to this operation, so that when n = 1 we are dealing with addition, when n = 2 with multiplication, and n = 3 with exponentiation. The higher-order superexponentiation operations, for n > 3, will be described inductively by stepping down to the (n – 1) case, and so on.

Since exponentiation for ladder numbers is not in general associative, that is, very often $(a\uparrow(b\uparrow c)) \neq ((a\uparrow b)\uparrow c)$, we have decided on a representation that describes a regular nesting of brackets, so that when this regular nesting occurs, we may dispense with brackets.

In particular, we introduce ${}^{n}\Uparrow$ to indicate nesting on the left, for example

$$(((a\ {}^{n}\Uparrow b)\ {}^{n}\Uparrow c) \ldots {}^{n}\Uparrow d) \equiv a\ {}^{n}\Uparrow b\ {}^{n}\Uparrow c \ldots {}^{n}\Uparrow d.$$

We introduce an alternative notation for the above, which is intended to be used sparingly, for example when n is a complicated expression, for emphasis, removing ambiguity, or calculation rather than display. This is

$<n\Uparrow$ for ${}^{n}\Uparrow$.

For nesting on the right, we introduce a completely analogous notation, namely

$$(a \Uparrow^{n} \ldots (b \Uparrow^{n} (c \Uparrow^{n} d))) \equiv a \Uparrow^{n} \ldots b \Uparrow^{n} c \Uparrow^{n} d,$$

and the equivalent non-superscript notation containing $a_h \Uparrow n> b_{h+1}$.

For a crude polymagma, the expression does not depend on parentheses, so we write

$$(a\ {}^{n}\Uparrow a\ {}^{n}\Uparrow a \ldots {}^{n}\Uparrow a) \text{ (m terms)} = (a \Uparrow^{n} \ldots a \Uparrow^{n} a \Uparrow^{n} a) \text{ (m terms)}.$$

Definintion 17.4.1. Let r, a and b belong to a field, novanion or zero algebra and $n \in \mathbb{N}$. A *left superexponential interpolation* ${}_{n}I(r, a, b) = a\ {}^{n}\Uparrow b$ when r = n. A *right superexponential interpolation* satisfies $I_n(r, a, b) = a \Uparrow^{n} b$ when r = n.

17.5. Superexponential algebras for n < 1.

We wish to introduce superexponentiation for n < 1. In the implementation we will give, all terms in an expression are significant in its evaluation, independently of local bracketing. For n = 0, we start off by defining

$$a\ {}^{0}\Uparrow a\ {}^{0}\Uparrow a \ldots {}^{0}\Uparrow a \text{ (m terms)} = a\ {}^{1}\Uparrow m = a + m, \qquad (1)$$

so that

$$a\ {}^{0}\Uparrow a = a + 2, \qquad (2)$$

with

$$(((a\ {}^{0}\Uparrow a)\ {}^{0}\Uparrow a) \ldots {}^{0}\Uparrow a) \text{ (m terms)} = a + 2(m – 1), \qquad (3)$$

$(a\ {}^0\!\Uparrow \ldots (a\ {}^0\!\Uparrow (a\ {}^0\!\Uparrow a)))$ (m terms) $= (m - 1)a$, (4)

giving for m terms

$(((a\ {}^0\!\Uparrow a)\ {}^0\!\Uparrow a) \ldots {}^0\!\Uparrow a) + (a\ {}^0\!\Uparrow \ldots (a\ {}^0\!\Uparrow (a\ {}^0\!\Uparrow a)))$

$= a\ {}^0\!\Uparrow a\ {}^0\!\Uparrow a \ldots {}^0\!\Uparrow a + (m - 1)a + m.$ (5)

The neutral element for the ${}^0\!\Uparrow$ operator satisfies

$a\ {}^0\!\Uparrow a = a,$ (6)

that is

$a + 2 = a.$ (7)

We have an implementation for this, arithmetic (mod 2), but now all a are neutral elements.

