CHAPTER VIII

Polynomial equations with complex roots

8.1. Introduction.

The objective of this chapter is to follow our history and study polynomial equations for complex variables under the assumed condition that Galois solvability restrictions hold. We will study the solution of polynomials in 'Bring-Jerrard' form, and also for degree ≤ 6 with dependent complex roots, which includes independent complex roots as a special case. A discussion of polynomial theory also for matrix variables and coefficients is given in chapters IX and X.

Complex polynomials where no known relationships between the zeros operate are in the domain of the theory of independent roots. This theory [Ed84], [Ga62], [Ro90], [St04] claims a no-go theorem for these types of solution by radicals of polynomials of degree greater than 4 with variables and coefficients in the complex number field. One way of looking at this is to show that the existence of a solution depends on inverting a set of polynomial equations, where such polynomial equations remain invariant under permutation of the variables. For permutations of five or more roots for the quintic or higher degree polynomials, Galois theory claims that the group structure of these permutations does not allow this inversion to take place by multiplicative algebraic operations on general complex numbers derived from the roots. A standard interpretation states no general inversion algorithm of this type exists.

Further, if a solution by radicals of the independent roots of a general polynomial equation of degree n greater than 4 were found 'by chance' within the complex number system, then this would imply the existence of such an inversion algorithm [Ar59].

The chapter gives a complete account of our theory of dependent roots. We treat roots as symbols which all differ, but which may have the same or related values. From the description of duplicate roots $(x + a)^2 = 0$ and antiduplicate roots satisfying $(x - a)(x + a) = 0$, we are able to describe roots depending on themselves linearly of the form $(x + a)(x + ha)$, and that if h is known in advance, the value of a in the general case can be determined. Thus it is possible to apply this process as an initial stage in the solution of a polynomial equation.

Finally, we show that a general complex polynomial of finite degree n can be embedded in another solvable polynomial of degree $k = \frac{n(n+1)}{2}$. This leads us to the general philosophy that the polynomial zeros we consider should be of this type, described by polynomial *entities*.

8.2. Reversible and irreversible algorithms (algorithmic thermodynamics).

If algorithms conform to the Galois theory of independent roots they have an interesting result. Since the general complex polynomial of degree > 4 is unsolvable according to this method, we will see there is a consequence for reversibility, which means that a polynomial in multiplicative form

$$(x - a_1)(x - a_2) \ldots (x - a_n) = 0 \qquad (1)$$

can be brought into additive form

$$x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots b_0 = 0 \qquad (2)$$

by the distributive axioms for a ring, but the transformation in the reverse direction from (2) to (1) cannot be effected for n > 4 by the algorithms operating within the theory. □

8.3. Polynomial equations with Gaussian rational number solutions.

A Gaussian rational number, g_1 or g_2 is of the form $g_1 = q_1 + r_1 i$, where $q_1, r_1 \in \mathbb{Q}$, the rational numbers, and factorises uniquely provided $g_1 g_2 \neq 0$, except that $g_1 g_2 = -g_1 \times -g_2$ and $i^2 = (-i)^2$. If a polynomial equation has all Gaussian rational solutions, say

$$(x + A)(x + B) \dots (x + D) = 0, \qquad (1)$$

expanded out as

$$x^n + (A + B + \dots + D)x^{n-1} + \dots + AB\dots D = 0, \qquad (2)$$

then its solution set can be determined by multiplying out coefficients to contain only Gaussian integers. Then all combinations of the leading term in x^n and the trailing term may be factorised and tested to see if they are roots. If we adjoin to (1) by multiplication *any* quartic polynomial in Gaussian rational coefficients, this also is solvable by solvability of the quartic, even though the roots of the quartic may not be Gaussian rational numbers. ☐

Analogous statements can be made when $\mathbb{Q}[1, i]$ is replaced by $aq_1 + br_1 i$, where a and $b \in \mathbb{A}$, the algebraic numbers consisting of a finite series of various mth roots of integers. Although such numbers may not have unique factorisation, the class number is finite, so that the factorisations are limited in number and can theoretically be determined algorithmically. ☐

8.4. The cubic $(x + A + B + C)(x + A + \omega B + \omega^2 C)(x + A + \omega^2 B + \omega C) = 0$.

We will solve the cubic in two ways and finally a third, where each occurrence of the variable x is represented by three linearly independent variables expressed as

$$(x + A + B + C)(x + A + \omega B + \omega^2 C)(x + A + \omega^2 B + \omega C) = 0, \qquad (1)$$

where ω is a cube root of unity:

$$\omega = e^{2\pi i/3} = \tfrac{1}{2}(-1 + \sqrt{3}).$$

On multiplying out, this equation reduces to

$$x^3 + 3Ax^2 + 3(A^2 - BC)x + A^3 + B^3 + C^3 - 3ABC = 0. \qquad (2)$$

If this equation is of the form

$$x^3 + qx + r = 0, \qquad (3)$$

then on putting $A = 0$, we obtain

$$q = -3BC$$

$$q^3 = -27B^3C^3, \qquad (4)$$

$$r = B^3 + C^3, \qquad (5)$$

so that we obtain the classical quadratic in B^3:

$$27B^6 - 27rB^3 - q^3 = 0,$$

with solution

$$B^3 = \frac{r}{2}\left[1 \pm \sqrt{1 + \frac{4q^3}{27r^2}}\right]. ☐ \qquad (6)$$

