CHAPTER VII

Ladder and complex algebra

7.1. Introduction.

Ladder algebra, which is described in [Ad14] is developed and includes a discussion of the uncountable continuum hypothesis, UCH, proving its incompatibility with the countability of the rational numbers, $\mathbb{Q}$, and a rule of induction for sets. We introduce real numbers with a transfinite number of terms. A comparison of our account with that of P. J. Cohen is given in volume III, chapter XIV. Complex numbers are applied to ladder analysis where we give examples employing the standard protocol, which defines ordinal infinity, and the strict transfer principle. Chapter XII discusses the axiom of choice.

We prove the fundamental theorem of algebra for complex roots using the topological idea of winding number. Fourier series, the complex Cauchy integral formula and the hyperintricate Cauchy-Riemann equations are represented in *Number, space and logic* [Ad18].

7.2. The inconsistency of the UCH property for $\mathbb{N}^{\mathbb{N}}$.

We repeat a small part of [Ad14] on countable and uncountable sets and ordinal infinities, and extend it, but first in this section we will find a rule for the conditions under which the principle of induction is valid for the properties of sets.

Doly García remarks that if a property holds for a set indexed by 1, n and n + 1, one cannot argue that it holds for the entire set, an instance being given by the set $A_1 = \{1\}$, with the property that $2 \notin A_1$, and in general for $A_n = \{1, 2, ... n\}$, where $n + 1 \notin A_n$, so that each complement exists for finite n, but not the entire set $\mathbb{N}$. This is part of the reason for us introducing ladder numbers, but we need to view this where nonstandard analysis is not used.

To respond to this criticism, if we look at the definition of the empty set just given, we notice that mZFC deals with predicates in a nonstandard way, in that the same sentence may range over true and false is admissible in the definition of a set, and false defines the void set in mZFC, and not otherwise. Then if we look at the complement of A_1, this is $\mathbb{N}\backslash\{1\}$, and the complement of A_n is $\mathbb{N}\backslash\{1, 2, ... , n\}$, so that applying the principle of induction, over the set $A_{\mathbb{N}}$ we are dealing with $\mathbb{N}\backslash\mathbb{N}$, which is the empty or void set, and thus the same property ranges over true and false in accordance with its principle of induction for sets. However
Proposition 7.2.1. *If the property for a set does not range over the void set*, *it does not become false*.

In mZFC, if a predicate holds for all finite $n \in \mathbb{N}$, either the predicate does not hold for all n, which could be the case for standard set theory, where the set of all n not including all finite n is empty, which satisfies a false predicate, or the predicate holds for all n, which is the case for nonstandard set theory, since now the set of all n not including all finite n has elements.

When mapping properties, for instance a bijection $\{\mathbb{V}, \emptyset\} \leftrightarrow \{\mathbb{V}, \emptyset\}$ restricted to $\mathbb{V} \rightarrow \mathbb{N}$, for standard set theory possible mappings are $\mathbb{V} \leftrightarrow \mathbb{V}$, $\mathbb{V} \leftrightarrow \emptyset$, $\emptyset \leftrightarrow \mathbb{V}$ and $\emptyset \leftrightarrow \emptyset$. Since $\mathbb{V}$ maps to true and $\emptyset$ to false, we may and will define that $\leftrightarrow$ satisfies the truth table for IF and only IF, written as $\Leftrightarrow$, with $A \leftrightarrow B \leftrightarrow C$ satisfying $(A \Leftrightarrow B)$ & $(B \Leftrightarrow C)$.

$A \Leftrightarrow B$	A	B
T	T	T
F	T	F
F	F	T
T	F	F

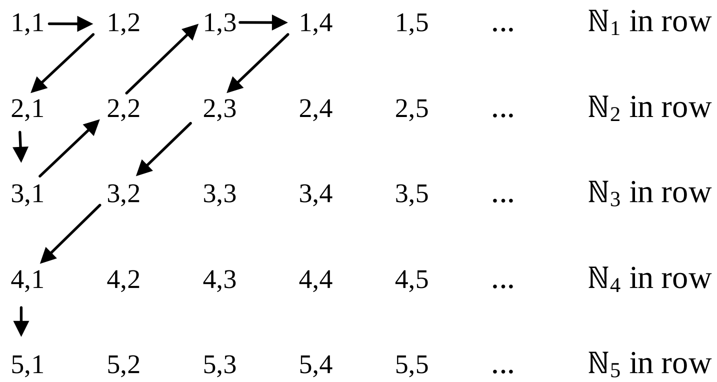
This allows us to state that the principle of induction applies to bijective properties in standard set theory. □

For the argument which follows there is no element where the bijection between the set of rationals, $\mathbb{Q}$, and what we define as $\mathbb{N}^{\mathbb{N}}$ does not hold; nowhere does the induction we have carried out range over the predicate for a void set, since its function of functions $\mathbb{N} \to \mathbb{N} \to ...$ mapped in sequence to $\mathbb{N} \to \mathbb{N} \times \mathbb{N} \to ...$ has a fixed and not exhausted function as its domain. This assertion is proved in chapter XIV section 6, which is developed from [5Co66], where cardinals defined by bijections are shown to be lowest ordinals, ordinals are countable if there is a bijection to $\mathbb{N}$, and it is proved that because a union of countably many countable sets is countable, exponential and superexponential operations on countable ordinals are countable.

For finite or countably infinite sets S, T or U we define $\equiv$ by the existence of at least one bijection within the natural numbers $\mathbb{N}$. This is an equivalence relation, in that $S \equiv S$, if $S \equiv T$ then $T \equiv S$, and if $S \equiv T$ and $T \equiv U$, then $S \equiv U$, so this forms a partition between those sets belonging to the equivalence class, and those outside it. Then for sets S_n, T_n, $n \in \mathbb{N}$, if for each n $S_n \equiv T_n$, then for the set of all S_n, $\{S_{nn}\} \equiv \{T_{nn}\}$, where this means if $S_n \equiv T_n$, then the bijection is maintained for $S_{n+1} \equiv T_{n+1}$, and the second subscript indicates a distinguished copy for $S_n \neq \varnothing$, defined inductively: $S_{n1} = \{S_n\}$, $S_{n2} = \{\{S_n\}\}$, $S_{n\,p+1} = \{S_{np}\}$.□

In this work we will adopt the argument of Cantor that *the set* $\mathbb{Q}$ *of rationals is countable*. Define the Cartesian product of all natural number pairs $\mathbb{N} \times \mathbb{N}$ as $\mathbb{N}^2$. Consider the rational numbers not in lowest terms given by the set $\mathbb{Q} \equiv \{\{1/n\}, \{2/n\}, \{3/n\}, . . .\} \equiv$ the unordered distinguished copies$\{\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3, . . .\} \equiv \mathbb{N} \times \mathbb{N}$ (a set of ordered pairs) which by the Cantor argument given next is $\equiv \mathbb{N}$. The mapping from $\mathbb{N}$ to $\mathbb{Q}$ is given in the following diagram.

