CHAPTER XVI

The $\mathbb{D}$w hyperintricate exponential algebras

16.1. Introduction.

We use the hyperintricate representation of matrices and explore exponentiation for these objects. Proofs by contradiction are employed to eliminate a number of possibilities for intricate exponentiation, and a novel hyperintricate exponential algebra is finally adopted.

16.2. The search for consistency for intricate exponential algebras.

For proposal $\mathbb{A}1$, the value of i^i is often derived as follows

$$i^i = [e^{i(\pi/2 + 2\pi z)}]^i = e^{-\pi/2 + 2\pi z}, \quad (1)$$

with $z \in \mathbf{Z}$, which is a real multifunction. In J-abelian format and the $\mathcal{JAF}$ format, for $\mathcal{J}^2 = -1$, $\mathcal{A}^2 = 1$ a general intricate number may be represented for $b^2 > c^2 + d^2$ by

$$[a + \mathcal{J}(b^2 - c^2 - d^2)^{½}]e^{\mathcal{J}\theta} \quad (2)$$

where $e^{\mathcal{J}\theta} = \cos\theta + \mathcal{J}\sin\theta$, and for $b^2 < c^2 + d^2$ by

$$[a + \mathcal{A}(-b^2 + c^2 + d^2)^{½}]e^{\mathcal{A}\theta} \quad (3)$$

where $e^{\mathcal{A}\theta} = \cosh\theta + \mathcal{A}\sinh\theta$. The determinant in cases (2) and (3) is $a^2 + b^2 - c^2 - d^2$, and may be negative for (3).

Putting $\theta = \pi/2$, taking the exponent of $\mathcal{J}$ of equation (2) and specialising to $\mathcal{J} = i$ indicates that (1) is the general form for i^i when $c = d = 0$. □

We will now investigate exponent zero solutions, and see whether they remain valid for intricate numbers under the equivalence class (3).

We call an evaluation of an expression derived from formulas f_1, ... f_n *inconsistent* if two separate instances of composite formulas applied to the expression, including possibly the identity formula, give no values for the expression equal to itself.

To investigate assignations of m^n where now m and n are intricate basis elements, consider proposals of type $\mathbb{A}$ under assumptions which are documented as we proceed

$$1^i = (e^0)^i = e^{0i} = e^0 = 1,$$

so that

$$1 = 1^i = (i.i.i.i)^i = (i^i)(i^i)(i^i)(i^i) = (i^i)^4,$$

giving proposals $\mathbb{A}2$ and $\mathbb{A}3$ respectively

$$i^i = \pm 1, \pm(ti + u\alpha + v\phi),$$

with $-t^2 + u^2 + v^2 = \pm 1$, or more generally for proposal $\mathbb{A}4$ a number $g = i^i$ satisfying

$$g^4 = 1 = 1^i = (i^i)^4. \ □$$

We eliminate the possibility $i^i = \pm 1$. If $i^i = 1$, then

$$1 = 1^i = (i^i)^i = i^{-1} = -i,$$

and if $i^i = -1$, then

$$-i = (i^i)^i = (-1)^i = (i^i)(i^i),$$

so

$$-i = 1.$$

We can eliminate the specific circumstances $i^i = \pm i$, $\pm\alpha$ and $\pm\phi$. If $i^i = i$, then

$$-i = i^{-1} = (i^i)^i = i^i = i,$$

if $i^i = -i$, then

$$-i = i^{-1} = (i^i)^i = (-i)^i = (i.i.i)^i = (i^i)(i^i)(i^i) = i,$$

if our choice were $i^i = \alpha$, then since $\alpha = \alpha^{-1}$

$$(i^i)^{-i} = (i^i)^i,$$

so

$$(i^i)^{-i} = i = (i^i)^i = -i,$$

where a similar comment can be made for $i^i = \phi$, likewise if $i^i = -\alpha$, since $(-\alpha) = (-\alpha)^{-1}$,

$$(i^i)^{-i} = (i^i)^i,$$

leading to the same contradiction, and also for $i^i = -\phi$. □

The inverse of

$$T = \pm(ti + u\alpha + v\phi),$$

is

$$T^{-1} = \pm[(ti + u\alpha + v\phi)]/(t^2 - u^2 - v^2),$$

where the sign for T^{-1} is dependent solely on $(t^2 - u^2 - v^2)$, and is thus effectively independent of T. Thus if $i^i = T$, then since $T = \pm T^{-1}$, if $T = T^{-1}$, we obtain

$$(i^i)^{-i} = (i^i)^i,$$
$$i = -i.$$

However for $T = -T^{-1}$, apart from the allocation $t = 1$, $u = v = 0$, which we have already discounted, there appears to be no obstruction. For example

$$i^i = T = -T^{-1}$$

implies

$$-i = (i^i)^i = (-1)^i[(i^i)^{-i}] = (i^i)(i^i)i = T^2 i,$$

but

$$T^2 = (-t^2 + u^2 + v^2) = -1,$$

which gives a consistent result.