We can, however, define ${}^0\!\Uparrow$ as a crude polymagma, with the m-fold operation defined by (1), and similarly for the equivalent operation $\Uparrow^0$. We now express $a\ {}^0\!\Uparrow a\ {}^0\!\Uparrow a \ldots {}^0\!\Uparrow a$ (m terms) as

$$\left[\sum_{r=1}^{m}\left(\frac{a}{m}\right)\right] + m = \sum_{r=1}^{m}\left[\left(\frac{a}{m}\right) + 1\right].$$

If we now interpret $a_1\ {}^0\!\Uparrow a_2\ {}^0\!\Uparrow a_3 \ldots {}^0\!\Uparrow a_m$ (m terms) as $\sum_{r=1}^{m}\left[\left(\frac{a_r}{m}\right) + 1\right]$ then

$$a\ {}^0\!\Uparrow b = \left(\frac{a}{2}\right) + \left(\frac{b}{2}\right) + 2. \quad (8)$$

Similarly

$$a\ {}^{-1}\!\Uparrow a\ {}^{-1}\!\Uparrow a \ldots {}^{-1}\!\Uparrow a \text{ (b terms)} = a\ {}^0\!\Uparrow b = \left(\frac{a}{2}\right) + \left(\frac{b}{2}\right) + 2, \quad (9)$$

thus when b = 2

$$a\ {}^{-1}\!\Uparrow a = a\ {}^0\!\Uparrow 2 = \left(\frac{a}{2}\right) + 3,$$

and again we interpret

$$a\ {}^{-1}\!\Uparrow b = \left(\frac{a}{4}\right) + \left(\frac{b}{4}\right) + 3,$$

so that

$$a_1\ {}^{-1}\!\Uparrow a_2\ {}^{-1}\!\Uparrow a_3 \ldots {}^{-1}\!\Uparrow a_m \text{ (m terms)} = \left[\frac{1}{2}\sum_{r=1}^{m}\left(\frac{a_r}{m}\right)\right] + \frac{m}{2} + 2,$$

and in general

$$a_1\ {}^{-n}\!\Uparrow a_2\ {}^{-n}\!\Uparrow a_3 \ldots {}^{-n}\!\Uparrow a_m \text{ (m terms)} = \frac{1}{2^n}\left[\sum_{r=1}^{m}\left[\left(\frac{a_r}{m}\right) + 1\right]\right] + \sum_{r=1}^{n}\left(\frac{1}{2^{r-1}}\right) + n. \ \square$$

17.6. The tetration and superexponential complex algebras.

The n = 4 operation is known as tetration. We desire to develop a superexponentiation algebra which like exponential algebra $\mathbb{D}1$ does not branch as real values derived from imaginary ones, except obtained from $e^{in\pi}$.

For n > 2 the exponential operations for the superexponentiation $\mathbb{D}w$ algebras we specify as satisfying the rules for a field and

$(a^{i\lambda})\Uparrow^n b = (a^{\lambda})\Uparrow^n ib,$

$(a^{i\lambda})\Uparrow^n ib = (a^{iw\lambda})\Uparrow^n b,$

and for left nesting

$(a^{\lambda})\ {}^n\!\Uparrow b = a^{(\lambda < n-1 \Uparrow b)},$

$(a^{i\lambda})\ {}^n\!\Uparrow b = a^{i(\lambda < n-1 \Uparrow b)},$

$(a^{\lambda})\ {}^n\!\Uparrow ib = a^{i(\lambda < n-1 \Uparrow b)},$

$(a^{i\lambda})\ {}^n\!\Uparrow ib = a^{iw'(\lambda < n-1 \Uparrow b)},$

where in general $w \neq w'$, and these may be complex numbers. $\square$

17.7. The lower $\mathcal{JAF}$ $\mathbb{D}$w exponential algebra.

For matrices in the intricate representation we employ the following assumptions.

(1) The binomial theorem applies. This means an intricate expression in $\mathcal{JAF}$ format

$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^{(h + j\mathcal{J})}$

is evaluated as

$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^h.(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})^{j\mathcal{J}}$,

where a, b, c, d, h and j $\in \mathbb{U}$.