$(m, n) \to m/n \in \mathbb{Q}_{\neq 0}$.



What is meant by the symbols . . . in the sets just given? This indicates that if the pth position is occupied, then a similar item exists at position p + 1, although we can remove . . . from the language and use the properties of $\mathbb{N}$ itself given in chapter III, section 3. Then by induction defined through the properties of $\mathbb{N}$, we have $\mathbb{N} \equiv \mathbb{N}^p$ for p a natural number, so that the set

$$\{\mathbb{N}_1, \mathbb{N}_2^{\,2}, . . . , \mathbb{N}_p^{\,p}, . . .\}$$
$$\equiv \{\mathbb{N}_1, \mathbb{N}_2 \times \mathbb{N}_2, . . , \mathbb{N}_p \times \mathbb{N}_p \times . . \text{(p terms)}, . . .\} \equiv \mathbb{N}, \quad (1)$$

contains by definition $\mathbb{N}^{\mathbb{N}} \equiv \{\mathbb{N} \times \mathbb{N} \times \ldots\}$. This is in violation of the assumptions of the uncountable continuum hypothesis (UCH) in set theory, that $\{0, 1\}^{\mathbb{N}}$ is uncountable. ☐

Repeated proof. By the definition of a union of sets of chapter III section 2 and induction

$$\bigcup_{\mathbb{N}} n \equiv \mathbb{N},$$

the union of all $n \in \mathbb{N}$ is $\mathbb{N}$.

When we go over to countably infinite sets, there is a property which does not hold for finite sets, that for distinct copies $\mathbb{N}_n$

$$\bigcup \mathbb{N}_n \equiv \mathbb{N}.$$

This is extended and is inherent in the Cantor argument for the countability of $\mathbb{Q}$, that there exists a constructible bijective mapping

$$\mathbb{N} \equiv \bigcup_{\mathbb{N}} \mathbb{N}_n \equiv (\bigcup_{\mathbb{N}} n) \times \mathbb{N} \equiv \mathbb{N} \times \mathbb{N}. \qquad (2)$$

If the size of the nth diagonal, up or down, in the previous diagram is n, and the sum of the diagonals is given by the arithmetic series

$$1 + 2 + 3 + \ldots + n = n(n + 1)/2,$$

then for each $m \in \mathbb{N}$ there is an $n(n + 1)/2$ and $t \in \mathbb{N}$ with the last diagonal of size n, $0 \leq t < n < n(n + 1)/2$, and by the Euclidean algorithm of chapter III section 9, a bijection

$$m \leftrightarrow n(n + 1)/2 + t.$$

This is a bijection

$$\mathbb{N} \times \mathbb{N} \leftrightarrow \mathbb{N},$$

or as we have written

$$\mathbb{N} \times \mathbb{N} \equiv \mathbb{N}.$$

Let $\mathbb{N}^1$ be $\mathbb{N}$, and $\mathbb{N}^{p-1} \times \mathbb{N}$ be $\mathbb{N}^p$. By the definition of the natural numbers (chapter III, section 3) the principle of induction states that because $\mathbb{N}^1$ is a subset of $\mathbb{N}$ containing 1 and $(n + 1)$ whenever $n \in \mathbb{N}$, and if $\mathbb{N}^{p-1} \leftrightarrow \mathbb{N}$ holds for 1 and $(p - 1) + 1$, then $\mathbb{N}^p \leftrightarrow \mathbb{N}$, so we have shown by the definition of $\mathbb{N}^p$,

$$\mathbb{N} \equiv \mathbb{N} \times \mathbb{N} \equiv \mathbb{N}^{p-1} \times \mathbb{N} \equiv \mathbb{N}^p.$$

There is a bijection

$$\mathbb{N}_n \leftrightarrow \mathbb{N}$$

obtained by stripping out all containing parentheses. Thus

$$\mathbb{N}_n \equiv \mathbb{N}, \qquad (3)$$

and we have proved the equivalence (2), where we can add the comment for any n,

$$n \times \mathbb{N} \equiv \mathbb{N}.$$

From the equivalence (3) we get

$$\mathbb{N}_p{}^p \equiv \mathbb{N}. \qquad (4)$$

Thus we have shown directly that equation (1) holds, where

$$\mathbb{N}^p \rightarrow \mathbb{N} \text{ is injective implies } \mathbb{N}^{p+1} \rightarrow \mathbb{N} \text{ is injective},$$

so, because there is no void set in the mappings, by induction we obtain its consequence

$$\mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N} \text{ is injective},$$

and since for the constant $\{1\}^{\mathbb{N}}$

$$\mathbb{N} \leftrightarrow \{1\}^{\mathbb{N}} \times \mathbb{N} \rightarrow \mathbb{N}^{\mathbb{N}} \text{ is injective}$$

we derive the result

$$\mathbb{N}^{\mathbb{N}} \equiv \mathbb{N}. \ \square \qquad (5)$$

Another way of stating this feature is that the uncountable continuum hypothesis is not an axiom that is independent of the countability of $\mathbb{Q}$ and this rule of induction, and the countability of $\mathbb{Q}$ naturally takes precedence over the continuum hypothesis.

Thus in a version of the second order logic developed here, all superexponential operations resulting in the construction of sets build countable sets from countable sets. This is a new conclusion, and must replace the uncountable continuum hypothesis. ☐

The previous proof raises the issue of the status of the Cantor diagonal argument on the uncountability of the real numbers. Before dealing with this, we address its counterpart for finite sets, where the Cantor argument does not work. We extend this idea to infinite but countable sets.

The Cantor diagonal argument is deconstructed as follows. First note the example of a finite set consisting of elements ordered as (a, b), where a and b $\in$ {0, 1}, can be described as two finite sets E, F where E = {(0, 0), (0, 1)} and F = {(1, 0), (1, 1)}. In this example we define *diagonal* E to be found from

$(\underline{0}, 0)$
$(0, \underline{1})$,

that is, (0, 1) as underlined, so NOT *diagonal* E = NOT (0, 1) = (1, 0). So E is finite, and the fact that NOT *diagonal* E $\notin$ E (but $\in$ F) does not show {E, F} is not a finite set. We have shown for a finite set, the diagonalisable elements form a subset of the finite set under an ordering.