We now have a proposal of type $\mathbb{A}3$ in which

$$i^i = \pm(ti + u\alpha + v\phi),$$

with $-t^2 + u^2 + v^2 = -1$ and $t > 1$ could be consistent. We must match this against what is known for the Euler formulas.

We note that

$$e^{TK} \neq e^{\pm tiK}e^{\pm u\alpha K}e^{\pm v\phi K},$$

since such exponential multiplication is non-commutative in general, and so does not map to addition in the usual sense. The correct argument is that $T^2 = -1$ implies

$$e^{T\theta} = \cos K + T\sin K,$$
$$= \cos K \pm (ti + u\alpha + v\phi)\sin K.$$

with T expressed in the first instance in terms of intricate variables as

$$T = i^i = \pm(ti + u\alpha + v\phi),$$

so $e^{p1+(qi + r\alpha + s\phi)}$ may be evaluated, for $K \in R$ and $j \in$ intricate И with $j \notin R$, as

$$q = tK,$$
$$r = uK$$

and

$$s = vK.$$

For either u = 0 or v = 0, $T^2 = -1$ ascribes the value (putting, say, u = 0)

$$v = \pm(t^2 - 1)^{1/2}.$$

Thus an allocation of i^i compatible with the Euler relations is

$$i^i = T = ti \pm (t^2 - 1)^{1/2}\phi,$$

with $|t| > 1$.

We will show that the choice $T = i^i = \pm(ti + u\alpha + v\phi)$, leads to multivalued functions. This is evident in the relation

$$T^T = [e^{T(\pi/2 + 2\pi z)}]^T$$
$$= e^{-(\pi/2 + 2\pi z)},$$

under the assumption that

$$(a^b)^c = a^{(bc)}.$$

Further, a real valued t commutes as an additive exponential, so that for a real variable $x = -t(\pi/2 + 2\pi z)$, then

$$e^{x + y} = e^x e^y.$$

Putting again u = 0, so that y above is a term in a single basis element, then

$$T^i = e^{(-t - v\alpha)(\pi/2 + 2\pi z)}$$
$$= e^{-t(\pi/2 + 2\pi z)}[\cosh -v(\pi/2 + 2\pi z) + \alpha \sinh -v(\pi/2 + 2\pi z)],$$

which is clearly multivalued. Moreover, it does not satisfy

$$T^i = (i^i)^i = -i. \ \square$$

16.3. The exponential algebra $\mathbb{A}4$ for $g = i^i$.

To generalise the Euler relations, define

$$\chi_{t,u} = \chi_{t,u}(\lambda) = \Sigma(n = 0 \text{ to } \infty)[\lambda^{tn+u}/(tn + u)!]$$

so that

$$\cos \lambda = \chi_{4,0} - \chi_{4,2},$$
$$\sin \lambda = \chi_{4,1} - \chi_{4,3},$$
$$\cosh \lambda = \chi_{4,0} + \chi_{4,2},$$
$$\sinh \lambda = \chi_{4,1} + \chi_{4,3}.$$

Taking $g^2 = J$, $J^2 = -1$, if

$$e^{g\lambda} = (\chi_{8,0} - \chi_{8,4}) + g(\chi_{8,1} - \chi_{8,5}) + g^2(\chi_{8,2} - \chi_{8,6}) + g^3(\chi_{8,3} - \chi_{8,7}),$$

then for $g^2 = -J$, $J^2 = -1$

$$e^{g\lambda} = (\chi_{8,0} - \chi_{8,4}) + g(\chi_{8,1} - \chi_{8,5}) - g^2(\chi_{8,2} - \chi_{8,6}) - g^3(\chi_{8,3} - \chi_{8,7}),$$

for $g^2 = J$, $J^2 = 1$

$$e^{g\lambda} = (\chi_{8,0} + \chi_{8,4}) + g(\chi_{8,1} + \chi_{8,5}) + g^2(\chi_{8,2} + \chi_{8,6}) + g^3(\chi_{8,3} + \chi_{8,7}),$$

and for $g^2 = -J$, $J^2 = 1$

$$e^{g\lambda} = (\chi_{8,0} + \chi_{8,4}) + g(\chi_{8,1} + \chi_{8,5}) - g^2(\chi_{8,2} + \chi_{8,6}) - g^3(\chi_{8,3} + \chi_{8,7}),$$

whereas for $g^4 = 0$

$$e^{g\lambda} = 1 + g\lambda + g^2\lambda^2/2 + g^3\lambda^3/6.$$

We will now explore the solution $g = i^i$ given in the third and fourth assignation of $e^{g\lambda}$ above, namely for $g^2 = \pm J$ and $J^2 = 1$. Clearly $g^4 = -1$ does not satisfy $(i^i)^4 = 1$.