(2) We emphasise that the upper component enclosed in brackets, $(h + j\mathcal{J})$, is formed by converting to intricate $\mathcal{JAF}$ format specifically for $\mathcal{J}$, and the lower term in $\mathcal{JAF}$ format, being $(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})$, includes the *same* term $\mathcal{J}$.

This is because for intricate i, α, ϕ

$a^{p1 + (qi + r\alpha + s\phi)t} \neq a^{p1}.a^{qti}.a^{rt\alpha}.a^{st\phi}$,

but with $\mathcal{J}^2 = (qi + r\alpha + s\phi)^2 = (-q^2 + r^2 + s^2) = \pm 1$ or 0,

$a^{p1 + t\mathcal{J}} = a^{p1}.a^{t\mathcal{J}}$.

(3) We form the 'lower algebra' evaluation of $\mathcal{JAF}$ exponentials:

$\mathcal{J}^{\mathcal{J}} = \mathcal{J}$, $\mathcal{A}^{\mathcal{J}} = \mathcal{A}$ and $\mathcal{F}^{\mathcal{J}} = \mathcal{F}$.

Once chosen, this evaluation is unique, including for intricate terms like

$(a + b\mathcal{J} + c\mathcal{A} + d\mathcal{F})\uparrow[(f + g\mathcal{J})\uparrow(h + k\mathcal{J})]$.

For the hyperintricate representation, when intricate A and B each in four variables form the hyperintricate A_B, which is called J-abelian, since a general hyperintricate X has 16 variables, A_B does not represent this form, so that additively we need to allocate a C_D with

$X = A_B + C_D$.

There can be no direct multiplicative representation, since

$E_FG_H = (EG)_{(FH)}$

is J-abelian. □

17.8. Inverse operations.

For addition in a field there exists what we will call a neutral element 0, satisfying

$a + 0 = a = 0 + a$,

and for multiplication a neutral element 1 with

$a \times 1 = a = 1 \times a$.

For exponentiation, this is not commutative, so the left neutral element v and the right neutral element w differ:

$a \uparrow w = a \uparrow 1 = a$

but

$v \uparrow a = ((a \uparrow (1/a)) \uparrow a) = a$,

so that the left neutral element v is $(a \uparrow (1/a))$, and the right neutral element w is 1.

There is the question of the value of the expression 0^0. For a field, 0^{-1} is not defined, and therefore neither is $0^1.0^{-1} = 0^0$, but for a zero algebra

$(a0)^0 = (a0)^1(a0)^{-1} = 1$.

A left neutral element, v_{an}, under a superexponential operation $<n\Uparrow$ satisfies

$$v_{an} {}^{n}\Uparrow a = a,$$

where v_{an} is in general dependent on a, and a right neutral element for the superexponential operation $<n\Uparrow$ given by w_{an} has

$$a {}^{n}\Uparrow w_{an} = a,$$

with similar cases for $\Uparrow n>$.

Then even in the nonassociative and noncommutative case we define the superexponential operation as satisfying a contravariant (order reversing) operation on a left inverse element $a_L^{n\sim}$

$$a {}^{n}\Uparrow a_L^{n\sim} = v_{an},$$

and for a right inverse $a_R^{n\sim}$

$$a_R^{n\sim} {}^{n}\Uparrow a = w_{an}. \square$$

17.9. Analogues of matrices.

Matrices $A = a_{ik}$ and $B = b_{ik}$ satisfy

$$A + B = a_{ik} + b_{ik}$$

and

$$AB = \sum_j a_{ij} b_{jk}.$$

Similarly we will say for n-superexponential operations that they satisfy

$$A <m\Uparrow B = a_{ik} <m\Uparrow\ b_{ik},$$

for $m < n$ and

$$A <n\Uparrow B = <n-1\Uparrow_j (a_{ij} <n\Uparrow b_{jk})$$

where $<n-1\Uparrow_j$ indicates that the operation $<n-1\Uparrow$ combines a_{ij} and b_{jk} in sequence over all values of j. $\square$