In the finite case the Cantor theorem holds that the power set (the set of subsets) of any set is larger than the set, but the NOT *diagonal* relation for finite sets does not give the correct semantic interpretation for infinity. The correct definition of a set S_m being finite is that it is empty or has all elements $(s_1, s_2, ... s_m)$ with fixed $m \in \mathbb{N}$, and an infinite set is not finite. ☐

The next definition is explained further in 7.3.1.
Definition 7.2.2. A *Eudoxus number*, the set of which we denote by $\mathbb{U}$, is a bounded number representable at most by a sum of a countably infinite set of rational numbers.

For the diagonal argument for infinite sets, consider the list of Eudoxus numbers denoted in binary and indexed by $2n_i$

$2\mathbb{N}$	The set of all diagonalisable Eudoxus numbers with respect to an ordering
$2n_i$	u_i

Firstly, we will consider a specific ordering of $\mathbb{U}$. By a similar argument to the finite case *diagonal* $\{u_i\}$ = {the Eudoxus number with the ith digit taken from u_i}, so NOT *diagonal* $\{u_i\}$ $\notin$ $\{u_i\}$ is consistent, since $\mathbb{N} \equiv 2\mathbb{N} \equiv (2\mathbb{N} + 1) \equiv 2\mathbb{N} \cup (2\mathbb{N} + 1)$, and all Eudoxus numbers are indexed by $n_i \in \mathbb{N} \equiv \mathbb{N}_{diagonal} \cup \{n_i\}_{nondiagonal}$. In this ordering $\{n_i\}_{nondiagonal}$ is one element appearing after the diagonal terms, or can be inserted in first to show the bijection with $\mathbb{N}$. Thus the utility of the diagonal argument disappears under a specific ordering.

We will now consider a generic ordering covering all possible orderings of $\mathbb{U}$. Then the larger set of $\{n_i\}_{nondiagonal}$ now corresponds to the set of possible $\mathbb{U}$, so each instance of this diagonal can appear, say, as the first element of $\mathbb{U}$ in a different ordering. Thus the utility of the diagonal argument now disappears under a generic ordering, and we are back to proving $\mathbb{N}^{\mathbb{N}} \equiv \mathbb{N}$, which we have already done, and so the Eudoxus numbers are countable. ☐

This theorem is in conflict with the assertion that $\{0, 1\}^{\mathbb{N}}$ is uncountable, which in turn uses the unacceptable semantic definition that the NOT *diagonal* relation for countably infinite sets acts as a criterion for uncountability. However as mentioned by P. J. Cohen in [1Co63], [1Co64] ‘one can construct models in which the set of constructible reals is countable’. ☐

7.3. Ladder algebra.

Definition 7.3.1. We adopt the *standard protocol* for ladder algebra:

$$\mathbf{\Omega}_{\mathbb{N}} = \sum_{\text{all } \mathbb{N}} 1. \quad (1)$$

For any $n \in \mathbb{N}$, $n < \mathbf{\Omega}_{\mathbb{N}}$, so $\mathbf{\Omega}_{\mathbb{N}} \notin \mathbb{N}$. We will treat $\mathbf{\Omega}_{\mathbb{N}}$ as being irreducible. This means we do not split $\mathbf{\Omega}_{\mathbb{N}}$ into noncontiguous components, or truncate or extend it. We adopt for $\mathbf{\Omega}_{\mathbb{N}}$ the negation shown below of a property attributed by Archimedes to Eudoxus of Cnidus for finite natural numbers: the ordinal infinity $\mathbf{\Omega}_{\mathbb{N}}$ is inaccessible with respect to n and obeys the rule

for every $m \in \mathbb{N}$ *and for* $\mathbf{\Omega}_{\mathbb{N}}$ *there does not exist an* $n \in \mathbb{N}$: $\mathbf{\Omega}_{\mathbb{N}} < mn$. ☐ (2)

Definition 7.3.2. We adopt the *strict transfer principle* for ladder algebra:

the axioms for a field or zero algebra hold with respectively $a\mathbf{\Omega}_{\mathbb{N}}$, $b\mathbf{\Omega}_{\mathbb{N}}$ *and* $c\mathbf{\Omega}_{\mathbb{N}}$ *replacing some or all of* a, b *and* c *in these axioms*. ☐

An example is $1\mathbf{\Omega}_{\mathbb{N}} = \mathbf{\Omega}_{\mathbb{N}}$.

Definition 7.3.3. *Ladder natural numbers* $\mathbf{L}_{\mathbb{N}\cup 0}$ are defined by

$$\mathbf{L}_{\mathbb{N}\cup 0} = \bigcup_m[\mathbb{N}_{\cup 0}(\mathbf{\Omega}_{\mathbb{N}})^m],\ m \in \mathbb{N}_{\cup 0}.$$

The algorithmic proof of a proposition by induction: choose a start value, assume for n and then prove for n + 1, now extends to $n \in \mathbf{L}_{\mathbb{N}\cup 0}$ under the strict transfer principle. However, the Peano axiom of induction states that the natural numbers are unique, so we must augment it for $\mathbb{N}_{\cup 0}$ by saying it contains no elements $(\mathbf{\Omega}_{\mathbb{N}})^m$. In ladder algebra the complement of all $\mathbb{N}$ is not empty, so an induction predicate can always be true, even in mZFC.

What is the status of finite proofs in this situation? We have seen by the axiom of the strict transfer principle above, that proofs by countable induction over $\mathbb{N}$ reduce to finite proofs. Thus there exist some finite proofs over countably infinite sets.

We argue in terms of states and not processes. For infinite processes, obtained values may oscillate infinitely. There is more than one approach that we can take to enable consistency.

The first method looks at logical deduction. Valid reasoning based on the evaluation of finitely determined states by processes which terminate finitely are retained. Otherwise valid infinite reasoning is restricted, so that we may be able to obtain consistent results previously unavailable. We can find *preferred evaluations* of these infinitely determined states or those with infinite processes so that the induction procedure is restricted for these types. We adopt the following method employing arguments under the strict transfer principle. For reasoning using all $\mathbb{N}$, start with the first element of $\mathbb{N}$, which is 1, and employ *linear induction mod* n by allocating either $\mathbb{N}$, or an ordered block of n elements which belong to $\mathbb{N}$, so that the same proof is valid in each block. *Nonlinear induction* can be defined for other partitions.