If

$g^i = (i^i)^i = i^{-1} = -i$

then

$(g^i)(g^i) = (\pm J)^i = -1,$

which gives

$[(\pm J)^i]^2 = [(\pm J)^2]^i = 1^i = 1,$

and finally we have obtained an allocation for which as yet we have not derived any inconsistencies. □

We need to be clear that $g^2 \neq -1$, which otherwise would be an assignation with model g = i. The model with g = i corresponds to the classical Euler relation

$e^{i\theta} = \cos\theta + i\sin\theta.$

Our solutions are non-classical, consistent with the Euler relations for cosh and sinh, and in particular we have selected a choice where

$i^i \neq [e^{i\pi/2}]^i = e^{-\pi/2 + 2\pi z}.$

16.4. A model for the intricate $\mathbb{A}4$ $g = i^i$ exponential algebra.

Let us assume there is a model where

$g = i^i = \alpha^{½},$

which corresponds to a model with $g^4 = 1$, $g^2 \neq \pm 1$. Possible hyperintricate matrices for $\alpha^{½}$ are given in chapter XII section 7. This gives directly

$i^\phi = \alpha^{-½\alpha}.$

The reasoning applied also holds under the substitutions

$\alpha \to [c\alpha - (1 - c^2)^{½}\phi],$

$\phi \to [(1 - c^2)^{½}\alpha + c\phi],$

for which the algebra of i, α, ϕ remains true.

Multiplying $i^i = (\alpha^i)(\phi^i) = \alpha^{½}$ on the left by α^i, or on the right by ϕ^i gives

$\phi^i = \alpha^i\alpha^{½},$

$\alpha^i = \alpha^{½}\phi^i.$

What is α^i in this model?

$\alpha^i = (\alpha^{½})^{2i}$

$= (i^i)^{2i}$

$= (i^{-1})^2$

$= -1.$

Thus

$\phi^i = -\alpha^{½}.$

Taking a power of ϕ

$\alpha^\alpha = (-1)^\phi$

$\phi^\alpha = (-\alpha^{½})^\phi$

and for a power of α

$\alpha^{-\phi} = (-1)^\alpha,$

so that

$\alpha^{\phi} = (-1)^{-\alpha}$
$\phi^{\phi} = (-\alpha^{-½})^{\alpha}$.

The i^i root under non-commutation is now
$(\alpha\phi)^i = \alpha^i\phi^i = (-\phi\alpha)^i = (-1)^i\phi^i\alpha^i$.
Thus if we maintain an allocation of
$(-1)^i = (i^i)(i^i) = g^2$
we obtain the contradiction
$g = (\alpha\phi)^i = \alpha^{½}$
$= g^2\alpha^{½} = g^3$.

There are various ways of formulating a remedy, but we could comment that since roots are not unique, the roots confined to non-commutation
$(\alpha\phi)^i = \alpha^i\phi^i = (-\phi\alpha)^i = (-1)^i\phi^i\alpha^i$
can be allocated the value
$(-1)^i = 1$.
This is not strictly forbidden, but it is inelegant. □

16.5. The partition of intricate numbers under $\mathcal{JAF}$ format.

Under $\mathcal{JAF}$ transformations $i \leftrightarrow \mathcal{J}$, $\alpha \leftrightarrow \mathcal{A}$ and $\phi \leftrightarrow \mathcal{F}$, each intricate number may be allocated to a $\mathcal{J}$, $\mathcal{A}$ or $\mathcal{F}$ equivalence class, or to $\mathcal{J}^2 = 0$, where $\mathcal{J}^2 = -1$, $\mathcal{A}^2 = 1$ and $\mathcal{F}^2 = 1$; to the equivalence class for $\mathcal{J}$ when Я = a + bi + cα + dϕ satisfies $-b^2 + c^2 + d^2 < 0$, or to $\mathcal{A}$ when $-b^2 + c^2 + d^2 > 0$ and say $c^2 > d^2$, and to $\mathcal{J}^2 = 0$ when $-b^2 + c^2 + d^2 = 0$.

Under the algebras so far considered, when Я has form 2.(2) or 2.(3), for partitions to the $\mathcal{J}$ equivalence class we obtain
$\mathcal{J}^{\mathcal{J}} = [e^{\mathcal{J}(\pi/2 + 2\pi z)\mathcal{J}}] = e^{(-\pi/2 + 2\pi z)}$,
a multivalued function. For the $\mathcal{A}$ equivalence class
$e^{\mathcal{A}\theta} = \cosh\theta + \mathcal{A}\sinh\theta$,
and we obtain here allocations of i^i under the $g^4 = 1$ format.