17.10. Commutators and associators.

For matrix multiplication we have already met the commutator as used in physics

$$[AB] = AB - BA,$$

where we will call the minus sign here the *external* operator. The equation satisfies the Jacobi identity

$$[A[BC]] + [B[CA]] + [C[AB]] = 0,$$

and also the Jacobi identity in the form

$$[[AB]C] + [[BC]A] + [[CA]B] = 0.$$

We can introduce the *anticommutator*

$$\{AB\} = AB + BA,$$

with an external operator of +. On combining the commutator and anticommutator we have

$$AB = ½[AB] + ½\{AB\}.$$

The *associator* is

$$<ABC> = (AB)C - A(BC)$$

and the *antiassociator* is

$$|ABC| = (AB)C + A(BC),$$

so that

$$(AB)C = ½<ABC> + ½|ABC|,$$

and the last identity gives

$$<ABC> = ½([AB]C – A[BC] + \{AB\}C – A\{BC\}).$$
$$|ABC| = ½([AB]C + A[BC] + \{AB\}C + A\{BC\}).$$

For exponentiation $A\uparrow B$, we can also introduce a type of commutator

$$[A\uparrow B] = (A\uparrow B) – (B\uparrow A).$$

We can also verify that this obeys the extended Jacobi identity

$$[A\uparrow[B\uparrow C]] + [B\uparrow[C\uparrow A]] + [C\uparrow[A\uparrow B]]$$
$$+ [[A\uparrow B]\uparrow C] + [[B\uparrow C]\uparrow A] + [[C\uparrow A]\uparrow B] = 0,$$

which follows from identities of the type

$$[A\uparrow[B\uparrow C]] = -[[B\uparrow C]\uparrow A].$$

For the same reason, these identities extend to the case where $\uparrow$ is replaced by $\upuparrows^n$ or ${}^n\upuparrows$.

We observe that if we represent the corresponding anticommutator

$$\{A\uparrow B\} = (A\uparrow B) + (B\uparrow A),$$

we now have the identity

$$\{A\uparrow\{B\uparrow C\}\} + \{B\uparrow\{C\uparrow A\}\} + \{C\uparrow\{A\uparrow B\}\}$$
$$– \{\{A\uparrow B\}\uparrow C\} – \{\{B\uparrow C\}\uparrow A\} – \{\{C\uparrow A\}\uparrow B\} = 0$$

from

$$\{A\uparrow\{B\uparrow C\}\} = \{\{B\uparrow C\}\uparrow A\},$$

etc., which again extends from the operation $\uparrow$ to $\upuparrows^n$ or ${}^n\upuparrows$.

We will also write

$[AB]$ as $[AB]_1$

where $\upuparrows 1>$ is the external operation of addition or subtraction,

$\{AB\}$ as $\{AB\}_1$,

$<ABC>$ as $<ABC>_1$,

$|ABC|$ as $|ABC|_1$.

Then for an external operation of matrix multiplication

$$[AB]_2 = AB/BA = (AB)(BA)^{-1} = ABA^{-1}B^{-1},$$

thus

$$[AB]_2[BA]_2 = ABA^{-1}B^{-1}BAB^{-1}A^{-1} = 1.$$

Likewise

$$\{AB\}_2 = ABBA,$$

so

$$\{AB\}_2\{(BA)^{-1}\}_2 = \{AB\}_2\{A^{-1}B^{-1}\}_2 = 1.$$

Similarly, since

$$[A\uparrow B]_2 = [A\uparrow B]/[B\uparrow A]$$

and

$$A^BB^{-A}.B^AA^{-B} = 1,$$

this is just

$$[A\uparrow B]_2[B\uparrow A]_2 = 1.$$

We are now interested whether

$$[A\upuparrows^n B]_2[B\upuparrows^n A]_2 = 1,$$

but

$$(B\upuparrows^n A)/(B\upuparrows^n A) = 1,$$

so this follows directly.