We cannot find an example giving different evaluations depending on n. So typically the case

$$1 – 2 + 3 – 4 + 5 – 6 + 7 – ...$$

has a series of terms -n (mod 2n). The sum for blocks of length 2n is then an invariant mean of -½, so that using the linear induction and strict transfer principles this carries over to the whole series $\mathbf{\Omega}_{\mathbb{N}}$, and the total sum is $-\frac{1}{2}\mathbf{\Omega}_{\mathbb{N}}$.

To evaluate this series an approach given in [SS03] is to define an *Abel sum* as a series

$$\mathrm{A(r)} = \sum_{k=0}^{\Omega} c_k r^k$$

where the c_k may be complex and

$$\lim_{r\to 1} \mathrm{A(r)} = s.$$

If a series converges to s for $0 \leq r < 1$ by analytic continuation A(1) has the value s, but the reverse implication may not apply. The authors claim that the Abel sum given above is one quarter, since they say A(r) is

$$A(r) = \sum_{k=0}^{\Omega} (-1)^k (k+1) r^k = 1/(1+r)^2.$$

We note that if this series were partitioned as

$$A(r) = 1 + (-2 + 3r)r + (-4 + 5r)r^3 + \ldots$$

the result is positive for r = 5/8, however if we put

$$A(r) = (1 - 2r) + (3 - 4r)r^2 + \ldots$$

this is negative for r = 5/8, so the formula is partition dependent. □

The next method extends Boolean logic. General multivalued logics are discussed in *Number, space and logic* [Ad18]. Thus, for example, for the infinite type of systems we have been considering, we could have multivalued logic operating on three states, 'true', 'false' and 'oscillates'. A particular type of multivalued logic discussed in chapter XIII of this volume is *probability logic*. A Boolean type logic with two states τ and υ can be extended so that its values are linear combinations of τ and υ, in particular retaining the boundary condition that both τ and υ exist within the logic. A linear probability logic contains the states $c\tau + (1 - c)\upsilon$ where c is, say, a real variable. It is a theorem that oscillating values may be allocated as the value $\frac{1}{2}\tau + \frac{1}{2}\upsilon$ in a linear probability logic.

The model used to interpret this logic is not here the statistical correlation approach given in chapter XIII, although its evaluation may be found as a limit process acting on its correlation with oscillating values.

In the 'paradox of the liar' the value of " 'A is valid' and 'A is invalid'" is false, and is not the same as " 'A is valid' and 'A is invalid' is false" (which is true) on keeping track of the level of nesting of the quotes and their implicit parentheses.

Since for us, logic also includes parentheses in its syntax, validity or invalidity of a formula may depend on the presence or absence of such parentheses, and this includes in recursive statements, which must have them.

X is invalid	means	X is invalid
(X is invalid) is valid	means	X is invalid,

where similar and further nestings are possible. For example there is a possible chain

(((X is invalid) is invalid) is invalid).

Thus there are an infinite number of states and their infinite countable evaluation

(((X is invalid) is invalid) is invalid) ...

is equivalent to the evaluation

$$Y = \prod_{\text{all natural numbers } \mathbb{N}} -1.$$

For example, using the strict transfer principle to evaluate

$$Y = \prod_{\text{all } \mathbb{N}} -1, \qquad (3)$$

partition ℕ into blocks of pairs starting from 1, then for each pair evaluate as a member of

$$W = (-1)_{\text{odd } \mathbb{N}} \times (-1)_{\text{even } \mathbb{N}},$$

and the product is always 1. Thus under the linear induction principle,

$$Y = 1,$$

and consequently the infinite countable evaluation equivalent to finding Y

(((X is invalid) is invalid) is invalid) ...

evaluates to X which is valid. Its linear probability logic evaluation is $\frac{1}{2}\tau + \frac{1}{2}\upsilon$. □

There exist ideals of these blocks, in which proofs of the above type live, for the example above 1.W and -1.W, and say cosets of the ideal 1.W

$0 + W$, and $-2 + W$,

where only the 0 coset is the preferred evaluation. It follows that the preferred evaluation of $\mathbf{\Omega}_{\mathbb{N}}$ obtained additively is even and not odd, and its linear probability evaluates to

½(even = τ)τ + ½(odd = τ)υ. ☐

We can express in this system the axiom that for any property P and for the set {x: P; x ∈ X}, there exists a set {y: NOT P; y ∈ Y}. Since by this means we can introduce uncountable sets, we can now establish uncountable induction.

We have seen that an uncountable set ℝ is not generated recursively by any means from ℕ, even when it has a countably infinite set embedded within it. We say $\mathbf{\Omega}_{\mathbb{R}}$ with uncountable ℝ is strictly less than ultrainfinity nЧ, n ∈ ℕ, whereas there exist a hierarchy of ultrainfinities $nЧ^m$, m ∈ ℕ, including superexponential types of this sort of expression, all of which are greater than $\mathbf{\Omega}_{\mathbb{R}}$. This arises because $(\mathbf{\Omega}_{\mathbb{R}})^{-1} = \boldsymbol{\varepsilon}_{\mathbb{R}}$ is not a multizero; no $a – a = \boldsymbol{\varepsilon}_{\mathbb{R}}$. ☐

We can further introduce a sequence

$$\mathbf{\Omega}_{\mathbb{N}} < \mathbf{\Omega}_{\mathbb{R}} < \mathbf{\Omega}_{\mathbb{R}'} < ...$$

in which there are no bijections of the type ℝ → ℝ′. It can now occur that, say, an algorithm halts in ℝ′ but not in ℝ, so ℝ′ is inaccessible with respect to ℝ. ☐

The allocation of two values, True and False, to logical evaluation may be extended to an n-fold set of values. There is a relation and a difference between these values and recursive procedures.

In recursion, if we apply the NOT operator to say the property of being countable for a set, then we have seen we can substitute the ordinal infinity $\mathbf{\Omega}_{\mathbb{N}}$ describing ladder numbers in the countably infinite set ℕ by the uncountable ordinal infinity $\mathbf{\Omega}_{\mathbb{R}}$ of ℝ described by the standard protocol

$$\mathbf{\Omega}_{\mathbb{R}} = \textstyle\sum_{\text{all } \mathbb{R}} 1. \qquad (4)$$

If we now apply recursively this procedure to $\mathbf{\Omega}_{\mathbb{R}}$, we get a higher order uncountability of a set ℝ′ with ordinal infinity $\mathbf{\Omega}_{\mathbb{R}'}$. There is then a countable or uncountable sequence

$$\mathbf{\Omega}_{\mathbb{N}} \rightarrow \mathbf{\Omega}_{\mathbb{R}} \rightarrow \mathbf{\Omega}_{\mathbb{R}'} \rightarrow ...$$

corresponding to sets with successively higher cardinality.

$$\mathbb{N} \rightarrow \mathbb{R} \rightarrow \mathbb{R}' \rightarrow ...$$

For the allocation of NOT to the value True or False, we merely switch their values. Nevertheless, we can consider this mapping as a recursion (mod 2). For an n-valued logic this may be considered as recursive (mod n).