Formulas of form 2.(1) have not disappeared, but we have now investigated solutions to i^i under the partition 2.(3).

16.6. Some further proposals on intricate exponential algebras.

The question we now wish to raise is: are multivalued functions a necessity for intricate exponential algebras? We review some other alternatives.

In order to meet consistency conditions under the above conditions of equality, we can make the following proposal – proposal $\mathbb{B}$. The relation of equality is not the same as equivalence. The relations deduced above under the = relation considered setting 1 = -1, i = -i, α = -α and ϕ = -ϕ under the exponential operations. Under an equivalence relation instead of these equalities, which corresponds to operations in the projective general linear group PGL(2), consistency returns.

In terms of intricate numbers, what is the identification algebra in $\mathbb{B}$? For a complex number, its norm is the determinant, and the complex number may be considered as a vector of unit norm, multiplied by its determinant, so that in reducing the complex number (mod 2), the norm (mod 2) is chosen which is multiplied by the vector of unit norm. For intricate numbers, the determinant may be zero or negative, but the same procedure works for nonzero norm. The operation is not defined for a singular matrix. ☐

For proposal $\mathbb{C}$, let Ш be the intricate number

$$Ш = p1 + qi + r\alpha + s\phi$$

so that its intricate conjugate is

$$Ш^* = p1 - qi - r\alpha - s\phi$$

so Ш** = Ш, in which

$$Ш^{-1} = Ш^*/(p^2 + q^2 - r^2 - s^2).$$

Then we define

$$Я^{nШ} = (Я^n)^{(Ш^*)}$$

where if $Я^{Ш}$ is wanted, the choice n = 1 will always be made, i.e. $(Я^1)^{Ш} = Я^{(Ш^*)}$.

As a consequence $(e^\theta)^{í} = e^{-i\theta}$. If n = 1, then $Я^1 = Я$, 1Ш = Ш and 1Ш* = Ш*, so the immediate question arises whether this is consistent. A type of nonassociativity is implied here with 1 as the middle term. Under what conditions is it meaningful to distinguish between $Я^{(nШ)}$ and $(Я^n)^{(Ш^*)}$? If we distinguish between the evaluation of $(e^\theta)^{í}$ and $e^{-i\theta}$, the leftmost expression involves evaluating a power of i, and the expression on the right evaluates a polynomial in natural number powers.

For the complex number

$$A = re^{i\theta}$$

we have

$$\log A = \log r + i\theta,$$

and thus for a complex number B under proposal $\mathbb{C}$

$$(A^1)^B = e^{B^* \log A} = e^{B^*(\log r + i\theta)}.$$

If B is allocated as C + iD, then under the proposal the formula for $(A^1)^B$ can be written more explicitly as

$$(r^C e^{D\theta})e^{i(-D\log r + C\theta)} = (r^C e^{D\theta})[\cos(-D \log r + C\theta) + i \sin(-D \log r + C\theta)].$$

The Euler formula is the basis of the conventional argument on the assignation of i^i, namely, for integer z,

$$e^{i[(\pi/2) + 2\pi z]} = i,$$

so introducing our proposal $\mathbb{C}$

$$\{e^{i[(\pi/2) + 2\pi z]}\}^i = e^{(\pi/2) + 2\pi z} = i^i,$$

and i^i is multivalued.

The previous assignations of $(i^i)^i = -i$ are now excluded. No consistent allocation of a particular value can be given in general, although it is possible to choose a principal value. If we view the solutions as complex multivalued functions, in which a formula satisfies a set of solutions under equivalence, then we can allocate, for a complex number G* with integer coefficients,

$$i^i = e^{G^*(\pi/2)}$$

to encompass all the above complex solutions under proposal $\mathbb{C}$. ☐

Proposal $\mathbb{D}1$ is the following.

$(e^{\lambda})^{\mu} = e^{\lambda\mu}$,
$(e^{i\lambda})^{\mu} = e^{i\lambda\mu}$,
$(e^{\lambda})^{i\mu} = e^{i\lambda\mu}$,
$(e^{i\lambda})^{i\mu} = e^{i\lambda\mu}$.

Once again, there are questions of consistency, and there are variants of the above proposal. To the question of how this algebra might be extended, finally, to intricate numbers, a response is to convert to $\mathcal{JAF}$ format, where the significant change from the standard contains

$[e^{\mathcal{J}\pi/2}]^{\mathcal{J}} = e^{\mathcal{J}\pi/2}$.