Alternatively, we can apply the symmetry-breaking Tschirnhaus substitution

$$x = y + h, \qquad (7)$$

so that

$$y^3 + 3(h + A)y^2 + 3[h^2 + 2Ah + (A^2 - BC)]y$$
$$+ h^3 + 3Ah^2 + 3(A^2 - BC)h + A^3 + B^3 + C^3 - 3ABC) = 0, \qquad (8)$$

and with $A + h = 0$

$$y^3 - 3BCy + B^3 + C^3 = 0, \qquad (9)$$

which gives the same as (4), (5) and (6). ☐

We might wish to find a practical method of finding these roots of a cubic equation in a recognisable form. We indicate how this can be done. If we choose as an example

$$x^3 - 2x + 1 = 0, \qquad (10)$$

we see that it has a solution x = 1. Factoring as

$$(x - 1)(x^2 + ax + b) = 0$$

$$(x - 1)(x^2 + x - 1) = 0 \qquad (11)$$

the above equation has real quadratic solutions

$$x = \frac{-1 \pm \sqrt{5}}{2}. \qquad (12)$$

With ω a cube root of unity, this corresponds to the cubic (1) written as equation (3).

Hence with q = -2 and r = 1

$$C^3 = \frac{1}{2}\left[1 \mp \sqrt{1 + \frac{4q^3}{27r^2}}\right] \qquad (13)$$

so that

$$B = \sqrt[3]{\frac{1}{2}\left(1 \pm \sqrt{\frac{-5}{27}}\right)} \qquad (14)$$

$$C = \sqrt[3]{\frac{1}{2}\left(1 \mp \sqrt{\frac{-5}{27}}\right)}. \qquad (15)$$

We do not immediately know which of the three possibilities in equation (1) is the root x = 1 or the other roots.

To find this, note that by the results of chapter XIV equation (14) can be represented by a complex number in polar form

$$B = \left(\rho e^{i\theta}\right)^{\frac{1}{3}} = \rho^{\frac{1}{3}} e^{\frac{i\theta}{3}}$$

$$= \rho^{1/3}(\cos(\theta/3) + i\sin(\theta/3)), \qquad (16)$$

and C by

$$C = \left(\rho e^{-i\theta}\right)^{\frac{1}{3}} = \rho^{\frac{1}{3}} e^{\frac{-i\theta}{3}} = \rho^{1/3}(\cos(\theta/3) - i\sin(\theta/3)), \qquad (17)$$

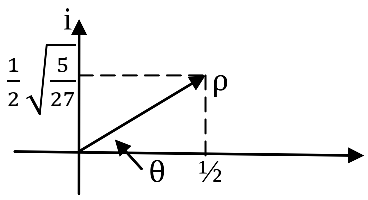
so that if for example we take the root

$$x = -B - C,$$

this is

$$x = -\rho^{1/3}(2\cos(\theta/3)). \qquad (18)$$

For the values (14) and (15), ρ and θ may be represented in the Argand diagram



where

$$\tan\theta = \sqrt{\frac{5}{27}}, \ \theta = 23.28373172°. \qquad (19)$$

By the Pythagoras theorem ρ satisfies

$$\rho^2 = \frac{1}{2^2}\left(1 + \left(\sqrt{\frac{5}{27}}\right)^2\right) = \frac{1}{4}\left(1 + \frac{5}{27}\right) = \frac{8}{27},$$

$$\rho^{1/3} = \sqrt{\frac{2}{3}} = 0.81649658 \quad (20)$$

and

$$x = -\sqrt{\frac{2}{3}}(2\cos(\theta/3)) = -1.618033986, \quad (21)$$

which is verified by a calculator to be without error analysis the root

$$x = \frac{\mathbf{-1 - \sqrt{5}}}{\mathbf{2}} = -1.618033989. \;\square \quad (22)$$

As a third method, it is possible instead of the allocation A = 0 in (2) to try C = 0. We obtain

$$(x + A)^3 + B^3 = 0. \quad (23)$$

Assuming a full expression

$$x^3 + px^2 + qx + r = 0, \quad (24)$$

equation (24) cannot be expressed with p = 0 if it is to describe (23), since then A = 0. However, we can set q to $\frac{p^2}{3}$. Then this non-classical substitution (try x = gy + 1) gives

$$3A = p,$$

$$A^3 + B^3 = \frac{p^3}{27} + B^3 = r. \;\square$$

8.5. The Vandermonde solution (of 1771) for the quartic.

For the quartic, we will consider

$$z^4 + Pz^2 + Qz + R = 0, \quad (1)$$

and write $\omega_4 = i = \sqrt{-1}$, which gives $\omega_4^2 = -1$ and $\omega_4^4 = 1$, so that on squaring, cubing and raising to the fourth power the independent coefficients of a, b and c in the multiplicative terms below, we obtain

$$(z + \omega_4 a + \omega_4^2 b + \omega_4^3 c)(z + \omega_4^2 a + b + \omega_4^2 c) \times$$
$$(z + \omega_4^3 a + \omega_4^2 b + \omega_4 c)(z + a + b + c) = 0. \quad (2)$$

Equation (2) is invariant under a swap of a and c, which allows us to compute some resulting coefficients. Alternatively and better, we can multiply the first and third expression in (2) to obtain

$$(z + \omega_4 a - b + \omega_4^3 c)\,(z + \omega_4^3 a - b + \omega_4 c) =$$
$$z^2 + [(\omega_4 + \omega_4^3)a - 2b + (\omega_4^3 + \omega_4)c]z$$
$$+ [a^2 + b^2 + c^2 - b(\omega_4 + \omega_4^3)a - b(\omega_4 + \omega_4^3)c + (\omega_4^2 + \omega_4^2)ac]$$
$$= z^2 - 2bz + a^2 + b^2 + c^2 - 2ac, \quad (3)$$

and the second and fourth expressions in (2) to get

$$(z - a + b - c)(z + a + b + c) = z^2 + 2bz - a^2 + b^2 - c^2 - 2ac. \quad (4)$$