This comparison we have been making has up to now been restricted to algebras which are abelian and associative. It is our thesis that these ideas can be extended to nonabelian and nonassociative n-fold logics. ☐

The axioms for exponentiation are given as a case of those for superexponentiation in chapter XVII. We extend the strict transfer principle to these axioms too. It follows that we may employ $(\mathbf{\Omega}_{\mathbb{N}})^2$ and $(\mathbf{\Omega}_{\mathbb{N}})^t$ for t an exponent ∈ 𝔽 or 𝕐. For t a positive number, $(\mathbf{\Omega}_{\mathbb{N}})^t$ becomes an example of a hyperinfinity.

We will denote $(\mathbf{\Omega}_{\mathbb{N}})^{-1}$ by $\boldsymbol{\varepsilon}_{\mathbb{N}}$. It follows from the property of $\mathbf{\Omega}_{\mathbb{N}}$ that the ordinal infinitesimal $\boldsymbol{\varepsilon}_{\mathbb{N}}$ satisfies

for every $m \in \mathbb{N}$ *and for* $\boldsymbol{\varepsilon}_{\mathbb{N}}$ *there does not exist an* $n \in \mathbb{N}$: $n\boldsymbol{\varepsilon}_{\mathbb{N}} > m$. (5)

Similarly $(\mathbf{\Omega}_{\mathbb{N}})^{-t} = (\boldsymbol{\varepsilon}_{\mathbb{N}})^{t}$ is a hyperinfinitesimal. ☐

If we define $\mathbf{\Omega}_{\mathbb{Z}} = \sum_{\text{all } \mathbb{Z}} 1$, then

$\mathbf{\Omega}_{\mathbb{Z}} = 1 + 2\mathbf{\Omega}_{\mathbb{N}}$. ☐ (6)

The set $\mathbb{Q}$ of zero or positive rational numbers where $q = m/n \in \mathbb{Q}$, $m \in \{\mathbb{N} \cup 0\}$, $n \in \mathbb{N}$, ignoring the equivalence relation where m/n is in lowest terms, may be mapped bijectively to the set of pairs {m, n}. Then under the axiom system we are adopting, derived from the strict transfer principle acting on all pairs bounded by m and n, the number of elements of $\mathbb{Q}$ in not necessarily lowest terms is $\mathbf{\Omega}_{\mathbb{N}}(\mathbf{\Omega}_{\mathbb{N}} + 1)$. ☐

Using the strict transfer principle we obtain the Euler formula

$e^{i\theta\mathbf{\Omega}} = \cos(\theta\mathbf{\Omega}) + i\sin(\theta\mathbf{\Omega})$. ☐

We may obtain integrals using this principle:

$\int_0^{\Omega} x^2\, dx = \frac{1}{3}\mathbf{\Omega}^3$. ☐

7.4. Algebraic and transcendental numbers.

If we consider $\sqrt{2}$, then this is not a rational number, for if it were $\frac{m}{n}$, with m and n natural numbers divided out to be in lowest terms, then m and n cannot both be even. So assuming

$$\sqrt{2} = \frac{m}{n}, \tag{1}$$

if m and n were both odd, then on squaring both sides of (1), multiplying by n^2 gives an odd number, m^2, equal to an even number, $2n^2$. If m is in any situation even, say $2m'$, then n^2 is $2m'^2$, so n^2 is even and so is n, a contradiction. Finally, if m is odd and n is even, then m^2 is even, which contradicts its assumption.

Our point of view can be that $\sqrt{2}$ is irreducible to a rational number, in the same way $\sqrt{-1}$ is. For linear induction and its preferred evaluation, or the linear probability evaluation, I am unclear what values, if any, can be assigned.

However, if $\sqrt{2}$ was a rational number plus an infinitesimal, with m, n natural numbers, and k a Eudoxus number, then

$$\sqrt{2} = \frac{m}{n} + k\boldsymbol{\varepsilon},$$

and on squaring

$$2 - \left(\frac{m}{n}\right)^2 = 2k\frac{m}{n}\boldsymbol{\varepsilon} + k^2\boldsymbol{\varepsilon}^2,$$

where the left hand side is a rational number, which contains on the right infinitesimals multiplied by Eudoxus numbers, which is a contradiction.

We have seen that if we represent $\sqrt{2}$ by

$$\sqrt{2} = a + \frac{b}{2} + \frac{c}{2^2} + \cdots + \frac{d}{2^n} + \cdots, \tag{2}$$

then a, b, ... , d can be represented in a consistent allocation (mod 2), that is, a, b, ... , d are all either even (0) or odd (1), and by the previous discussion of the principle of induction, if there is no infinitesimal term to the right of (2) the property of their sum (mod 2) exists.

For ladder numbers, $\mathbf{\Omega} = 0 \pmod 2$ is a consistent type of preferred evaluation, and $\boldsymbol{\varepsilon} = \mathbf{\Omega}^{-1} = 0 \neq 1 \pmod 2$ is also.

Definition 7.4.1. An *algebraic number* $x \in \mathbb{A}$ satisfies a polynomial in x of degree $n \in \mathbb{N}_{\cup 0}$ with Gaussian number (complex integer) coefficients.

Definition 7.4.2. A *transcendental number* is a number with the imaginary part removed which is not algebraic.

Possible transcendental numbers may satisfy polynomial equations of degree $n \notin \mathbb{Z}$, for example of non-finite degree.