The multiplicative odd ($\mathcal{O}$)/even ($\mathcal{E}$) algebra satisfies

.	$\mathcal{O}$	$\mathcal{E}$
$\mathcal{O}$	$\mathcal{O}$	$\mathcal{E}$
$\mathcal{E}$	$\mathcal{E}$	$\mathcal{E}$

This Z_2 type of multiplication is precisely the type of exponential structure for proposal $\mathbb{D}1$, with 1 for $\mathcal{O}$ and i for $\mathcal{E}$.

As a variant, we can allocate, for basis elements m, n, p $\in\{1, i, \alpha, \phi\}$,

$(m^{n})^{\theta p} = (m^{\theta n})^{p} = m\uparrow[\theta(n^{p})]$,

with θ real. This differs in practice from proposal $\mathbb{B}$ where $(i^{i})^{1} \neq (i^{1})^{i}$, in that now

$(i^{i})\uparrow 1 = i\uparrow(i^{1}) = i$

and

$(i^{1})\uparrow i = i\uparrow(1^{i}) = i$.

Directly under proposal $\mathbb{D}1$ we obtain the following relations in $\mathcal{JAF}$ format

$$\begin{aligned}
[e^{k\pi/2}.e^{\mathcal{J}\pi/2}]^{\mathcal{J}} &= e^{(k+1)\mathcal{J}\pi/2} \\
&= e^{k\mathcal{J}\pi/2}.\mathcal{J} \\
&= [\cos(k\pi/2) + \mathcal{J}\sin(k\pi/2)].\mathcal{J} \\
&= -\cos[(k-1)\pi/2] - \mathcal{J}\sin[(k-1)\pi/2] \\
&= -e^{(k-1)\mathcal{J}\pi/2}.
\end{aligned}$$

Putting k = 1 gives

$-e^{0} = -1 = \cos\pi + \mathcal{J}\sin\pi = -1$,

so this aspect is consistent. □

We also introduce the $\mathbb{D}2$, $\mathbb{D}3$ and $\mathbb{D}4$ exponential algebras. For $\mathbb{D}2$

$[e^{\mathcal{J}\pi/2}]^{\mathcal{J}} = e^{\mathcal{J}\pi}$,

for $\mathbb{D}3$

$[e^{\mathcal{J}\pi/2}]^{\mathcal{J}} = e^{\mathcal{J}3\pi/2}$,

and for $\mathbb{D}4$

$[e^{\mathcal{J}\pi/2}]^{\mathcal{J}} = e^{-\mathcal{J}2\pi}$.

These are the only examples of allocations which are nonstandard. □

16.7. The $\mathbb{E}1$, $\mathbb{E}2$ and $\mathbb{E}3$ intricate exponential algebra proposals.

It is possible to introduce proposals where $(e^{i\lambda})^{i\mu} \neq e^{i\lambda\mu}$. These are of type $\mathbb{E}1$, where $(e^{i\lambda})^{i\mu} = e^{\alpha\lambda\mu}$ or $(e^{i\lambda})^{i\mu} = e^{\phi\lambda\mu}$, or involve hyperintricate variables, say allocating all initial variables as hyperintricates with a lower layer of 1, and an exponential of this as $e\uparrow[i_1\lambda 1_i\mu] = e\uparrow[i_i\lambda\mu]$. These are proposals of type $\mathbb{E}2$, a later modification being $\mathbb{E}3$, where the latter gives indications on the extension to an intricate algebra.

Proposals under the heading $\mathbb{E}1$ contain multifunctions since $(e^{i\lambda})^{i\mu} = e^{\alpha\lambda\mu}$ implies

$$i^i = \cosh(\pi/2 + 2\pi z) + \alpha \sinh(\pi/2 + 2\pi z). \square$$

Proposal $\mathbb{E}3$ is to consider exponential variables $e^{i\lambda}$ as elements of a 2-hyperintricate algebra with trailing layer 1 and that $(e^{i\lambda})^{i\mu}$ occurs where the second occurrence, the $i\mu$ variable, is represented by $1_i\mu$ with leading layer 1. Then

$$(e^{i\lambda})^{i\mu} = e\uparrow[(i_i)\lambda\mu]$$
$$= \cosh\lambda\mu + (i_i)\sinh\lambda\mu,$$

since $(i_i)^2 = 1$.

If it is understood that on forming expressions to the power i, all terms will be converted to exponential format for their evaluation, then we obtain multifunctions:

$$(-1)^i = (e^{i\pi + i2\pi z})^i = \cosh(\pi + 2\pi z) + i_i\sinh(\pi + 2\pi z)$$
$$= (i^i)(i^i) = e^{(i\pi/2 + i\pi/2)i} = \cosh^2(\pi/2) + \sinh^2(\pi/2) + 2i_i\cosh(\pi/2)\sinh(\pi/2). \square$$

The modification, $\mathbb{E}3$, we now introduce is formed from exponential variables $e^{i\lambda}$ as elements of a 3-hyperintricate algebra with trailing layers 1 and that the expansion of $(e^{i\lambda})^{i\mu}$ is implemented where the second occurrence, the $i\mu$ variable, is represented by $1_{ii}\mu$, which has a leading layer of 1. Then

$$(e^{i\lambda})^{i\mu} = e\uparrow[(i_{ii})\lambda\mu]$$
$$= \cos\lambda\mu + (i_{ii})\sin\lambda\mu,$$

since $(i_{ii})^2 = -1$.