Multiplying (3) and (4) together gives (2) and we finally get

$$z^4 + (-4b^2 + 2b^2 - 4ac)z^2 + 4(a^2 + c^2)bz - a^4 + b^4 - c^4 + 2a^2c^2 - 4b^2ac = 0,$$
$$z^4 + (-2b^2 - 4ac)z^2 + 4(a^2 + c^2)bz - a^4 + b^4 - c^4 + 2a^2c^2 - 4b^2ac = 0. \quad (5)$$

On putting

$$P = -2b^2 - 4ac, \quad (6)$$
$$Q = 4(a^2 + c^2)b, \quad (7)$$
$$R = -a^4 + b^4 - c^4 + 2a^2c^2 - 4b^2ac, \quad (8)$$

and squaring (7)

$$Q^2 = 16(a^4 + 2a^2c^2 + c^4)b^2, \quad (9)$$

thus on combining (8) and (9)

$$Q^2 = 16(b^4 + 4a^2c^2 - 4b^2ac - R)b^2,$$

and from (6)

$$Q^2 = (16b^4 + 4(P^2 + 4Pb^2 + 4b^4) + 16(P + 2b^2)b^2 - 16R)b^2,$$

that is

$$16b^6 + 8Pb^4 + (P^2 - 4R)b^2 - (Q^2/4) = 0, \qquad (10)$$

a solvable cubic in b^2. To find this solution use the substitution

$$b^2 = b'^2 - \frac{P}{6}, \qquad (11)$$

so

$$b^4 = b'^4 - \frac{P}{3}b'^2 + \frac{P^2}{36},$$

$$b^6 = b'^6 - \frac{P}{2}b'^4 + \frac{P^2b'^2}{12} - \frac{P^3}{216},$$

giving

$$16b'^6 + 8(-P + P)b'^4 + \left(\frac{16P^2}{12} - \frac{8P^2}{3} + (P^2 - 4R)\right)b'^2$$

$$+ \left(-\frac{16P^3}{216} + \frac{8P^3}{36} + (P^2 - 4R)\left(-\frac{P}{6}\right) - \frac{Q^2}{4}\right) = 0,$$

or

$$16b'^6 - \left(\frac{P^2}{3} + 4R\right)b'^2 - \frac{P^3}{54} + \frac{2PR}{3} - \frac{Q^2}{4} = 0. \qquad (12)$$

Looking at section 4, equation (3) with variables q and r for which we substitute x by b'^2 becomes equation (12) above. Thus

$$q = -\left(\frac{P^2}{48} + \frac{R}{4}\right), \qquad (13)$$

$$r = -\frac{P^3}{54\times 16} + \frac{PR}{24} - \frac{Q^2}{64}, \qquad (14)$$

and the solutions are

$$b'^2 = -B - C,$$

$$-\omega_3B - \omega_3{}^2C$$

$$-\omega_3{}^2B - \omega_3C,$$

where ω_3 is a cube root of unity, and in section 4 we found

$$B^3 = \frac{r}{2}\left[1 + \sqrt{1 + \frac{4q^3}{27r^2}}\right],$$

$$C^3 = \frac{r}{2}\left[1 - \sqrt{1 + \frac{4q^3}{27r^2}}\right].$$

Hence on combining (6) and (7) so that

$$4a^2Q = 16a^4b + (P + 2b^2)^2b,$$

$$a^4 - (Q/4b)a^2 + (P + 2b^2)^2/16 = 0, \qquad (15)$$

using (15) we can obtain a and thereby c, thus solving (1) from the roots in (2). □

8.6. Ascending solutions of the quadratic and cubic.

A quadratic equation is usually assumed to possess a unique type of solution, but we may multiply the quadratic

$$x^2 + Qx + R = 0$$

by (x + h) and solve as a cubic (although the cubic equation is derived via an intermediate solution of the quadratic). Thus the formulas for the quadratic are not unique, if we allow adjoined roots, although by the fundamental theorem of algebra its *values* are unique for x a complex number. A similar comment may be made about adjoining a root to the cubic. □

8.7. Complex polynomials of degree n.

For a polynomial of degree n, let

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0, \qquad (1)$$

where x is a complex number and the a_i, i = 0 … (n – 1), are fixed complex numbers.

Suppose we set for complex b_1

$$a_{n-1}x^{n-2} + a_{n-2}x^{n-3} + \dots + a_2x + b_1 = 0, \qquad (2)$$

then for any substitution of the variable x, a value of b_1 can be found.

Suppose a polynomial equation is in *Bring-Jerrard* form

$$x^n + (a_1 - b_1)x + a_0 = 0. \qquad (3)$$

Then if (3) is solvable we can find a b_1 so that (1) holds. Conversely, if we can solve (1) then we can find a b_1 such that (3) obtains.

