Superexponential algebra
Volume II

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Jim H. Adams is a researcher in the concepts of mathematics and their explicit representation. This work is a guide to some of these ideas, innovations or developments. It aims to teach creative mathematics rather than mathematical performance. It is also an encouragement to the aspiring mathematician to get involved in mathematical research at an early stage, and continually.

After a career in IT, Jim joined New Music Brighton as a composer and performer of his own works. He has been extensively involved in mathematical research.

The eBooks published in 2014 were *The climate and energy emergencies* and *Innovation in mathematics*. Volumes I and II of *Superexponential algebra* were mainly created throughout 2015 and volume III was completed in early 2017. A mathematical sequel will be *Number, space and logic*.

Keywords: matrices, hyperintricate number, uncountable continuum hypothesis, solvability, ring automorphisms, Galois theory, quintic polynomial, variety, Groebner basis, Zorn's Lemma, probability sheaves.

Foreword

It's mathematics, Jim, but not
as we know it! – Graham Ennis

Superexponential algebra combines a textbook at undergraduate level with a research project. The reader not ready to start a course on mathematics at university could first look at J.H. Conway and R.K. Guy's *The book of numbers,* in reference [1CG00]. Our eBook may be thought of as a commentary and development of the mathematics of the 20th and 21st centuries in the same way as our *Innovation in mathematics* [Ad14] has been for the 19th century. The eBook is in three volumes which cover and generalise all of mathematics. The first part deals with collections of objects called sets, sets with rules for addition and multiplication called rings and further extensions of this idea to 'nonassociative algebras', the second part studies multipolynomials, say the quadratic in two variables $x^2 + 2xy + 3y^2$, and the third tackles mathematical development beyond addition and multiplication to that of exponentiation, then superexponentiation, a complete generalisation beyond exponentiation. All volumes say something about the theory of reasoning called logic. The work *Innovation in mathematics*, completed in 2014, has already covered some aspects of this programme.

It is our programme that geometry in its entirety can be replaced by algebra, and that logic which can be represented by a general theory of sets can also be described by the same algebra. The most general type of algebra we will be considering is superexponential algebra, where higher order operations, say exponentiation, can be found by repetition of lower order operations, say multiplication. The algebra is often noncommutative, so the order of terms is important, and nonassociative, meaning the position of brackets matters. This causes the transformations from algebra back to geometry and logic to share these properties.

Our investigations use detailed algebra, and will be extended to an in-depth investigation of the rules of logic and the structure of spaces. Thus the eBook provides the specifics of what could be expanded by others to cover mathematics in a wide generality, and as far as I can presently think, of the widest generality.

This eBook progresses from the particular to the general. We distinguish two main aspects of our discussion, generalisations of number, which includes but is not limited to the theory of arrays of numbers in rows and columns known as matrices, and repeated operations between these general numbers, in the first case addition and multiplication, then exponentiation and finally the superexponential operations which are the generalisation of these.

The purpose of this eBook has become progressively wider. Its former title, *Hyperintricate matrices*, showed us looking at the theory of matrices in a representation we will introduce called the hyperintricate representation, but we go beyond matrices to cover non matrix algebras, and we study operations beyond those of addition and multiplication, in general covering superexponential operations in a nonstandard way. The sequel is a more advanced work of extended scope, *Number, space and logic* [Ad18].

As a minimum, a course based on the book might include chapters I and II of volume I with 1.9, 1.12, 2.4 and 2.13 omitted. Chapter III contains basic mathematical culture up to, say, 3.7. After that it should be possible to select what you like, but chapter XV of volume III logically comes before chapter XVI and the introduction of chapter XVII and section 17.4 are useful for the sequel [Ad18].

Our results

In setting out, our intention was to probe the correctness of various mathematical results, but not part of the original plan was the need to state that some of our conclusions have become revolutionary. The reader wishing to be presented with opinion is referred to the appendices.

By stages in this eBook we generalise the idea of numbers, and extend and compare the operations on them. We introduce a representation for matrices having two rows and two columns. For these 2×2 matrices this is called the *intricate* representation (otherwise known as the split-quaternions or coquaternions). This matrix algebra contains a subalgebra in an equivalent form to the complex numbers. For $2^n \times 2^n$ matrices a corresponding representation is called the *hyperintricate* representation. This representation is presented more fully than in the eBook *Innovation in mathematics*, [Ad14], although the first two chapters in this eBook are derived from it. The coming together of the two ideas – complex numbers and matrices – allows the development of mathematics from both these subject areas and their interaction.