The exponential operations involving i less than twice are defined by

$$(e^{\lambda})^{\mu} = e^{\lambda\mu},$$
$$(e^{i\lambda})^{\mu} = e\uparrow i_{11}\lambda\mu,$$
$$(e^{\lambda})^{i\mu} = e\uparrow i_{11}\lambda\mu,$$

so that $(e^{\lambda})^{i\mu}$ operates with identical formulas to $(e^{i\lambda})^{\mu}$.

Subsequent exponentiations have leading layer 1

$$\{(e^{i\lambda})^{i\mu}\}^{i\nu} = \{e\uparrow[(i_{ii})\lambda\mu]\}\uparrow i\nu$$
$$= e\uparrow\{[(i_{ii})\lambda\mu][1_{ii}\nu]\}$$
$$= e\uparrow[(i_{11})\lambda\mu\nu].$$

On forming expressions to the power i, all terms are converted to exponential format for their evaluation, so that for example

$$(-1)^i = (e^{i\pi + i2\pi z})^i = \cos(\pi + 2\pi z) + i_{ii}\sin(\pi + 2\pi z) = -1$$
$$= (i^i)(i^i) = e^{(i\pi/2 + i\pi/2)i} = \cos^2(\pi/2) - \sin^2(\pi/2) + 2i_{ii}\cos(\pi/2)\sin(\pi/2). \square$$

The extension to a general hyperintricate algebra is obtained under the relations

$$(\alpha_{\alpha\alpha})^2 = (\phi_{\phi\phi})^2 = (\alpha_{\phi\phi})^2 = (\phi_{\alpha\alpha})^2 = (\alpha_{ii})^2 = (\phi_{ii})^2 = 1$$

and

$$(i_{ii})^2 = (i_{\alpha\alpha})^2 = (i_{\phi\phi})^2 = -1,$$

with the understanding that

$$(e^{i\lambda})^{\alpha\mu} = e\uparrow[(i_{\alpha\alpha})\lambda\mu]$$
$$= \cos\lambda\mu + (i_{\alpha\alpha})\sin\lambda\mu,$$

and similarly for instance

$$(e^{\alpha\lambda})^{\phi\mu} = e\uparrow[(\alpha_{\phi\phi})\lambda\mu]$$
$$= \cosh\lambda\mu + (\alpha_{\phi\phi})\sinh\lambda\mu,$$

so that the complete $\mathbb{E}3$ exponential algebra can be generated. □

The reason we have discussed $\mathbb{E}3$ is that the intricate algebra for $\mathbb{D}1$ is now indicated – compact the two lower layers together to 1, or effectively nullify them. Thus the intricate basis elements chosen in $\mathbb{D}1$ are the ones that are presented first in the exponentiation.

The algebra with these hyperintricate basis elements is of cosh/sinh form where the squares of these basis elements are 1, and of cos/sin form where the squares are -1, and a J-abelian 3-hyperintricate number has basis elements as a summation of these types, or of types with 1 in some layers. These representations may also be converted to more general $\mathcal{JAF}$ format.

Division is not in general available. This implies roots are not always present.

Under proposal $\mathbb{E}3$ we now obtain the following relations in $\mathcal{JAF}$ format

$$[e^{k\pi/2}.e^{\mathcal{J}\pi/2}]^{\mathcal{J}} = e^{k\mathcal{J}\pi/2}[e\uparrow\mathcal{J}_{\mathcal{JJ}}\pi/2]$$
$$= e^{k\mathcal{J}\pi/2}.\mathcal{J}_{\mathcal{JJ}}$$
$$= [\cos(k\pi/2) + \mathcal{J}_{11}\sin(k\pi/2)].\mathcal{J}_{\mathcal{JJ}}$$

which is not a contradiction, as is the similar case under proposal $\mathbb{D}1$. □

16.8. The $\mathbb{D}$1 and $\mathbb{D}$w J-abelian exponential algebras.

A natural solution we adopt is to pursue the analogy of left nested exponentiation, ↑, with multiplication and define a type of ↑ distributivity, with the exponential term e, as follows:

$$[e\uparrow(ti + u\alpha + v\phi)]\uparrow(xi + y\alpha + z\phi) =$$
$$e\uparrow[t(x + y + z)]i + [u(x + y + z)]\alpha + [v(x + y + z)]\phi. \quad (1)$$