A polynomial equation in Bring-Jerrard form (3) may be brought into reduced Bring-Jerrard form under the substitution x = ry as

$$y^n + my + m = 0. \square \qquad (4)$$

8.8. The zeros of the complex sextic Bring-Jerrard polynomial.

We provide an instance of a typical feature of the sextic, unsolvable according to Galois theory. The equation

$$v^2 + vu + u^2 = 0 \qquad (1)$$

may be multiplied by (v – u) to form

$$v^3 - u^3 = 0 \qquad (2)$$

with solution (ω is a cube root of unity)

$$v = \omega u \qquad (3)$$

provided we exclude

$$v = u.$$

Consider

$$(x^3 + ax^2 + bx + c)^2 + (x^3 + ax^2 + bx + c)(x^3 + dx^2 + ex + f)$$
$$+ (x^3 + dx^2 + ex + f)^2 = 0. \qquad (4)$$

By the equivalence of (4) and (1), we have solutions

$$x^3 + ax^2 + bx + c = \omega(x^3 + dx^2 + ex + f) \qquad (5)$$

with

$$\omega = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}, \qquad (6)$$

and (5) is solvable by solvability of the cubic.

Expanding out (4) by using

$$(x^3 + ax^2 + bx + c)^2 = x^6 + 2ax^5 + (2b + a^2)x^4 + (2ab + 2c)x^3$$
$$+ (2ac + b^2)x^2 + 2bcx + c^2, \qquad (7)$$

and similarly for $(x^3 + dx^2 + ex + f)$, we obtain

$$3x^6 + (3a + 3d)x^5 + (3b + 3e + a^2 + ad + d^2)x^4 + (2ab + ae + bd + 2de + 3c + 3f)x^3 + (2ac + af + cd + 2df + b^2 + be + e^2)x^2 + (2bc + bf + ce + 2ef)x + c^2 + cf + f^2 = 0. \quad (8)$$

We will put the coefficients of x^5 to $x^2 = 0$, so that equation (4) is in Bring-Jerrard form

$$3x^6 + Kx + L = 0. \quad (9)$$

Equating coefficients of (8) and (9), we now have

$$a = -d \quad (10)$$

$$e = -b - \frac{a^2}{3} \quad (11)$$

$$f = -c + \frac{a}{3}(e - b) \quad (12)$$

$$a(f - c) = b^2 + be + e^2, \quad (13)$$

so from (11) and (12) in (13)

$$a(-2c + \frac{a}{3}(-2b - \frac{a^2}{3})) = b^2 - b^2 - \frac{ba^2}{3} + b^2 + \frac{2a^2b}{3} + \frac{a^4}{9},$$

giving

$$\frac{2a^4}{9} + a^2b + b^2 + 2ac = 0. \quad (14)$$

We want

$$K = 2bc + b(-c + \frac{a}{3}(-2b - \frac{a^2}{3})) + c(-b - \frac{a^2}{3}) + 2(-b - \frac{a^2}{3})(-c + \frac{a}{3}(-2b - \frac{a^2}{3}))$$

$$= 2bc + \frac{a^2c}{3} + \frac{2ab^2}{3} + \frac{5a^3b}{9} + \frac{2a^5}{27} \quad (15)$$

and using (14)

$$K = 2bc - \frac{a^2c}{3} + \frac{ab^2}{3} + \frac{2a^3b}{9}, \quad (16)$$

whereas

$$L = c^2 + cf + f^2 \quad (17)$$

$$= c^2 + c(-c + \frac{a}{3}(-2b - \frac{a^2}{3})) + c^2 - \frac{2ac}{3}(-2b - \frac{a^2}{3})) + \frac{a^2}{9}\left(4b^2 + \frac{4ba^2}{3} + \frac{a^2}{9}\right),$$

or

$$L = \frac{2abc}{3} + c^2 + \frac{a^2b}{9}\left(\frac{7b}{2} - \frac{5a^2}{6}\right). \quad (18)$$

So we can solve for a, b and c satisfying (14)

$$3x^6 + (2bc - \frac{a^2c}{3} + \frac{ab^2}{3} + \frac{2a^3b}{9})x + \frac{2abc}{3} + c^2 + \frac{a^2b}{9}\left(\frac{7b}{2} - \frac{5a^2}{6}\right) = 0. \quad (19)$$

Indeed, putting

$$b = a^2\delta \quad (20)$$

gives from (14)

$$c = -a^3\left(\frac{1}{9} + \frac{\delta}{2} + \frac{\delta^2}{2}\right), \quad (21)$$

and equation (19) becomes

$$3x^6 + a^5\left(-\delta^3 + \frac{\delta^2}{2} + \frac{11\delta}{18} + \frac{1}{27}\right)x + a^6\left(-\frac{\delta^3}{3} + \frac{5\delta^2}{9} + \frac{14\delta}{27} + \frac{1}{9}\right) = 0, \quad (22)$$

solvable for chosen a and δ. □

8.9. Polynomial equations of degree ≤ 6 with roots (x + a)(x – a) = 0.

The method we develop in the present section is the most practical for obtaining iterated roots of polynomial equations. However, historically the approach using dependent root techniques was first developed using the results of sections 8.10 and 8.11, when it was conjectured from the book by Netto [Ne1892] that non degree conserving techniques allowed an escape clause from Galois theory.

We will first take the case of the sextic and study the equation with roots (x + a)(x – a) = 0:

$$(x^2 - a^2)(x^4 + bx^3 + cx^2 + dx + e) = 0. \quad (1)$$

If this is put in the form

$$x^6 + Px^5 + Qx^4 + Rx^3 + Tx^2 + Ux + V = 0, \quad (2)$$

then there is a computable mapping between (2) and (1).

