CHAPTER XV

Exponential algebra

15.1. Introduction.

We develop some elementary features of the intricate and hyperintricate *exponential algebra* and in particular the bosonic and fermionic Euler relations – an extension of the $e^{i\theta}$ expansion idea. Hyperintricate roots and the hyperintricate binomial theorem are also addressed.

15.2. The intricate Euler relations.

We will define

$$\cos\theta = \frac{e^{-i\theta} + e^{i\theta}}{2}, \tag{1}$$

$$\sin\theta = i\frac{e^{-i\theta} - e^{i\theta}}{2}, \tag{2}$$

$$\cosh\theta = \frac{e^{-\theta} + e^{\theta}}{2}, \tag{3}$$

$$\sinh\theta = \frac{-e^{-\theta} + e^{\theta}}{2}, \tag{4}$$

so that

$$\cos^2\theta + \sin^2\theta = 1, \tag{5}$$

$$\cosh^2\theta - \sinh^2\theta = 1. \tag{6}$$

Complex numbers satisfy the *bosonic Euler relation* with positive determinant

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{7}$$

which can be obtained from formulas (1) and (2) or using a Taylor series expansion, where

$$e^{\lambda} = 1 + \lambda + \lambda^2/2 + \lambda^3/3! + \ldots \tag{8}$$

For intricate basis elements α and ϕ, we might think a similar argument gives

$$e^{\alpha\theta} = \cosh\theta + \alpha\sinh\theta, \tag{9}$$

$$e^{\phi\theta} = \cosh\theta + \phi\sinh\theta. \tag{10}$$

However, we will see for instance that if

$$e^{\alpha\theta} = \alpha(\cosh\theta + \alpha\sinh\theta), \tag{11}$$

so that, in effect, we are defining

$$e^{\alpha\lambda} = \alpha(1 + \lambda + \lambda^2/2 + \lambda^3/3! + \ldots), \tag{12}$$

then in the case of equation (12), equation (6) still holds.

We have seen in chapter I, section 6, that the determinant of an intricate number

$$(a1 + bi + c\alpha + d\phi)(a1 - bi - c\alpha - d\phi) = a^2 + b^2 - c^2 - d^2. \tag{13}$$

But using (8) and an intricate number $e^{\alpha\lambda}$ multiplied by its complex conjugate $e^{-\alpha\lambda}$ is $e^{\alpha\lambda - \alpha\lambda} = e^0 = 1$, which is positive, and cannot represent (13) in the case considered when b = d = 0, and also $a^2 < c^2$. This forces us either to admit that there exist values of intricate numbers not represented by exponentials, or to adopt equations like (11) and (12). ☐

I want to add an observation here, that formula (9) or respectively (11), together with formula (8) or (12) do not work in a region greater or equal to zero and less than one, under the assumption that the absolute values of $\cosh\theta = \frac{e^{-\theta} + e^{\theta}}{2} > \sinh\theta = \frac{-e^{-\theta} + e^{\theta}}{2}$. This is true under the standard criterions of arithmetic we are using, and cannot be transferred to nonstandard numbers to allow for absolute values $\sinh\theta > \cosh\theta$. These are the reasons for the extension of (11) discussed in sections 4 and 5.

We now introduce an alternative formula to equation (7), the *fermionic Euler* relation

$$e^{\pi - i\theta} = \cos\theta + i\sin\theta, \quad (14)$$

so that now we are redefining cos θ as –(cos θ) in the previous allocation. This clearly still satisfies equation (5).

There are many formulas derivable from the Euler relations. Since

$$\cosh\theta = 1 + \theta^2/2 + \theta^4/4! + \ldots$$
$$\sinh\theta = \theta + \theta^3/3! + \theta^5/5! + \ldots$$

we also have

$$\cos\theta = \cos\alpha\theta = \cos\phi\theta,$$
$$\cosh\theta = \cosh\alpha\theta = \cosh\phi\theta,$$

and for instance, using $\alpha\phi = i$, and adopting equation (11) valid for determinants, we can set

$$\alpha = \pm(e^{\alpha\theta} - 1)/2^{½},$$
$$\phi = \pm(e^{\phi\theta} - 1)/2^{½},$$