This noncommutative ring with unit under + and × is explicit enough to generate all the relations we need. There is no division operation for ↑, because if a real value is absent, ↑ cannot generate 1. The hyperintricate exponential algebra becomes available through this structure by the means described subsequently. □

An important question is now whether it is decidable that the above suggestion is consistent. All binomial type exponential operations are generated and defined via

$$[e^{p1 + (qi + r\alpha + s\phi)K}]\uparrow(a1 + bi + c\alpha + d\phi),$$

so we conclude the operation ↑ is as consistent as rings in general. □

The equation (4) is not $\mathcal{JAF}$ invariant. If we transform $i \to \mathcal{J}$, $\alpha \to \mathcal{A}$ and $\phi \to \mathcal{F}$, so that

$$ti + u\alpha + v\phi = t'\mathcal{J} + u'\mathcal{A} + v'\mathcal{F}$$

and

$$xi + y\alpha + z\phi = x'\mathcal{J} + y'\mathcal{A} + z'\mathcal{F}, \quad (2)$$

then if the right hand side of (1) remains invariant,

$$ti + u\alpha + v\phi(x + y + z) = (t'\mathcal{J} + u'\mathcal{A} + v'\mathcal{F})(x' + y' + z') \quad (3)$$

so we have

$$x + y + z = x' + y' + z', \quad (4)$$

but taking the intricate conjugate of (2)

$$-x^2 + y^2 + z^2 = -x'^2 + y'^2 + z'^2,$$

for which (4) by the binomial theorem does not hold for

$$x = 1,\ y = z = 0,$$

$$x' = \sqrt{(1 - y'^2 - z'^2)},$$

with x′, y′ and z′ positive. □

J-abelian hyperintricate exponentiation is the natural extension of the intricate case. For example for 4 x 4 matrices we have 16 basis elements, 8 of which can be put in one combined J-abelian representation.

Since the $\mathcal{JAF}$ ring transformations map $i \to \mathcal{J}$, $\alpha \to \mathcal{A}$ and $\phi \to \mathcal{F}$, but do not provide an invariant description of $\mathbb{D}1$ exponential and superexponential algebras, there is a spectrum of possible $\mathbb{D}1$ algebras which may be chosen, and we will present the $\mathbb{D}1$ exponential algebra spectrally, under the understanding that some specific $\mathcal{JAF}$ basis must be chosen throughout to maintain consistency.

For a hyperintricate number

$$\sum_j [\underline{\vee}_k(a_{kj} + b_{kj}J_{kj})] = \sum_j [\underline{\vee}_k e\uparrow(\rho_{kj} + \sigma_{kj}J_{kj})],$$

where $J_{kj}^2 = 0$ or ± 1, we evaluate

$$[e\uparrow(\rho + \sigma J)]\uparrow(\rho' + \sigma' J')$$

as the lower $\mathbb{D}1$ exponential algebra expression

$$e\uparrow(\rho\rho' + \sigma\rho' J + \sigma'\rho J' + \sigma\sigma' J),$$

and the Σ terms by a binomial expansion under this rule. Note that the last term for the exponentiated sum is $\sigma\sigma' J$ and not $\sigma\sigma' JJ'$ – this is the specific difference between the $\mathbb{D}1$ exponential algebra, which is minimally branched, and the standard exponential algebra.

As a $\mathcal{JAF}$ ring we select the *lower expansion*

$$e^v = [e\uparrow(r + s\mathcal{J} + t\mathcal{A} + u\mathcal{F})]\uparrow(r' + s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}'),$$

where

$$v = rr' + r(s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}') + r'(s\mathcal{J} + t\mathcal{A} + u\mathcal{F})$$
$$+ (ss' + st' + su')\mathcal{J} + (ts' + tt' + tu')\mathcal{A} + (us' + ut' + uu')\mathcal{F}. \quad (5)$$

This is related to an *upper expansion* which is different from (5), where we choose

$$v = rr' + r(s'\mathcal{J}' + t'\mathcal{A}' + u'\mathcal{F}') + r'(s\mathcal{J} + t\mathcal{A} + u\mathcal{F})$$
$$+ (ss' + ts' + us')\mathcal{J}' + (st' + tt' + ut')\mathcal{A}' + (su' + tu' + uu')\mathcal{F}'. \ \square \quad (6)$$

If we denote the exponential operator by ↑, then for J-abelian 2-hyperintricate numbers with selected intricate basis element components x, y and z, we define

$$(x_y)\uparrow z = (x\uparrow z)_{(y\uparrow z)},$$

with natural extensions to the J-abelian n-hyperintricate case. This structure is the $\mathbb{D}1$ hyperintricate exponential algebra.