17.11. Analogues of differentiation and integration.

The difference operator acting on a function f(x) for fields is the expression

$$\frac{f(x+\delta)-f(x)}{\delta},$$

and the commutative and associative differentiable operator for fields is defined by

$$\lim_{\delta\to 0}\frac{f(x+\delta)-f(x)}{\delta},$$

where we define this to be the evaluation firstly of the numerator divided by the denominator, then all terms varying with $\delta \neq 0$ are suppressed. "Evaluation" here includes equating all δ / δ to 1, and "suppress", which follows evaluate, includes setting all terms containing δ in positive powers to zero. If terms with δ in negative powers are set to zero, the derivative then specifies its convergent part.

For a superexponential operator $<n\Uparrow$, with the limit tending to the left neutral element its analogue is

$$\lim_{\delta\to v_{\delta\, n-1}} [f\big(x <n{-}1\Uparrow \delta\big) <n{-}1\Uparrow f(x)_L^{n-1\sim}] <n\Uparrow\delta_L^{n\sim}, \qquad (1)$$

and for the limit tending to the right neutral element, the derivative is

$$\lim_{\delta\to w_{\delta\, n-1}} \delta_R^{n\sim}\Uparrow n>[f(x)_R^{n-1\sim}\Uparrow n{-}1>f\big(x\Uparrow n{-}1> \delta\big)]. \qquad (2)$$

For example, let us choose n = 3. For the neutral element

$$v_{\delta 2} = 1 = w_{\delta 2},$$

and

$$v_{\delta 3} = (\delta\uparrow(1/\delta)),$$

whereas

$$w_{\delta 3} = 1.$$

Then

$$\delta\ {}^2\!\Uparrow 1 = \delta\times 1 = \delta = 1\times\delta = 1 \Uparrow^2 \delta,$$

giving

$$\delta_L^{2\sim} = (1/\delta) = \delta_R^{2\sim},$$

and $\delta_L^{3\sim}$ satisfies

$$\delta\ {}^3\!\Uparrow \delta_L^{3\sim} = \delta\uparrow\delta_L^{3\sim} = v_{\delta 3} = (\delta\uparrow(1/\delta)),$$

and in a similar way

$$\delta_R^{3\sim} \Uparrow^3 \delta = w_{\delta 3} = 1,$$

with

$$\delta_R^{3\sim} = 1 \uparrow (1/\delta).$$

Thus the differential of equation (1) is evaluated in the case n = 3 as

$$\lim_{\delta\to 1}[f(x\delta) / f(x)]^{1/\delta},$$

and the differential of equation (2) as

$$\lim_{\delta\to 1}(1^{1/\delta})[f(x\delta) / f(x)].$$

We will look at the function $f(x) = x^x$. Then

$$f(x\delta) / f(x) = \frac{(x\delta)^{x\delta}}{x^x} = \frac{x^{x\delta}\delta^{x\delta}}{x^x}.$$

It is clear that in the limit $\delta\to 1$ this evaluates to 1. Thus in this example the left derivative given by equation (1) for $f(x) = x^x$ is 1. For the right derivative of equation (2) we have

$$\lim_{\delta\to 1}(1^{1/\delta}) = 1,$$

and thus in this case the left and right superexponential derivatives are the same.

A similar argument for the function f(x) = x or f(x) = 1 gives a superexponential n = 3 derivative of 1. Thus in these cases of superexponentiation the derivative is trivial.

The equivalent of the difference operator for superexponential operators, which is obtained by removing the limit in equations (1) and (2), does however lead to nontrivial algebra. □

17.12. Analogues of varieties.

The superexponential analogue of a variety, called a *supervariety*, contains a string with a number of matching interior brackets set equal to zero, where each term contains within the brackets constants and variables in any defined order separated by superexponential operations of arbitrary degree for which the interpretation of the terms is given uniquely.

Superpolynomials, which when set to 0 we call supervarieties, are the basic building blocks of superexponential algebra. For logic, they are related to the colour sets of *Number, space and logic* [Ad18]. For the branched spaces of [Ad18], they describe an extended idea of the Euler-Poincaré characteristic.

17.14. Exercises.

Develop a difference algebra which is the finite case of differentiation in section 17.11.