in this equation with $\theta = \sinh^{-1}\pm 1 = \pm\cosh^{-1}\pm 2^{1/2}$. □

If we choose to represent

$$e^h = e^{a1 + bi + c\alpha + d\phi} = e^{a1}e^{bi}e^{c\alpha}e^{d\phi}, \quad (15)$$

then multiplicative noncommutativity, for example

$$e^{bi}e^{c\alpha} \neq e^{c\alpha}e^{bi}$$

from the Taylor series expansions, immediately tells us that for abelian addition (15) cannot hold, although a1 commutes. In the example above the noncommutation expressed as $e^{bi}e^{c\alpha} - e^{c\alpha}e^{bi}$ depends on ϕ, b and c only. This observation is unaltered if we substitute $\alpha e^{c\alpha}$ for $e^{c\alpha}$.

The square of $J = (bi + c\alpha + d\phi)$ is $(-b^2 + c^2 + d^2)$. On setting $z = a1 + JK$, for K real, when $J^2 = -1$ the Taylor expansion gives

$$e^z = e^{a1 + (bi + c\alpha + d\phi)K} = e^{a1+JK}$$
$$= e^{a1}(\cos K + J\sin K), \quad (16)$$

when $J^2 = +1$ assuming a Taylor expansion similar to equation (12)

$$e^z = e^{a1}J(\cosh K + J\sinh K), \quad (17)$$

and when $J^2 = 0$

$$e^z = e^{a1}(1 + JK + J^2K^2/2 + J^3K^3/3! + ...) = e^{a1}(1 + JK). \quad (18)$$

We note for representations of this form that

$$e^{a1 + JL + JM} = e^{a1}e^{JL}e^{JM} \text{ or } e^{a1}Je^{JL}e^{JM}. \text{ □} \quad (19)$$

It is useful to classify solutions. On putting

$$E = r\alpha + s\phi$$

so that

$$E^2 = r^2 + s^2,$$

then when

$$b^2 = E^2 + 1$$

we say the solution is trigonometric, implying we use trigonometric functions, when

$$b^2 = E^2,$$

the solution is linear and when

$$b^2 = E^2 - 1$$

the solution is hyperbolic, employing hyperbolic functions.

Incidentally, the expression $e^{p1+(qi+r\alpha+s\phi)K}$ is equal to

$$\tfrac{1}{2}e^{p1}[(1-q-s\alpha+r\phi)e^{-iK}+(1+q+s\alpha-r\phi)e^{iK}]$$

when $J^2 = -1$,

$$\tfrac{1}{2}e^{p1}J[(1-qi-r\alpha-s\phi)e^{-K}+(1+qi+r\alpha+s\phi)e^{K}]$$

when $J^2 = 1$, and evaluates as $e^{p1}[1+(qi+r\alpha+s\phi)K]$ when $J^2 = 0$. □

The inverse of $e^{p1+(qi+r\alpha+s\phi)K}$ is

$$e^{-p1-(qi+r\alpha+s\phi)K} = e^{-p1}[\cos K-(qi+r\alpha+s\phi)\sin K]$$

for $J^2 = -1$, with corresponding expressions for $J^2 = 1$ and $J^2 = 0$ (for the intricate conjugate we multiply by e^{p1} rather than e^{-p1}). Further, for $J^2 = -1$

$$e^{p1+(qi+r\alpha+s\phi)K} = -[e^{p1+(qi+r\alpha+s\phi)(K+\pi)}],$$

and similar considerations give

$$\sin K-(qi+r\alpha+s\phi)\cos K = e^{(qi+r\alpha+s\phi)(K+\pi/2)}.$$ □

We can determine trigonometric products or products of other type, so that

$$\{e\uparrow[a1+(bi+c\alpha+d\phi)K_1]\}\{e\uparrow[p1+(qi+r\alpha+s\phi)K_2]\} =$$
$$\tfrac{1}{2}e^{a+p}\{(1+M)\cos(K_1+K_2)+(1-M)\cos(K_2-K_1)$$
$$+[(q+b)i+(r+c)\alpha+(s+d)\phi]\sin(K_1+K_2)$$
$$+[(q-b)i+(r-c)\alpha+(s-d)\phi]\sin(K_2-K_1)\},$$

where

$$M = (bq-cr-ds)+(cs-dr)i+(bs-dq)\alpha+(-br+cq)\phi.$$ □

Chapter XVI introduces g, where $g^2 = \pm J$, $J^2 = \pm 1$. The assignation $g^4 = 1$ with $g = i^i$ provides a non-multifunction complex algebra for exponentiation, part of a study of hyperintricate exponentiation.