With no ladder numbers, the definition of the empty set in mZFC and its properties under induction previously discussed show it is possible for finite rational numbers to have empty infinitesimal sets whilst at the same time infinite sequences of rationals can contain sets of infinitesimals in the complement of $\varnothing$, satisfying the same predicate. We refer to *Innovation in mathematics* [Ad14] for other work showing that representable infinitesimals exist. ☐

The transcendental number π may be represented by

$$\pi = q + \frac{r}{p} + \frac{s}{p^2} + \cdots + \frac{t}{p^n} + \cdots \qquad (3)$$

Since practical evaluations of π are generated by countable algorithms and in no other way, we are justified in representing π by a collection of numbers differing by a Eudoxus number times an infinitesimal

$$\pi^* = q + \frac{r}{p} + \frac{s}{p^2} + \cdots + \frac{t}{p^n} + \cdots + m\boldsymbol{\varepsilon}. \qquad (4)$$

In the same way as it is possible to change the base point of a vector in a vector space, it is possible to change the value of m in (4). For instance, we could have m = 0. For this value of π at m = 0, its evaluation (mod p^n) is possible if and only if we ignore subsequent terms in the series which relative to p^n contain terms $\frac{u}{p^v}$, where the denominator is equivalent to dividing by zero (mod p^n). However, it is possible to evaluate an approximation to π (mod p), and this holds for fractions in p for all primes. The limit of this fraction differs from the selected value of π by at most an infinitesimal. ☐

We can extend this discussion and introduce the hypothesis that $\sqrt{2}$ and π are *real numbers* defined by at most transfinite polynomials. We do this as follows.

Definition 7.4.3. The *unbounded real numbers* $\mathbb{R}$ are complex numbers with no imaginary parts derived from solutions of polynomial equations of degree $m \in \mathbb{M}_{\cup 0}$ with the properties

(i) m obeys the Peano axioms (but both $\mathbb{N}$ and $\mathbb{M}$ satisfy induction)
(ii) $\mathbb{N} \subset \mathbb{M}$
(iii) $\mathbb{M}$ is not bijective to $\mathbb{N}$
(iv) there exists no proper subset with the above properties satisfying $\mathbb{N} \subset \mathbb{M}' \subset \mathbb{M}$.

Further, there exist numbers (1/m) which are smaller than any countable infinitesimal. The preferred evaluation of $\mathbb{M}$ is even. ☐

So it is possible to discuss ‘transfinite rationals’ using $\mathbb{M}$ and similarly ‘transfinite algebraic numbers’. We mention an application. The proof of the general Riemann hypothesis then proceeds as an extension of the case for ‘local function fields’ [Ad18]. ☐

7.5. Norms of complex and intricate numbers.

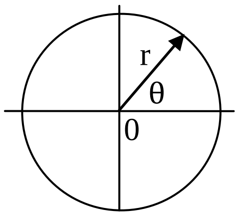
The field of complex numbers is represented by the symbol ℂ. A complex number may be represented as

$$x = a + bi = r(e{\uparrow}i\theta) = re^{i\theta}, \qquad (1)$$

with r real. By a theorem of Euler proved in chapter XV

$$re^{i\theta} = r(\cos\theta + i\sin\theta), \qquad (2)$$

shown in the Argand diagram given next



The absolute value of x, which can be more generally defined in terms of a norm, is

$$||x|| = +\sqrt{a^2 + b^2}. \qquad (3)$$

Then $\cos^2\theta + \sin^2\theta = 1$, and $\cos\theta + i\sin\theta$ can have negative values in the real and imaginary parts. We apply a convention for the representation of these numbers to give a nonnegative value to r

$$||x|| = ||r|| = r. \qquad (4)$$

For another complex number, $x' = a' + b'i = r'e^{i\theta'}$,

$$||x||.\ ||x'|| = ||xx'|| \qquad (5)$$

and

$$e^{i\theta}.e^{i\theta'} = e^{i(\theta + \theta')}. \qquad (6)$$

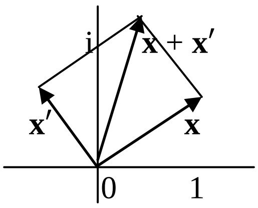
The absolute value satisfies

$$||x|| > 0 \text{ unless } ||x|| = 0, \text{ when } x = 0, \qquad (7)$$

and the 'triangle inequality' for vectors holds

$$||x + x'|| \leq ||x|| + ||x'||, \qquad (8)$$

shown below.



The properties of the absolute value are very similar to the properties of the distance function d(x, y) between two points x and y in a *metric space*:

$$d(x, y) > 0 \text{ unless } d(x, y) = 0, \text{ when } x = y$$
$$d(x, z) \leq d(x, y) + d(y, z),$$

where for complex numbers we have taken the origin, or base point, to be x = 0.

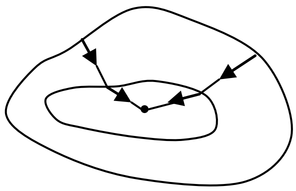
The space, or manifold, of the complex number field has the following special properties:

(1) It is 2 dimensional
(2) Angle is defined
(3) It is locally and globally flat – we will discuss this later
(4) It has an orthogonal distance function defined by the absolute value, or norm. ☐

The norms squared we have so far been considering are nonnegative, as they are for n-dimensional *Banach spaces*, where the norm is Eudoxus. The restriction of the norm squared of an intricate number to the complex case, which in this case is the determinant, is of this form. However, we have seen in chapter I that the intricate norm squared of the intricate number a1 + bi + cα + dϕ is $a^2 + b^2 – c^2 – d^2$, so that for example the norm squared of an actual number is $a^2 – c^2$, which being hyperbolic, can take on positive and negative values, and so does not satisfy directly the properties above of a metric space. We will see in [Ad18] the metric function can be redefined to be positive. A hyperimaginary number is J-abelian, so its norm squared is obtained directly from a positive determinant, but we cannot say this in general for a hyperintricate number. ☐

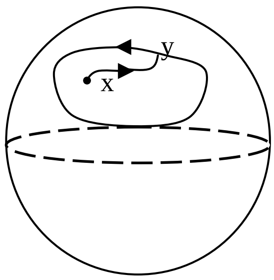
7.6. The idea of winding numbers.

Consider [CS66] a two dimensional surface in one piece, and a loop γ on the surface with no self-intersections. Then, if the loop can be contracted to a point (a point is not a loop), where the contracted set of loops do not intersect, the original loop divides the surface into two pieces, the contracted set and its complement.



contracting a loop to a point

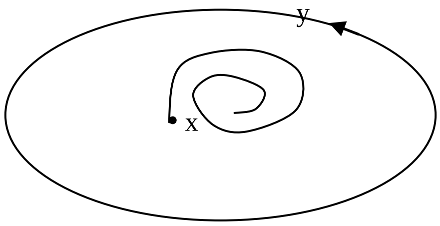
We may choose a point x on the surface but not on the loop, and a continuous line from x to a point y on the loop.



a loop γ on the surface of a sphere –
x is a point and a line to y on γ is shown

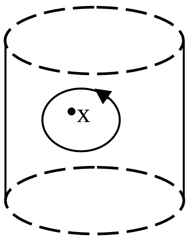
This defines locally a set of vectors at each point of the line going in the direction of x to y. Likewise, we may give the loop an orientation, say as a set of vectors going anticlockwise around the loop.