This algebra is in accordance with what we would expect under the compression ring epimorphism κ of x_y, that is in this case

$$[\kappa(x_y)]\uparrow z = (x\uparrow z)(y\uparrow z).\ \square$$

For lower expansion $\mathbb{D}w$ exponential algebras, $w \in \mathbb{N}$, we adopt

$$[e^{\mathcal{J}^s}]^{\mathcal{J}} = [e^{\mathcal{J}^s}]^{\mathcal{A}} = [e^{\mathcal{J}^s}]^{\mathcal{F}} = e^{\mathcal{J}^{ws}},$$
$$[e^{\mathcal{A}^s}]^{\mathcal{J}} = [e^{\mathcal{A}^s}]^{\mathcal{A}} = [e^{\mathcal{A}^s}]^{\mathcal{F}} = e^{\mathcal{A}^{ws}},$$
$$[e^{\mathcal{F}^s}]^{\mathcal{J}} = [e^{\mathcal{F}^s}]^{\mathcal{A}} = [e^{\mathcal{F}^s}]^{\mathcal{F}} = e^{\mathcal{F}^{ws}},$$

and related extensions in the J-abelian case. □

16.9. Further reasoning on intricate binomials and Euler relations.

We will assume the *complex binomial theorem* is of the form

$$(a + ib)^{(c + id)} = (a + ib)^c(a + ib)^{id},$$

where $(a + ib)^{id}$ may also be expressed as $a^{id}(1 + i(b/a))^{id}$ or $(ib)^{id}(1 – i(a/b))^{id}$. Consequently there exist convergent solutions.

There are at least two ways of expanding out $(a + ib)^{id}$. The first is

$$(a + ib)^{id} = \{a^{id} + id(a^{id-1})ib + [id(id – 1)/2](a^{id-2})i^2b^2 + ...\},$$

and the second is

$$(ib + a)^{id} = \{(ib)^{id} + id((ib)^{id-1})a + [id(id – 1)/2]((ib)^{id-2})a^2 + ...\},$$

the first when $b \leq a$ and the second when $a \leq b$.

The binomial theorem may be converted to exponential algebra form under the allocation, for r Eudoxus, of $r^i = (e^{\lambda})^i = e^{i\lambda}$. For -r to an ith power this is $[(e^{\pi i})(e^{\lambda})]^i = -e^{i\lambda}$ with the warning, for example, that $[(e^{\pi i})(e^{\lambda\alpha})] \neq e^{\pi i + \lambda\alpha}$. Thus where terms are multiplied as e to powers, the result is not in general e to the sum, and multiplication and the multiplicative order have to be retained.

For 2-hyperintricates $(A + B)_{(C + D)} = A_C + A_D + B_C + B_D$, but $A_C + B_D$ is not of the form G_H in general, and thus we are reduced to considering general terms $E_F + G_H$ which cannot be further reduced. To apply the binomial theorem for hyperintricate exponentials, apply it to $E_F + G_H$. Because the hyperintricate algebra we have introduced can evaluate $E_F{}^L$ we look at

$$(E_F + G_H)^L = \{E_F(1 + E^{-1}{}_{F-1}G_H)\}^L. \qquad (1)$$

We now hit a problem. Since for general matrices

$$(AB)^1 = A^1B^1$$

but

$$(AB)^{-1} = B^{-1}A^{-1}$$

we cannot expect in all circumstances that

$$(E_F + G_H)^L = E_F{}^L(1 + E^{-1}{}_{F-1}G_H)^L,$$

so we can introduce the hypothesis that for $L = M + N$ with M having positive hyperintricate coefficients and N having negative ones, that

$$(E_F + G_H)^L = \{E_F{}^M(1 + E^{-1}{}_{F-1}G_H)^M\}\{(1 + E^{-1}{}_{F-1}G_H)^N E_F{}^N\} \qquad (2)$$

but this must be in accordance with what we have already said for J-abelian exponentiation. We know how to obtain the inverse E^{-1} and F^{-1} from chapter II, sections 15 and 16. Select a $\mathcal{JAF}$ basis and express $E_F{}^M$ and $(1 + E^{-1}{}_{F-1}G_H)^M$, etc. within it. Irreducible expressions will in general remain irreducible under these transformations. By use of the multinomial theorem of chapter XV section 9 we are now able to evaluate (2). □

16.10. Exercises.

(A) Show for a positive number a that

$(a^{b})^{c} = a^{(bc)}$,

$(a^{ib})^{c} = a^{(ibc)}$,

$(a^{b})^{ic} = a^{(ibc)}$,

$(a^{ib})^{ic} = a^{(iwbc)}$,

holds in general if and only if the same equations hold with the exponential number e replacing a.

Show that this also holds when a is negative if and only if it holds for (-e).