15.3. Comparing exponential products and the Euler relations.

If we wish to express

$$e^{p+(bi+c\alpha+d\phi)K} = e^p[\cos K+(bi+c\alpha+d\phi)\sin K]$$

where $-b^2+c^2+d^2 = -1$ as $e^w e^{xi} e^{y\alpha} e^{z\phi}$, then we can expand this out using the Euler relations 15.2.(8) for e^{xi}, and 15.2.(12) for $e^{y\alpha}$ and $e^{z\phi}$, as

$$e^w(\cos x+i\sin x)\alpha(\cosh y+\alpha\sinh y)\phi(\cosh z+\phi\sinh z)$$
$$= e^w(\cos x+i\sin x)\alpha\phi(\cosh y-\alpha\sinh y)(\cosh z+\phi\sinh z)$$
$$= ie^w(\cos x+i\sin x)(\cosh y-\alpha\sinh y)(\cosh z+\phi\sinh z).$$

Putting $X = \cos x$, $Y = \cosh y$ and $Z = \cosh z$ and equating intricate parts, we obtain

$$ie^w[XYZ+(1-X^2)^{1/2}(1+Y^2)^{1/2}(1+Z^2)^{1/2}] = e^p\cos K, \quad (1)$$
$$ie^w[(1-X^2)^{1/2}YZ-X(1+Y^2)^{1/2}(1+Z^2)^{1/2}] = e^p b\sin K, \quad (2)$$
$$ie^w[-X(1+Y^2)^{1/2}Z+(1-X^2)^{1/2}Y(1+Z^2)^{1/2}] = e^p c\sin K, \quad (3)$$
$$ie^w[XY(1+Z^2)^{1/2}+(1-X^2)^{1/2}(1+Y^2)^{1/2}Z] = e^p d\sin K. \quad (4)$$

Squaring and adding these terms, but for (3) and (4) subtracting, gives

$$-e^{2w} = e^{2p}, \quad (5)$$

whereas squaring (1) and substituting from (5) gives

$$\sin^2 K = X^2-Y^2-Z^2+X^2Y^2+X^2Z^2-Y^2Z^2-2N, \quad (6)$$

where

$$N^2 = X^2Y^2Z^2(1-X^2)(1+Y^2)(1+Z^2), \quad (7)$$

so correspondingly (1) and (2) squared and added give with (5)

$$(b^2 - 1)\sin^2 K = Y^2 + Z^2 + 2Y^2Z^2, \qquad (8)$$

and similarly for (2) squared, subtracted from (3) squared we obtain

$$(1 + d^2)\sin^2 K = X^2 - Y^2 + 2X^2Y^2. \qquad (9)$$

We can devise solutions by expressing Z in terms of c, d, K, X and Y in the following linking equation. Put

$$T = (1 + c^2)\sin^2 K - X^2,$$
$$U = (1 + d^2)\sin^2 K - X^2 + Y^2,$$

then eliminating Y^2Z^2 and X^2Y^2 terms in (6) and (7) from (8) and (9) gives

$$[4U(2 - U) - (1 - 2X^2)^2]Z^4 + [8U(1 - X^2 + Y^2) - 2T(1 - 2X^2)]Z^2 - T^2 = 0. \;\square$$

15.4. Intricate zero determinants and negative determinants.

That the solutions originally developed are not the most general may be deduced from the observation that the determinant of $e^{a1 + (bi + c\alpha + d\phi)K}$ was given by an expression of the form 15.2.(8) or .(10) multiplied by its intricate conjugate, and in each case this is e^{2a1}, which cannot be zero or negative.