Indeed, we have

$$P = b \quad (3)$$
$$Q = -a^2 + c$$
$$R = -a^2b + d$$
$$T = -a^2c + e$$
$$U = -a^2d$$
$$V = -a^2e,$$

and the equations given by (3) can be directly inverted:

$$b = P \quad (4)$$
$$a^4P + Ra^2 + U = 0$$

with the constraint

$$T = -a^2(Q + a^2) - V/a^2$$

and so

$$a^2 = \frac{-R \pm\sqrt{R^2 - 4UP}}{2P}$$
$$b = P$$
$$c = Q + \frac{-R \pm\sqrt{R^2 - 4UP}}{2P}$$
$$d = \frac{3R \pm\sqrt{R^2 - 4UP}}{2}$$
$$e = V/[\frac{R \pm\sqrt{R^2 - 4UP}}{2P}].$$

We may now directly solve (1), knowing the classical solution of the quartic. ☐

According to Galois theory, if we then consider the Tschirnhaus substitution

$$x = y + h, \quad (5)$$

so that in effect the polynomial is a completely general one, the equation

$$y^6 + P'y^5 + Q'y^4 + R'y^3 + T'y^2 + U'y + V' = 0, \quad (6)$$

is then unsolvable directly by the common method, that is, the mapping

$$(h, a, b, c, d, e) \rightarrow (P', Q', R', T', U', V')$$

cannot be inverted by usual Galois techniques of steady descent to equations of lower degree.

In the case of duplicate roots, which follows in the next section, the duplicate roots maintain their status under Tschirnhaus substitutions. ☐

8.10. Polynomials of degree ≤ 6 with duplicate zeros.

We will next take the case of the sextic and study the equation with duplicate roots

$$(x + a)^2(x^4 + bx^3 + cx^2 + dx + e) = 0. \qquad (1)$$

If this is put in the form

$$x^6 + Px^5 + Qx^4 + Rx^3 + Tx^2 + Ux + V = 0, \qquad (2)$$

then again there is a computable mapping between (2) and (1).

This time we have

$$P = 2a + b \qquad (3)$$
$$Q = 2ab + a^2 + c$$
$$R = 2ac + a^2b + d$$
$$T = 2ad + a^2c + e$$
$$U = 2ae + a^2d$$
$$V = a^2e.$$

The value of a in terms of P, Q, R, T, U and V can be obtained from the next section, 8.11, and so b, c, d and e can be determined.

We may now directly solve (1), knowing the classical solution of the quartic. □

We can consider the Tschirnhaus substitution

$$x = y + h', \qquad (4)$$

when (4) retains duplicate roots in y and the modified equation (2)

$$x^6 + P'x^5 + Q'x^4 + R'x^3 + T'x^2 + U'x + V' = 0, \qquad (5)$$

is then solvable by the same method, that is, the mapping

$$(h', a, b, c, d, e) \to (P', Q', R', T', U', V')$$

may be inverted. □

8.11. A differential condition for the detection of duplicate roots.

Theorem 8.11.1. *Select a quadratic factor* $g(x) = 0$ *of a polynomial equation* $F(x) = 0$. *There exists a unique transformation* $x \to x + h$, *such that either*

$g(x)$ *represents* $(x + a)^2 = 0$ *or* $g(x)$ *represents* $x^2 - a^2 = 0$.

Proof. Consider the product $(x + b)(x + c) = 0$. If $b = c$, then this is the first case, and if $b \neq c$, $h = -(b + c)/2$. □

Definition 8.11.2. Let x be real or complex, then $\Delta x^m = mx^{m-1}$ if $m \geq 1$, otherwise $\Delta x^m = 0$.

Remark 8.11.3. We are not employing Cauchy-Riemann complex differentiation.

Theorem 8.11.4. *If*

$$F(x) = \sum_{m=0}^{n} a_m x^m = 0 \qquad (1)$$

has duplicate roots, *then at this root*

$$\Delta F(x) = 0. \qquad (2)$$

Proof.

$$\Delta[f(x)g(x)] = f(x)\Delta g(x) + g(x)\Delta f(x).$$

Thus if f(x) corresponds to an arbitrary polynomial and g(x) to the duplicate zero $(x + a)^2$, then

$$\Delta g(x) = 2(x + a),$$

so that

$$\Delta[f(x)g(x)] = (x + a)\times(\text{a polynomial}). \ \square$$

Equation (2) is of the form

$$\Delta F(x) = \textstyle\sum_{m=1}^{n} m a_m x^{m-1} = 0, \qquad (3)$$

so that multiplying the above equation by x and subtracting mF(x) we get a distinct equation in the power x^{m-1}. Combining this equation with (3), we get an equation in x^{m-2}, so that we have obtained a further descent of the degree, and this process may be iterated. At each stage we retain the root (x + a) = 0, thus at the penultimate resolution we obtain a linear equation in x, which must be the root, and finally we obtain an equation in x^0 which corresponds to a constraint on the coefficients of (1).

By this technique the zero (x + a) may be obtained, and we may proceed by methods already introduced. □

8.12. Polynomial equations of degree $\leq$ 6 with roots (x + a)(x + ha) = 0.

If we know a relationship between two zeros of the sextic, say if one zero is a then the other zero is ha, then from the observation that a polynomial containing these zeros satisfies

$$(x + a)(x + ha) = K(x + da)^2 + (1 - K)(x^2 - d^2a^2), \qquad (1)$$

with

$$2Kd = h + 1 \qquad (2)$$

and

$$(2K - 1)d^2 = h, \qquad (3)$$

there are two solutions to (2) and (3), corresponding to d = 1 and d = h. Choosing d = 1 gives

$$(x + a)(x + ha) = \left(\frac{1+h}{2}\right)(x + a)^2 + \left(\frac{1-h}{2}\right)(x^2 - a^2), \qquad (4)$$

so that the sextic equation can be split into two parts, one with duplicate roots, and one with antiduplicate roots, both parts of which we have previously solved, represented by

$$\left(\frac{1+h}{2}\right)(x^6 + Px^5 + Qx^4 + Rx^3 + Tx^2 + Ux + V)$$
$$+ \left(\frac{1-h}{2}\right)(x^6 + P'x^5 + Q'x^4 + R'x^3 + T'x^2 + U'x + V') = 0. \qquad (5)$$