For a surface and a loop within it, a line going from x may never reach the loop, even when we might think the point x is inside the loop, so we will consider only straight lines from x to y. Straight lines are defined in the complex number field as having a unique local minimum distance measured in the standard way between two points.



part of an infinite spiral
(which could converge to an interior circle)

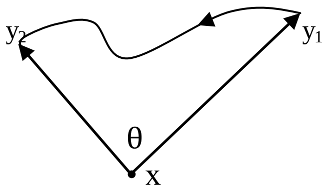
Now consider a complex number field. This is a locally flat surface, because all triangles with straight lines within a contractible loop have interior angles summing to π radians. This also occurs outside the loop.



a cylinder with infinite height –
a locally flat surface that could contain
an uncontractible loop around its girth

The complex number field contains no loops γ which cannot be contracted to a point – this is derived from the fact that it is globally flat.

If along a segment of the loop γ the angle from x to the loop is always increasing (going anticlockwise) in the direction of the arrows along the loop or always decreasing (going clockwise), then we may compose the loop as a number of similar such segments joined together.

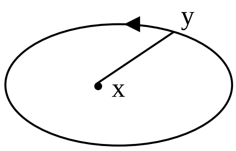


If, going round the loop once the sum of these positive and negative angle differences is 2π radians, we say the *winding number* is

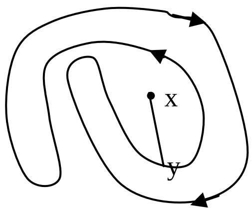
$$w(\gamma) = +1 = \frac{\text{sum of angle differences}}{2\pi}.$$

If the angle differences sum to a clockwise direction for the loop γ, the winding number is -1.

A loop without intersections has an *interior*, which is all points x not on the loop for which the winding number of the entire loop is $w(\gamma) = \pm 1$. Then points not on the loop for which the winding number of the entire loop is 0 are *exterior* to the loop. An infinite straight line is not a loop. We define its winding number not through x to be ±½.



x in interior of loop



x exterior to loop

Because the loop divides the Argand diagram into parts which have separate winding numbers, we can prove

{points in interior} ∪ {points in exterior} ∪ {points on the loop}
= {set of all points in the Argand plane}.

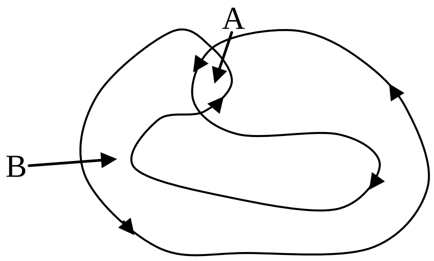
Example 7.6.1. Choose x at the origin. The winding number w(γ) where γ is constant, not passing through the origin, and so is a point always subtending an angle θ, is given by

$$w(\gamma) = 0 = \frac{\theta - \theta}{2\pi}.$$

A loop which self-intersects finitely may be thought of as non-intersecting loops joined at n points, where these points are not continuously connected to themselves. Since they define no angle differences, the winding numbers of these points are zero.

A loop which self-intersects continuously may be thought of as non-intersecting loops joined at separated points, together with segments which intersect infinitely and continuously, where these other points are connected to themselves. Since they define angle differences, the winding numbers of these segments are not necessarily zero.

If the interior of a loop is included in or equals the interior of another loop, and they are both part of a self-intersecting loop, then the winding number for the first loop is defined to be the sum of the winding numbers of the two loops, and so on recursively for more loops interior or equal to others.



A is included in B: $A \subset B$

Winding number in A = +2,
in B but not in A = +1

Example 7.6.2. Choose x at the origin. The winding number $w(\gamma)$, where $\gamma(\theta)$ is $e^{in\theta}$, with n an integer and θ varying from 0 to 2π, so $\gamma(\theta)$ self-intersects continuously, is given by

$$w(\gamma) = n = \frac{2n\pi - 0}{2\pi}.$$

Theorem 7.63. *Choose* x *at the origin. Let* γ_ε *be a continuous family of maps where no* γ_ε *meets the origin, defined by a parameter* $\varepsilon \in [0, 1]$. *For all* ε *the value of* $w(\gamma_\varepsilon)$ *is the same.*

Proof. The winding numbers are well defined, as no γ_ε meets the origin. The value of $w(\gamma_\varepsilon)$ varies continuously with ε. Because it is an integer, it must therefore be constant. ☐

7.7. The fundamental theorem of algebra for complex roots. [BM69], [St04]

Theorem 7.7.1. *Every polynomial* f(x) *of positive degree with complex coefficients has a zero with* x *complex.*

Proof. Let x be complex with $x = re^{i\theta}$ and

$$f(x) = \sum_{k=0}^{n} a_k x^k \qquad (1)$$

with $a_n \neq 0$. If f(x) = 0, this has a zero

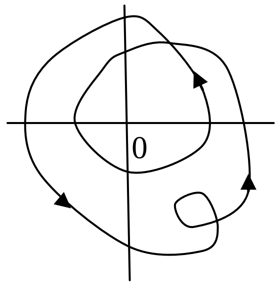
$$\sum_{k=0}^{n} \left(\frac{a_k}{a_n}\right) x^k = 0,$$

so we will only consider the case where the leading coefficient is 1.

Equation (1) is now

$$f(x) = r^n e^{in\theta} \left(1 + \sum_{k=1}^{n} a_{n-k} r^{-k} e^{-ik\theta}\right). \qquad (2)$$

Equation (2) defines a mapping $x \rightarrow f(x)$. For each finite r > 0 the function $f(re^{i\theta})$ defines a closed curve of radius r and centre 0.



If the curve goes through the origin, the theorem is proved. Otherwise consider the integral

$$\psi(r, \theta) = \int_0^{\theta} d\theta$$

around the closed curve, $\psi(r, 2\pi)$, which is $2\pi n$, where n is the number of times the curve winds anticlockwise round the origin. If $a_0 = 0$, then 0 is a root, and we are done. If $a_0 \neq 0$ we will assume that f(x) has no zeros and prove a contradiction.