Since a determinant is a multiplicative function, that is for matrices C and D

$$\det CD = \det C \det D,$$

a representation with zero or negative determinant may be obtained on multiplying by an intricate number which itself has a zero or negative determinant. An intricate representation extending this type is

$$W = -e^{x1 + J_1L} + \Delta.e^{y1 + J_2M}, \qquad (1)$$

with $\det \Delta = -1$.

When $J^2 = -1$, the determinant $\det J = b^2 - c^2 - d^2 = +1$, and $Je^{a + JK} = e^{a + J(\pi/2 - K)}$ has positive determinant. The intricate number $Je^{a + JK}$ has respectively zero or negative determinant when det J is zero or negative respectively.

We note that, provided we use the zero algebra, we can still use (1) where the first term on the right is zero, since $-e^{-1ч}$ can be interpreted as a new type of multizero.

Consider two intricate numbers, U and V, where J is fixed and of the form $J^2 = 1$, where

$$U = -e^{x_1 + JK_1} + Je^{y_1 + JL_1},$$
$$V = -e^{x_2 + JK_2} + Je^{y_2 + JL_2},$$

then U and V commute, whereas if $J^2 = 0$ and

$$U = -e^{x_1}(1 + JK_1) + Je^{y_1}(1 + JL_1)$$
$$= -e^{x_1} + J(e^{x_1}K_1 + e^{y_1}),$$
$$V = -e^{x_2} + J(e^{x_2}K_2 + e^{y_2}),$$

then again U and V commute. In neither case is the form for U and V the most general, so that the representations we have chosen, for fixed J, are abelianisations of the general case.

We now force a non-commutative algebra where $\mathcal{J} = bi + c\alpha + d\phi$, under the bijective mapping $\mathcal{J} \leftrightarrow i$, $\mathcal{A} \leftrightarrow \alpha$ and $\mathcal{F} \leftrightarrow \phi$ of chapter I. The form is now

$$W = -e^{x_1 + \mathcal{J}K_1} + \mathcal{J}e^{x_2 + \mathcal{J}K_2} + \mathcal{A}e^{x_3 + \mathcal{J}K_3} + \mathcal{F}e^{x_4 + \mathcal{J}K_4}.$$

This form is not restrictive up to sign since $e^{\mathcal{J}K} = -[e^{\mathcal{J}(K + \pi)}]$, $\mathcal{J} = -(bi + c\alpha + d\phi)$ also satisfies $\mathcal{J}^2 = -1$, and $\mathcal{A} \to -\mathcal{A}$, $\mathcal{F} \to -\mathcal{F}$ complies. However, there are two types of algebras here, with non-isomorphic handedness, that is chirality, for the triple $\mathcal{J}$, $\mathcal{A}$, $\mathcal{F}$.

The form is now surjective to a general intricate number, which can have positive, zero or negative determinant. ☐

15.5. J-abelian intricate powers.

Let J_n for a variable n, where the J_n are distinct with $J_n^2 = 0$ or ±1, satisfy

$$U_n = -e^{x_1 + J_nK} + J_ne^{y_1 + J_nL}. \quad (1)$$

Distinct U_n are non-commutative. This equation is of the most general form, since the determinant of U_n for $J_n^2 = -1$ is positive. Under i ↔ J_n equivalence the determinant of p1 + qi + rα + sϕ is greater than zero with value $p^2 + q^2 – r^2 – s^2$ for $J_n^2 = -1$. ☐

Binomial expansions of (1) for real powers are abelian for fixed J using the binomial theorem for powers of intricate numbers, under i, α or ϕ ↔ J equivalence if $J^2 \neq 0$. ☐

15.6. The hyperintricate Euler relations.

Part of the material of this section has been simplified and extended in chapter II, where the J-layered approach to hyperintricate representations is discussed.

An n-hyperintricate number cannot be represented in general by a product of (n – 1)-equivalent hyperintricates, since to take the case n = 2, we may represent a product by

$$Я_2 = (A_1 + B_1i_1 + C_1\alpha_1 + D_1\phi_1)(T_1 + U_11_i + V_11_\alpha + W_11_\phi) \quad (1)$$

and this has 8 variables whereas $Я_2$ has 16. The two factors above commute. For a full hyperintricate number, $Я_n$ can never be represented in the above way. However, we can represent $Я_2$ by a double sum of products

$$Я_2 = \Sigma(r = 1, 2)(A_r + B_ri_1 + C_r\alpha_1 + D_r\phi_1)(T_r + U_r1_i + V_r1_\alpha + W_r1_\phi). \quad (2)$$

Extended alternatives to the form (1) are that T, U, V and W are complex of type T = (a + bi), or respectively of type T = (a + cα) or T = (a + dϕ). These examples of interior coefficient algebras may be considered as a restatement of the form (2).