If Galois theory were to hold then the splitting of solutions for unknown h cannot be obtained algorithmically by radicals, since this corresponds to independent roots, but if h is known then the methods already found for polynomial equations work for factorisation not just equated to zero, and (5) can be obtained. □

8.13. Solvable polynomials containing duplicate and antiduplicate zeros.

Consider the polynomial in multiplicative form

$$F(x) = (x + a)^2(x + b)^2(x + c)^2(x + d)^2G(x) \qquad (1)$$

where G(x) has zeros independent from (x + a)(x + b)(x + c)(x + d), in the arbitrary general case of degree $\leq$ 4.

Then

$$\Delta F(x) = (x + a)(x + b)(x + c)(x + d)H(x) \qquad (2)$$

where H(x) is another polynomial not reducible to subzeros of (1).

Solving (1) and (2) in additive form simultaneously gives by descent a polynomial in additive form of degree 4 containing the zeros (x + a)(x + b)(x + c)(x + d). Further reduction beyond these solutions gives no further information, that is, 0 = 0.

It is then possible to divide (1) by $(x + a)^2(x + b)^2(x + c)^2(x + d)^2$ using the algebra of chapter VII, and obtain G(x) which can be solved separately. The degree of the roots G(x) combined with (x + a)(x + b)(x + c)(x + d) is 8. Similar independent types of reduction are available for antiduplicate roots, upping this degree to 12, and the dependent roots studied in 8.14 up this further to 20. This is referred to in section 17.

Theorem 8.13.1. *Let a polynomial have degree* n, *with* g *pairs of duplicate zeros and* h *pairs of antiduplicate zeros, where* $g \le 4$ *and* $h \le 4$, *then the polynomial is solvable when*

$$n - 2g - 2h \le 4.$$

Proof. Since n is finite, it is possible to determine the number of possible combinations of g and h, and discard those solutions which do not correspond with its implementation in the polynomial.

For given g and h, to solve for 1 instance of say g, which by previous methods we have already obtained, reduces the remaining polynomial to be solved to degree n – 2, and for g occurrences by n – 2g. If we apply this also for h, we get a combined remaining polynomial of degree n – 2g – 2h, which we know is solvable for a remaining degree ≤ 4. ☐

Example 8.13.2. The decic

$$x^{10} + Kx^9 + Lx^8 + Mx^7 + Nx^6 + Px^5 + Qx^4 + Rx^3 + Tx^2 + Ux + V = 0$$

with g + h = 3, for example 3 pairs of antiduplicate roots, is solvable. This is then a septic (degree 7) with three adjoined roots, which we can find but not specify beforehand. ☐

We state an alternative philosophy on the solution of polynomials, at this stage in one variable, x. We will call a polynomial with duplicate, antiduplicate, or zeros of the type to be given in section 8.14, an *entity*. Theorem 8.13.1 can now be described as: *any polynomial of degree* $\le 20 = (4 \times 4) + 4$ *in one variable can be embedded in a solvable polynomial entity of degree* $\le 36 = (2 \times 4 \times 4) + 4$ *in one variable*. We will study such entities.

8.14. Solvable polynomial equations with roots (x + a)(x ± 1/a) = 0.

We form the hypothesis that if we adjoin to the permutations for the quartic the permutation

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix} \text{ or } \begin{pmatrix} a & -a \\ -a & a \end{pmatrix}, \qquad (1)$$

this is the identity transformation considering each of the permutations in (1) as one item, and this internal symmetry is the key to the results we have obtained.

This hypothesis is predictive, since we expect to find the same type of circumstance for the permutations of the pair of roots

$$\begin{pmatrix} a & 1/a \\ 1/a & a \end{pmatrix} \text{ and } \begin{pmatrix} a & -1/a \\ -1/a & a \end{pmatrix}. \tag{2}$$

Then to solve for a in (2) corresponds to solving two polynomials, one in x, and one under the substitution $x \to (1/x)$ or $x \to (-1/x)$, converted to one in proper degree n by multiplying by x^n. On eliminating x^n this decrements a sextic polynomial to a quintic, but we have seen that a sextic polynomial with an identical solution to a quintic can be solved by descent.

We also know that to convert an equation with root a to an equation with root -a, all we need do is perform the substitution $x \to -x$. Thus all polynomials of type (2) can be manipulated to determine the zero a. □

We will now obtain solutions of the first occurrence of (2) directly. Suppose

$$(x + a)(x + 1/a)(x^4 + bx^3 + cx^2 + dx + e) = 0 \tag{3}$$

then

$$x^6 + [b + (a + 1/a)]x^5 + [c + (a + 1/a)b + 1]x^4 + [d + (a + 1/a)c + b]x^3$$
$$+ [e + (a + 1/a)d + c]x^2 + [d + (a + 1/a)e]x + e = 0,$$

giving for

$$x^6 + Px^5 + Qx^4 + Rx^3 + Tx^2 + Ux + V = 0,$$
$$P = b + (a + 1/a) \tag{4}$$
$$Q = c + (a + 1/a)b + 1 \tag{5}$$
$$R = d + (a + 1/a)c + b \tag{6}$$
$$T = e + (a + 1/a)d + c \tag{7}$$
$$U = (a + 1/a)e + d \tag{8}$$
$$V = e. \tag{9}$$