Galois theory concerns the solution of polynomial equations using groups to describe them. These groups are sets with only one operation, say multiplication. It is distinct from Galois *representation* theory, a topic in multiplicative number theory. Transformations of a group to itself, called group automorphisms, are not the same kind of animal as complex ring automorphisms, with two operations, + and ×. We find ring transformations which swap roots do not in general leave a third root intact, and so cannot be described by permutations. Thus the Galois model is false, which implies the classical theory of Jordan, Hölder, Schreier and E. Artin linking groups to polynomial equations cannot stand. So, for a universal theory we must separate group theory from this theory of polynomials. When dependencies between roots are known, then Galois end results that there are no solutions by radicals of degree greater than four can be violated, but hold under the condition of ‘killing central terms’ of a polynomial equation. A question arises whether there are solutions of greater degree without killing central terms. There are. This issue is discussed in chapter XI and solved in [Ad18].

Innovation in mathematics, without violating the results of K. Gödel or P. J. Cohen, proved that some properties of the real numbers cannot hold. In our work the natural whole numbers are extended to include infinities, and the resulting theory is called nonstandard. We argue against one of the rules of standard set theory, the continuum hypothesis, so all derived sets are countable, implying the same for nonstandard set theory. We then extend number theory to consider an infinite set which cannot be put in correspondence with the natural numbers, otherwise sharing their properties. This set is outside of nonstandard set theory, and we use it to redefine real numbers. Thus we can define ‘transfinite rationals’ and ‘transfinite algebraic numbers’ and prove the Riemann hypothesis for these as an extension of the case for ‘local function fields’. An account is given in *Number, space and logic* [Ad18].

We investigate methods of proof based on ‘diagonal arguments’ appearing in proofs showing the undecidability of some problems in arithmetic, and in Gödel’s incompleteness theorem, which states that a system cannot be proved consistent within itself. For the undecidability result we are interested in the principle of induction for proofs, which indicates how we cover all cases for a proof and find ways to prove a theorem correct. We are in particular interested in procedures known as recursive, defined in the text. We find a function $f(x) = 0$ is recursive, so is $f(x) = 1$, but joining the function $f(x) = 1$ to $f(x) = 0$ is inconsistent and thus not recursive, but ‘diagonal arguments’ ignore this fact and so are erroneous.

Apart from related cases, the mathematical edifice investigated here everywhere remains, but in the process new ways of thinking have been developed, new objects of study and proofs are introduced, and interpretations of standard results have been looked at in new ways.

In Volume I the properties of intricate and hyperintricate numbers are first developed for the operations of addition and a generalised multiplication. A new proof of Wedderburn's little theorem in ring theory is provided. We define alternative types of multiplication for matrices, introduce Lie algebras, representable from matrices A and B by the Lie product AB – BA, and then describe some cases where the objects to be investigated (hyperduplicate numbers and n-novanions) are not representable by matrices. We discuss hyperintricate numbers for finite arithmetic (called congruence arithmetic), in particular the study extended to matrices of an important result in number theory, Fermat's little theorem.

In Volume II we use a modification of the rules, called axioms, of set theory and show that the countability of the rational numbers is not consistent with the uncountable continuum hypothesis, an axiom of set theory, which is explained in chapter VII. We investigate a generalised extension of algebra to include infinitesimal and infinite numbers, which here and in [Ad14] is called ladder algebra. We describe polynomial theory, polynomial equations for duplicate and related roots, where the roots are solutions to a polynomial when it equals zero, and then for matrix roots. We discuss ring automorphisms, which deconstructs the Galois theory of group automorphisms. Polynomial equations in more than one variable, known as varieties, are discussed. We prove the descending unsolvability of polynomial equations of degree greater than the quartic by radicals, a nondescending radical attempt at the sextic, and discuss 'QR' approximation solutions for polynomial equations. Probability sheaves are investigated. These are spaces generalising logic, and contain values between true and false.

In Volume III, for algorithms and logic, we discuss questions of computability, consistency and proof theory. We have seen the Gödel idea gives us no information on noncomputable functions or undecidability. A result of Gentzen states that using infinite proofs, we can prove this consistency. We discuss hyperintricate exponentiation and develop new $\mathbb{D}$w exponential algebras. Finally we study superexponential operations, including those of $\mathbb{D}$w type.

Volume II in more detail

In chapter VII we restate the results of *Innovation in mathematics* [Ad14] on ladder numbers, which use an algebra of infinities and infinitesimals. Zermelo-Fraenkel (ZF) set theory contains the axioms of the continuum hypothesis (CH) and the axiom of choice (AC). The arguments we give on the inconsistency of the real number system constructed from countable Cauchy sequences relative to ZF and uncountable CH can now be restated as their mutual inconsistency under the additional assumption that the rationals are countable. Some finite analogues of proofs originating from the work of Cantor, who first developed set theory, are shown to be false. We introduce the idea of winding number and use it to prove the fundamental theorem of algebra, that complex roots of a polynomial equation always have a zero.

We discuss matrix polynomials. A historical review shows studying discriminants is useful, so we do so. For Galois theory, the abstract Jordan-Hölder and Galois connections theories are omitted as not giving the correct picture for polynomial rings. This heresy will soon be explained.