We now obtain the n-hyperintricate Euler relations. The number of n-hyperintricate basis elements is 4^n, which may be represented as $(-1 + 5)^n = (1 + 3)^n = 1 + 3m$ for some m. So if n is even, m is divisible by 5, and if n is odd, (m – 1) is divisible by 20. We will use this to allocate an n-hyperintricate number as a real part and 3m other parts.

Let L range over the 3 values {i, α, ϕ}, $Ч_{rL}$ be an n-hyperintricate basis element, all layers of which are 1 except those taken from a triple L in position r, and h_{qrL} be a subscripted coefficient for $Ч_{rL}$. We will use u for r in context.

Then using the floor function $\lfloor\ \rfloor$, or integer part, for $Я_n$ such a product of intricates has 4n variables, therefore there must be $N = \lfloor 1 + [(4^{n-1} – 1)/n] \rfloor$ sums of these products, and if n is not a power of 2, the last product can contain less terms than the previous N – 1.

Thus an n-hyperintricate number may be represented by

$$Я_n = \Sigma(q = 1 \text{ to } N)\ \Pi(r = 1 \text{ to } n)[p_{qr} + \Sigma(\text{over } L)h_{qrL}Ч_{rL}]$$
$$= \Sigma(q = 1 \text{ to } N)\{\Pi(r = 1 \text{ to } n)p_{qr} + \{\Sigma(\#s = 1 \text{ to } n)$$
$$\Pi(\text{over } t = n \text{ except for } \#s \text{ subselections})p_{qt}$$
$$\Pi(\text{over } u = n \text{ except for } n – \#s \text{ subselections})[\Sigma(\text{over } L)h_{quL}Ч_{uL}]\}\},$$

where in products different values of u give commuting $Ч_{uL}$, and

$$[\Sigma(\text{over L})h_{quL}Ч_{uL}]^2 = [-h_{qui}^2 + h_{qu\alpha}^2 + h_{qu\phi}^2].$$

We will write

$$[-h_{qui}^2 + h_{qu\alpha}^2 + h_{qu\phi}^2] = J_{qu}^2K_{qu}^2,$$

where $J_{qu}^2 = 0$ or ±1. Then when the product $\Pi_uJ_{qu}^2 = 0$

$$e\uparrow(\Pi_uJ_{qu}K_{qu}) = [1 + \Pi_uJ_{qu}K_{qu}],$$

when the product $\Pi_uJ_{qu}^2 = -1$

$$e\uparrow(\Pi_uJ_{qu}K_{qu}) = [\cos(\Pi_uK_{qu}) + \Pi_uJ_{qu}\sin(\Pi_uK_{qu})]$$

and when the product $\Pi_uJ_{qu}^2 = 1$, to maintain determinants

$$e\uparrow(\Pi_uJ_{qu}K_{qu}) = \Pi_uJ_{qu}[\cosh(\Pi_uK_{qu}) + \Pi_uJ_{qu}\sinh(\Pi_uK_{qu})].$$

Then for q = 1 to N, r = 1 to n

$$Я_n = \Sigma_q\{\Pi_rp_{qr} + \Sigma_{\#s}[\Pi_tp_{qt}][e\uparrow\Pi_u(J_{qu}K_{qu})]\}.$$

These are the n-hyperintricate Euler relations. □

Also for n = 2 a full representation is

$$Я_2 = A + J_1B + 1_{J'}C + J'''_{J''}D + J''''_{J''}E, \qquad (3)$$

with 16 components, for which in general $J \neq J' \neq J'' \neq J''' \neq J''''$, where the first three terms amount to 7 variables and the last two to 9. We allocate $J_1^2 = 0, \pm1$, $(1_J)^2 = 0, \pm1$, $(J'''_{J''})^2 = 0, \pm1$, with $(J''')^2 = G'''$, $(J'')^2 = H$, $G'''H = 0, \pm1$ and $(J''')^2 = 0, \pm(J'''')^2$.