On eliminating all variables on the right except (a + 1/a), we obtain

$$Q – T + V – 1 + (a + 1/a)[-R + (a + 1/a)(T – V)]$$
$$= (a + 1/a)[-2 + (a + 1/a)^2](U – V(a + 1/a)), \tag{10}$$

and

$$P – R + (a + 1/a)(T – V – 1) = [-1 + (a + 1/a)^2](U – V(a + 1/a)), \tag{11}$$

where we can solve (10) and (11) directly, the two formulas between them giving a constraint on the coefficients. □

Indeed, more simply by an alternative method, if we wish to obtain solutions to the quintic

$$(x + a)(x + 1/a)(x^3 + bx^2 + cx + d) = 0 \tag{12}$$

which we represent as

$$x^5 + Px^4 + Qx^3 + Rx^2 + Tx + U = 0, \tag{13}$$

then under the substitution $x \to 1/x$, we can represent the transformed equation with a leading term 1 (called a *monic* polynomial)

$$x^5 + (T/U)x^4 + (R/U)x^3 + (Q/U)x^2 + (P/U)x + 1/U = 0, \tag{14}$$

so that subtracting (14) from (13)

$$[P – (T/U)]x^4 + [Q – (R/U)]x^3 +[R – (Q/U)]x^2 + [T – (P/U)]x + [U – 1/U] = 0, \tag{15}$$

gives a solution to (x + a)(x + 1/a) from the solutions of the quartic, equation (15). □

The number of independent variables in the coefficients of x in our sextic was 5, but if we extend the number of variables to 6 by the substitution

$$x \to gx,$$

then these solutions are of the same +/- sign, those of opposite sign being obtained from the zeros

$$(x + a)(x - 1/a) = 0.$$

Galois theory claims that we cannot obtain a general solution of the sextic or quintic by these methods. ☐

8.15. A solvable sextic equation with one internal constraint.

We will consider the complex polynomial equation

$$(x^3 + ax + b)(x^2 + cx + d)(x + e) = 0, \qquad (1)$$

set equal to the polynomial

$$x^6 + px^5 + qx^4 + rx^3 + tx^2 + ux + v = 0. \qquad (2)$$

Expanding out (1) and equating to (2)

$$p = e \qquad (3)$$
$$q = a + d + ec$$
$$r = de + ac + ae + b$$
$$t = ace + ad + be + bc$$
$$u = ade + bce + bd$$
$$v = bde.$$

On eliminating e from (3)

$$q = a + d + pc \qquad (4)$$
$$r = dp + a(c + p) + b$$
$$t = acp + ad + b(c + p)$$
$$u = adp + b(cp + d)$$
$$v = bdp. \qquad (5)$$

Then, successively eliminating a, then b and putting w = (pc + d)

$$r = \frac{v}{dp} + dp + (q - w)\left(\frac{w}{p} + p - \frac{d}{p}\right) \qquad (6)$$

$$t = (q - w)w + \frac{v}{dp}\left(\frac{w}{p} + p - \frac{d}{p}\right) \qquad (7)$$

and

$$u = (q - w)dp + \frac{v}{dp}w. \qquad (8)$$

We will use (6) and (7) to eliminate w^2. Then

$$(2q - p^2 + d - \frac{v}{dp^2})w - pr + dp^2 + q(p^2 - d) + t + \frac{v}{p^2} = 0. \qquad (9)$$

Using (8) and (9) we can eliminate w, giving a cubic in d:

$$p^3d^3 + (-rp^2 + tp - u + \frac{v}{p})d^2$$
$$+ (up^2 - vp)d + v(-qp + r - \frac{t}{p} + \frac{u}{p^2} - \frac{v}{p^3}) = 0. \qquad (10)$$

There is also a *constraint* on p, q, r, t, u and v. Equation (10) is solvable for d. Therefore using (8) we can obtain w, and hence c, and from (3), (4) and (5) obtain e, a and b. ☐

8.16. General polynomial equations can be embedded in a solvable entity.

Consider the product

$$\prod_{j=1}^{n}\left(x^j - a_j\right) = 0, \qquad (1)$$

for example

$$(x^5 - a_5)(x^4 - a_4)(x^3 - a_3)(x^2 - a_2)(x - a_1) = 0,$$

that is

$$x^{15} - a_1x^{14} - a_2x^{13} - (a_3 - a_2a_1)x^{12} - (a_4 - a_3a_1)x^{11} - (a_5 - a_4a_1 - a_3a_2)x^{10} \dots = 0.$$

Then in general on putting $k = \frac{n(n+1)}{2}$

$$x^k - a_1x^{k-1} - a_2x^{k-2} - \dots + \left[\sum \prod_{\sum j=h}\left(-a_j\right)\right]x^{k-h} + \dots + \prod_{j=1}^{n}\left(-a_j\right) = 0,$$

gives the coefficients $a_1, \dots a_n$. □

Put $a_j = (b_j)^j$, so that $\left(x^j - a_j\right) = 0$ contains the root $\left(x - b_j\right) = 0$. We get

Corollary 8.17.1. *All finite polynomial equations may be embedded in solvable polynomial entities of type* (1). □

8.17. The theory of dependent roots of a polynomial equation.

A question of interest concerning the solvability of polynomial equations is: what is the general backdrop to the explicit solutions for solvable entities in one variable? We will treat this question under the heading of dependent roots of polynomial equations, and answer it as a reduction procedure, followed by solvability algorithms if available.