Let the value of r vary continuously, dependent on a parameter, $\varepsilon \in [0, 1]$, where

$$r = \frac{\varepsilon}{1 - \varepsilon} \text{ for } \varepsilon < 1.$$

and let us choose the ε dependent loop with desirable properties

$$\gamma = \frac{f(re^{i\theta})}{r^n + 1} \text{ for } \varepsilon < 1,$$

and when $\varepsilon = 1$ define

$$\gamma = f(re^{in\theta}).$$

The function $f \rightarrow \gamma$ is continuous, which can be checked for $0 \leq \varepsilon < 1$ and as we now show for $\varepsilon = 1$. Then from equation (2)

$$\frac{f(re^{i\theta})}{r^n + 1} = \frac{r^n}{r^n + 1} e^{in\theta} + \frac{\sum_{k=1}^{n} a_{n-k} r^{n-k} e^{i(n-k)\theta}}{r^n + 1},$$

where the first term on the right tends to $e^{in\theta}$ as $r \rightarrow \infty$ ($\varepsilon \rightarrow 1$), and the second term to zero.

Since at $\varepsilon = 0$, γ is constant, which satisfies the condition of example 7.5.1, so its winding number is zero, and the limit as $\varepsilon \rightarrow 1$ satisfies the condition of example 7.5.2, with winding number n, whereas theorem 7.6.3 for the continuous function $f \rightarrow \gamma$ shows these two winding numbers are identical, this is the required contradiction. □

Theorem 7.7.2. *A polynomial* f(x) *is divisible by* (x – a) *if and only if* f(a) = 0, *where*

$$f(x) = (x - a)g(x)$$

for some polynomial g(x).

Proof. Let

$$f(x) = \sum_{k=0}^{n} c_k x^k,$$

then

$$\begin{aligned} f(x) - f(a) &= \sum_{k=0}^{n} c_k (x^k - a^k) \\ &= \sum_{k=0}^{n} c_k (x - a)(x^{k-1} + x^{k-2}a + \dots + a^{k-1}) \\ &= (x - a)g(x). \end{aligned}$$

Conversely, if f(x) = (x – a)g(x), substituting a for x gives f(x) = 0. □

A polynomial is *reducible* if it can be factored into polynomials of lower degree, otherwise it is irreducible.

Theorem 7.7.3. *The only irreducible polynomials over the complex numbers* ℂ *are linear*.

Proof. By theorem 7.7.1, there exists a complex b_1 such that $f(b_1) = 0$, so by theorem 7.7.2, we can write $f(x) = (x – b_1)r(x)$. If the degree of r(x) is greater than 1, by recursion this also has a complex root. Thus f(x) can be written in the form

$$f(x) = c(x – b_1)(x – b_2) ... (x – b_n). \quad \square \qquad (3)$$

Theorem 7.7.4. (uniqueness of polynomial representations). *Any complex polynomial with complex coefficients is unique up to order of factors in the form* (3), *and additively*.

Proof. This follows multiplicatively from the unique factorisation theorem of chapter III applied to polynomial factorisation of symbol terms instead of prime factorisation and that a product of complex numbers vanishes if and only if one of its factors is zero.

In more detail, the polynomial (x – a) has unique factorisation as itself. Let n factors

$$(x – a_1) ... (x – a_n), \qquad (4)$$

have unique factorisation up to order of these factors. Introduce a further factor

$$(x – a_1) ... (x – a_n)(x – a_{n+1}). \qquad (5)$$

Then, formally, we know what is meant by 'multiplication' of a factor, so we can 'divide' by the factor $(x – a_{n+1})$ to return to the original n factors. From uniqueness of complex division this division is unique, since provided the leading coefficient of x is 1, then $(x – a_{n+1})/(x – d) = 1$ if and only if $a_{n+1} = d$. This is valid for $(x – a_{n+1}) \neq 0$ and is allocated by analytic continuation at $(x – a_{n+1}) = 0$.

If we have case (5) also equal to the product of factors

$$(x – a'_1) ... (x – a'_n)(x – a'_{n+1}), \qquad (6)$$

where at least one $a_i \neq a'_i$ under all orderings, then dividing (5) by $(x – a_{n+1})$ we have value v, and comparing with (6) divided by $(x – a'_{n+1})$, our induction hypothesis says the result is again v. This means that to choose representative values, say

$$(x – a'_n)(x – a_{n+1}) = (x – a_n)(x – a'_{n+1}). \qquad (7)$$

Because x and a_{n+1} are complex numbers the multiplicative form of a product of two roots expressed additively is unique up to order of factors of the roots, that is

$$(x + b)(x + c) = x^2 + (b + c)x + bc = (x + b')(x + c') = x^2 + (b' + c')x + b'c' \qquad (8)$$

implies if the additive form is unique

$$b + c = b' + c' \text{ and } bc = b'c'$$

giving

$$-c^2 + (b' + c')c = b'c',$$

so by the standard solution of the quadratic equation

$$b = b' \text{ and } c = c', \text{ or } b = c' \text{ and } c = b'. \qquad (9)$$

We need to prove uniqueness by an induction procedure also for the polynomial in additive format. We say that a polynomial of degree n is *uniquely expressed* if there is one and only one form of its multiplicative representation up to order of factors and this expression maps identically to one and only one set of coefficients for the polynomial in additive format. We need to show that if the representation of two polynomials is not identical, then both their multiplicative and additive forms differ between one polynomial and the other.

Assume that all polynomials of degree less than n are uniquely expressed. We will prove this for degree n.

Then the permutation of two roots is equivalent to the same additive form for two roots. But an arbitrary permutation of n objects may be obtained as a series of swaps of two objects.

Thus for the final swap of two roots, up to order of factors a multiplication of the remaining roots in additive format times the additive form of the two roots is unique. If the induction assumes the remaining roots are uniquely expressed additively, so this is a bijective mapping between the additive and multiplicative representation of these remaining roots, additive non uniqueness for a general polynomial of degree n implies the coefficient of x^k for some $k < n$ is not uniquely expressed, so its factorisation as a product of multiplicative polynomials up to an order of factors derived from the product of an additive and a quadratic polynomial is not unique, otherwise they are both unique and so is their product, a contradiction. Conversely, by the distributive rule if the multiplicative form is not unique up to order of these factors, nor is its additive form. ☐

7.8. Exercises.

(A) Using the strict transfer principle, show

$$\Sigma_{\text{all } n \in \mathbb{N}} (n) = \tfrac{1}{2}\mathbf{\Omega}_{\mathbb{N}}(\mathbf{\Omega}_{\mathbb{N}} + 1)$$

and

$$\Pi_{\text{all } n \in \mathbb{N}} (n) = \mathbf{\Omega}_{\mathbb{N}}!$$