For fixed J_1, $1_{J'}$, $J'''_{J''}$, $J''''_{J''}$ and real variables A, B, C, D and E there is no analogue of J-abelian; for two such 2-hyperintricate numbers usually

$$Я_2Я'_2 \neq Я'_2Я_2,$$

since J_1 and $J'''_{J''}$, J_1 and $J''''_{J''}$, $1_{J'}$ and $J'''_{J''}$, $1_{J'}$ and $J''''_{J''}$, and $J'''_{J''}$ and $J''''_{J''}$ do not generally commute.

For n-hyperintricate numbers the representation related to (3) has 4^n components in $Я_n$, and the number of combinations of various J's is greater than this for n > 8, so such a representation exists either with multiple J's for specific indices as in (3) for $n \leq 8$ or without them for n > 8. □

15.7. Roots of intricate basis elements.

The multiplicative inverses of the intricate basis elements are as follows.

$$1^{-1} = 1,\ i^{-1} = -i,\ \alpha^{-1} = \alpha,\ \phi^{-1} = \phi.$$

We shall see that for $\alpha^{1/2}$ and $\phi^{1/2}$, square roots of intricate basis elements can be represented by hyperintricate numbers. For square roots

$$1^{1/2} = \pm1 \text{ or } \pm(ui + v\alpha + w\phi),$$

with $-u^2 + v^2 + w^2 = 1$, allowing us to expand the list of *possibilities* below, also

$$i^{1/2} = \pm(1 + i)/\surd2,$$

giving the dependent relation (with the same positive or negative sign)

$$i^{-1/2} = \pm(1 - i)/\surd2,$$

with

$$\alpha^{1/2} = \pm \begin{Vmatrix} 1 & 0 \\ 0 & \pm i \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \alpha & 0 \\ 0 & \pm i \end{Vmatrix}$$

or

$$\pm \begin{Vmatrix} \phi & 0 \\ 0 & \pm i \end{Vmatrix}$$

and

$$\phi^{1/2} = \pm(1/\surd 2) \begin{Vmatrix} i^{-1/2} & i^{1/2} \\ i^{1/2} & i^{-1/2} \end{Vmatrix}$$

For integers m and n we give the following roots.

$1^{1/(2n+1)} = e^{i2\pi m/(2n+1)}$
$i^{1/(2n+1)} = e^{i\pi(4m+1)/[2(2n+1)]}$
$\alpha^{1/(2n+1)} = \alpha$
$\phi^{1/(2n+1)} = \phi$.

$1^{1/(2n)} = e^{i\pi m/n}$
$i^{1/(2n)} = e^{i\pi(4m+1)/(4n)}$

with

$$\alpha^{1/(2n)} = \pm \begin{Vmatrix} 1 & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \alpha & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$$\pm \begin{Vmatrix} \phi & 0 \\ 0 & \pm i^{1/n} \end{Vmatrix}$$

$\phi^{1/(2n)}$ may be obtained recursively from roots with smaller n. For example, since

$$\phi^{1/2} = \pm(1/\surd 2)\, i^{1/2} \begin{Vmatrix} -i & 1 \\ 1 & -i \end{Vmatrix}$$

we have

$$\phi^{1/4} = \pm(1 \text{ or } i)[i^{1/4}/2^{1/4}] \begin{Vmatrix} a & b \\ b & a \end{Vmatrix}$$

where (see the next paragraph for an indication of how to obtain this)

$a^2 = (-½ \pm 1/\surd 2)i$
$b^2 = -i - a^2$.

More generally, for natural numbers $t = 2^n$ and $u = 2^{n-1}$, if

$$\phi^{1/t} = e^{i2\pi m/t}[i^{1/t}/2^{1/t}] \begin{Vmatrix} P & Q \\ Q & P \end{Vmatrix}$$

where we have previously determined that

$$\phi^{1/u} = e^{i2\pi m/u}[i^{1/u}/2^{1/u}] \begin{Vmatrix} p & q \\ q & p \end{Vmatrix}$$

then on squaring the matrix in P and Q

$P^2 + Q^2 = p$

$2PQ = q$

which reduces to a solvable quadratic equation, say in P^2, so $\phi^{1/t}$ is determined.