As accountancy we can consider root symbols all different, but possibly with related values.

When identical elements occur, the usual terminology is to say the polynomial is *inseparable*, and if this does not occur, to call the polynomial *separable*. For my part, when there is a function h_{ab} which is known or can be found so that

$$u_b = h_{ab}(u_a)$$

then I will say the roots $(x - u_a)$ and $(x - u_b)$ are *dependent*, and if there is no such known function, that the roots are *independent*. Thus an inseparable polynomial is a special case of a polynomial with dependent roots.

In this chapter we have discussed internal dependencies maintaining for each variable within the dependency an *involution*, $\mathbb{I}$, which by definition satisfies

$$\mathbb{I}^2 = 1.$$

The duplicate roots we came across are usually described by the word *inseparable*. These we describe by the involution

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix}, \qquad (1)$$

whereas for antiduplicate roots, if w is chosen as an arbitrary constant, the permutation

$$\begin{pmatrix} a & w - a \\ w - a & a \end{pmatrix}, \qquad (2)$$

gives an additive type of involution, the other involution types studied being

$$\begin{pmatrix} a & w/a \\ w/a & a \end{pmatrix}, \qquad (3)$$

where the values $w = 0$ for (2) and $w = \pm 1$ for (3) were chosen.

General dependencies may be classified so that they include cases (1), (2) and (3) above. Strictly speaking, this classification may be subdivided into the absolute case, where we know a specific set of roots, in which case the remaining equation is obtained by dividing out these roots, so this is fairly trivial, and the relative case, where with respect to a set of roots we specify dependencies between them.

Let a general polynomial be of the form

$$x^n + Px^{n-1} + ... + V = 0. \qquad (4)$$

To consider the relative case, we select a root if there is one for which a dependency is to be specified. If this root is a, then if there is a binary dependency so that a further root is ha, then we will find that there is an extension of the argument of section 12, so that

$$(x + a)(x + ha) = \left(\frac{1+h}{2}\right)(x + a)^2 + \left(\frac{1-h}{2}\right)(x^2 - a^2), \qquad (5)$$

where the polynomial equation can be split into two parts, one with duplicate roots, and one with antiduplicate roots, both parts of which we must ensure we can solve in the general case, representing (4) by (5) multiplied by a common polynomial as

$$\left(\frac{1+h}{2}\right)(x^n + P'x^{n-1} + ... + V') + \left(\frac{1-h}{2}\right)(x^n + P''x^{n-1} + ... + V'') = 0, \qquad (6)$$

where

$$\left(\frac{1+h}{2}\right)(P') + \left(\frac{1-h}{2}\right)(P'') = P \qquad (7)$$

is uniquely expressed with h known and likewise for the other coefficients.

We have already indicated the method of solution for the $(x + a)^2$ roots which occupy the $\left(\frac{1+h}{2}\right)$ term in (6), explained in sections 10 and 11 as a method by descent.

Now consider the $\left(\frac{1-h}{2}\right)$ term in (6), which contains $(x + a)(x - a)$. We will look at the monic function

$$x^n + p_1x^{n-1} + p_2x^{n-3} + ... + p_{n-1}x + p_n, \qquad (8)$$

which contains this $(x^2 - a^2)$ term. Writing this as

$$(x^2 - a^2)(x^{n-2} + b_3x^{n-3} + b_4x^{n-4} + ... + b_{n-2}x^2 + b_{n-1}x + b_n), \qquad (9)$$

expanding out and comparing with (8) gives

$$p_1 = b_3$$
$$p_2 = b_4 - a^2$$
$$p_3 = b_5 - a^2b_3$$
$$...$$
$$p_{n-r} = b_{n-r-2} - a^2b_{n-r}$$
$$...$$
$$p_{n-2} = b_n - a^2b_{n-2}$$
$$p_{n-1} = -a^2b_{n-1}$$
$$p_n = -a^2b_n. \qquad (10)$$

These terms may be split into two sets of equations in p_k, with k even or odd. For instance we have $p_1 = b_3$ and $p_3 = b_5 - a^2b_3$ give $p_3 = b_5 - a^2p_1$, and then $p_5 = b_7 - a^2b_5$ determines the value of b_7 in $p_5 = b_7 - a^2(p_3 - a^2p_1)$, etc.

By these means we obtain two equations in powers of a^2, one in values p_k with k odd, and another in p_k with k even. By a method perfectly analogous to that of descent for $(x + a)^2$, we may solve these two equations in a^2 when $(x + a)(x - a)$ are the roots.

Having done this, we can extract out the roots a and ha, so that the degree of the resulting equation is decremented by two. If this is the only dependency, we can amalgamate the left and right terms before zero in (6) as the same equation, which is now subject to standard solvability criterions.

If there are further dependencies related to the root a, and they are all of a binary form, we can reintroduce the root a and now its dependency with another root, say h′a. Since we have reintroduced a, the result on repeating this method is now only to reduce the degree by one, and we can continue in this fashion until we are finished.

If further dependencies are not related in any way to the root a, we can introduce roots b and h″b, and proceed as before, the first time decrementing the degree by two, and subsequently by one.

These dependencies we have assumed so far are linear, and can be manipulated and solved by linear algebra, but the involution (3) is not inherently linear. In general dependencies will be polynomial dependencies between roots, so that the problem has to be solved recursively, if that is possible. ☐

That Galois theory or its replacement is a theory of algorithms and not states can be deduced by specifying a dependency between roots. When this is known, sometimes solutions can be found, but when this is unknown solutions can disappear under Galois restrictions. ☐