If a natural number $T = ty$, with y an odd number, is encountered instead of t, the determination of $\phi^{1/T}$ can be found from the relation, valid for natural numbers y and t

$$\phi^{1/T} = [\phi^{1/y}]^{1/t} = \phi^{1/t}. \square$$

The results of section 2, where an intricate number may be represented by e^z for a positive determinant, can be used to give a root $e^{z/n}$, although for $J^2 = 1$, cosh cannot take the value zero.

These ideas may be combined. For example, to get around the cosh restriction on representations of α, we may write using an exterior coefficient algebra

$$\alpha^{1/n} = \left\| \begin{matrix} e^{2\pi mi/n} & 0 \\ 0 & e^{\pi(2m+1)i/n} \end{matrix} \right\|$$

$$\equiv ½(e^{2\pi mi/n} + e^{\pi(2m+1)i/n})1_1 + ½(e^{2\pi mi/n} - e^{\pi(2m+1)i/n})\alpha_1. \square$$

Further considerations are given in Chapter XVI.

15.8. The intricate binomial theorem for real powers.

Suppose $w = 2m + s$, *with* m *and* k *natural numbers and* w *real*. *Let* int *be the integer part of a real number and* $0! = 1$. *Then*

$$Я_1^w = [a^2 - b^2 + c^2 + d^2 + 2a(bi + c\alpha + d\phi)]^m Я_1^s.$$

This gives

$$Я_1^w = \sum(k = 0 \text{ to } m)[m!/(m-k)!k!]$$
$$\{2^k a^k (a^2 - b^2 + c^2 + d^2)^{m-k}(-b^2 + c^2 + d^2)^{\text{int}(k/2)}(bi + c\alpha + d\phi)^{k - 2\text{int}(k/2)}\}Я_1^s.$$

Proof. By the binomial theorem, we use the fact that real numbers commute and the relation

$$(bi + c\alpha + d\phi)^2 = \{i(b1 + c\phi - d\alpha)\}^2$$
$$= -(b1 - c\phi + d\alpha)(b1 + c\phi - d\alpha)$$
$$= -b^2 + c^2 + d^2. \square$$

15.9. The n-hyperintricate binomial theorem for real powers.

Now consider the 2-hyperintricate

$$Я_2 = A + P + Q + R,$$

in which typically a_{1i} is a real coefficient and 1_i is an indexed basis element, where

$$A = a_{11}1_1,$$
$$P = a_{1i}1_i + a_{1\alpha}1_\alpha + a_{1\phi}1_\phi,$$
$$Q = b_{i1}i_1 + c_{\alpha 1}\alpha_1 + d_{\phi 1}\phi_1$$

and

$$R = b_{ii}i_i + b_{i\alpha}i_\alpha + b_{i\phi}i_\phi + c_{\alpha i}\alpha_i + c_{\alpha\alpha}\alpha_\alpha + c_{\alpha\phi}\alpha_\phi + d_{\phi i}\phi_i + d_{\phi\alpha}\phi_\alpha + d_{\phi\phi}\phi_\phi.$$

A is commutative with respect to P, Q and R, *also products are*:

$$A^2 = a_{11}^2 1_1,$$
$$P^2 = (-a_{1i}^2 + a_{1\alpha}^2 + a_{1\phi}^2)1_1,$$
$$Q^2 = (-b_{i1}^2 + c_{\alpha 1}^2 + d_{\phi 1}^2)1_1,$$

$R^2 = H + 2L,$

where

$H = (b_{ii}{}^2 - b_{i\alpha}{}^2 - b_{i\phi}{}^2 - c_{\alpha i}{}^2 + c_{\alpha\alpha}{}^2 + c_{\alpha\phi}{}^2 - d_{\phi i}{}^2 + d_{\phi\alpha}{}^2 + d_{\phi\phi}{}^2)1_1,$

$L = \{(b_{ii}c_{\alpha\alpha} - b_{i\alpha}c_{\alpha i})\phi_\phi - (b_{ii}c_{\alpha\phi} - b_{i\phi}c_{\alpha i})\phi_\alpha + (b_{i\phi}c_{\alpha\alpha} - b_{i\alpha}c_{\alpha\phi})\phi_i$

$- (b_{ii}d_{\phi\alpha} - b_{i\alpha}d_{\phi i})\alpha_\phi + (b_{ii}d_{\phi\phi} - b_{i\phi}d_{\phi i})\alpha_\alpha - (b_{i\phi}d_{\phi\alpha} - b_{i\alpha}d_{\phi\phi})\alpha_i$

$+ (c_{\alpha\alpha}d_{\phi i} - c_{\alpha i}d_{\phi\alpha})i_\phi - (c_{\alpha\phi}d_{\phi i} - c_{\alpha i}d_{\phi\phi})i_\alpha + (c_{\alpha\alpha}d_{\phi\phi} - c_{\alpha\phi}d_{\phi\alpha})i_i\},$

$PQ = (a_{1i}b_{i1}i_i + a_{1i}c_{\alpha 1}\alpha_i + a_{1i}d_{\phi 1}\phi_i$

$+ a_{1\alpha}b_{i1}i_\alpha + a_{1\alpha}c_{\alpha 1}\alpha_\alpha + a_{1\alpha}d_{\phi 1}\phi_\alpha$

$+ a_{1\phi}b_{i1}i_\phi + a_{1\phi}c_{\alpha 1}\alpha_\phi + a_{1\phi}d_{\phi 1}\phi_\phi)$

$= QP,$

$F = (PR + RP)/2$

$= (-a_{1i}b_{ii} + a_{1\alpha}b_{i\alpha} + a_{1\phi}b_{i\phi})i_1$

$+ (-a_{1i}c_{\alpha i} + a_{1\alpha}c_{\alpha\alpha} + a_{1\phi}c_{\alpha\phi})\alpha_1$

$+ (-a_{1i}d_{\phi i} + a_{1\alpha}d_{\phi\alpha} + a_{1\phi}d_{\phi\phi})\phi_1$

and

$G = (QR + RQ)/2$

$= (-b_{i1}b_{ii} + c_{\alpha 1}c_{\alpha i} + d_{\phi 1}d_{\phi i})1_i$

$+ (-b_{i1}b_{i\alpha} + c_{\alpha 1}c_{\alpha\alpha} + d_{\phi 1}d_{\phi\alpha})1_\alpha$

$+ (-b_{i1}b_{i\phi} + c_{\alpha 1}c_{\alpha\phi} + d_{\phi 1}d_{\phi\phi})1_\phi,$

so with different coefficients, F acts like Q, G like P, and L and PQ like R, then for example

$FG = GF.$ □

Thus

$Я_2{}^2 = A^2 + P^2 + Q^2 + R^2 + 2\{A(P + Q + R) + PQ + F + G\},$

and separating commutative and non-commutative terms

$Я_2{}^w = \sum(k = 0 \text{ to } m)[m!/(m - k)!k!]$

$\{2^k(A^2 + P^2 + Q^2 + H)^{m-k}[(A(P + Q + R) + PQ + F + G + L]^k\}Я_2{}^s.$ □

The n-hyperintricate binomial theorem for real powers may also be considered. From

$Я_n{}^w = \{\Sigma(q = 1 \text{ to } N)\,\Pi(r = 1 \text{ to } n)[p_{qr} + \Sigma(\text{over } L)h_{qrL}Ч_{rL}]\}^w$

which is of the form

$Я_n{}^w = \{\sum(v = 1 \text{ to } N)[a_vA_v]\}^w,$

where the a_v are real, the terms A_v involve a real term and terms $\Sigma(\text{over } L)h_{qrL}Ч_{rL}$ which commute for different r, or correspond with $Ч_{rL}$ to a power for the same r. Therefore on taking the power of w, we can apply the multinomial theorem to obtain

$Я_n{}^w = \sum(\text{for } k_1 + k_2 + \ldots + k_N = w)[w!/\,k_1!k_2!\,\ldots\,k_N!]\Pi(1 \le t \le N)[a_tA_t]\uparrow k_t.$ □