We document the development of our thought, presenting the theory of complex polynomial equations with independent roots assuming Galois theory describes solvability restrictions, then by appending roots, and for dependent roots. For dependent roots we find there exists a polynomial entity which contains a general complex polynomial, but this entity is solvable.

The theory is extended to noncommutative algebras, essentially for matrix variables, where the Cayley Hamilton theorem for companion matrices always gives solutions for complex polynomials. For independent roots, usual results carry over to the 'J-abelian' case of $2^n \times 2^n$ matrices for hyperintricate polynomials with coefficients all of the same J-abelian type.

Polynomial equations may be written in multiplicative format as a product of roots, or expanded out in additive format. Galois theory is the multiplicative theory of roots, described by group automorphisms. For polynomial rings, we show ring automorphisms are a different kind of animal to those for group automorphisms. We give a concrete model for ring automorphisms, showing that each ring automorphism permutes two roots, in which any such automorphism shifts all roots by the same amount and so does not leave the remaining roots intact. Because ring automorphisms are involutions on pairs of roots not leaving other roots fixed, ring automorphisms are not described by general root permutations, so that the former Galois solvability criterions for normal subgroups now act on commutative elements and cannot be reinterpreted. Neither group nor ring automorphism theory usually describe linear transformations, and the former does not correctly describe the theory of dependent roots. However, a ring automorphism representation is the same as a multiplicative representation, and linear transformations may be applied externally to it.

Chapter XI develops the theory of complex varieties for the extended multidimensional case of polynomials. We investigate the solutions of general complex polynomial equations using linear and polynomial transformation algorithms in the context of varieties, and give a proof of Galois theory end results in this special case when killing central terms, and its extension: the descending unsolvability theorem, which states that polynomials are unsolvable by radicals if they descend through the quintic. We attempt to solve the sextic by radicals using 'comparison methods' which avoid further descent through the quintic. For algorithms we discuss the computational method of QR algorithms for the solution of polynomial equations.

We develop some features of symbolic logic to describe how the axiom of choice, the well-ordering principle and Zorn's lemma are related, and discuss further aspects of rings and ideals, the Hilbert basis theorem, which uses Zorn's lemma, the Hilbert Nullstellensatz, which states that a set of polynomials can be put in a form equivalent to them all being zero or a combination equal to 1, with nothing else possible, and study Gröbner bases, which tells us how to divide polynomials with integer coefficients by others in a consistent way.

The hyperintricate research program intersects with what appears to be some new ideas on probability sheaves (at least on publication – they were available thirty years ago), discussed as topology and logic.

The reader and the author

A central objective of this work is to encourage the mathematical beginner to produce mathematical ideas, and to give the reader a lead by the production of examples. There is the thinking current in much mathematical work that mathematics is about proofs, and the aim of teaching is to prepare the student to construct proofs, or in a more limited way to apply results. But a proof is the final stage in the presentation of a mathematical idea. Proofs are necessary, but not sufficient.

The objective of creative mathematics is the production of ideas. These need caring support, and if the student is to create ideas this also needs long training by doing. Ideas may be formed from many directions, including by study of mathematical masters of the subject throughout the history of mathematics. They may need successive refinement, a transfer from invalid to valid thinking and an investigation of the boundary between them. They are dependent on the understanding of meanings rather than the mere (or difficult) syntactic application of rules of deduction in proofs.

This eBook was put together not without struggle, conceptual, computational and checking for errors. In this I have often started with a result derived intuitively, so that hypotheses have been tested, with the end result sometimes obtained by a repeated process of successive approximation. I have used external references only after failure of my own attempts, and almost always I have proved these external results again by my own methods so that I am satisfied as to their truth.

The work is designed to be understood by those starting mathematical study, whilst at the same time containing ideas of interest to the expert. I have tried at all stages to be clear, and the language used is not an elevated one, because I think sophisticated technical language does not help the beginner. The intrepid A-level student should find the work accessible, but it contains much computation, and usually I have not backed off from computations when they seemed to me to be necessary, or even just to fill out the subject. The end result is that the work could be a challenge, but not one that cannot be overcome, even for the beginner exploring the subject for the first time.

I am aware that the first language of the reader may not be English, so I have looked at the text and put easier words in place of less usual ones.

In preparing this eBook I have decided to avoid 'ordering lunch using categorical language' and support those starting out by minimising its use. Category theory deals with morphisms (an example of a morphism is a transformation, called a mapping or function), where the positions of the brackets do not matter, in other words associative mappings

$$a(bc) = (ab)c,$$

which are not general enough for what we want. Where associativity holds we use a typical example without employing technical categorical language.

An insight when looking at a mathematical work is that if the reader understands *why* a calculation is being made, this will readily give an understanding of the subject, but if he or she restricts knowledge to *what* is being proved, this is more limiting, and the reasons why a particular line of reasoning is used may not be clear. It can be the case that all stages of a proof may follow symbolically correct rules, but at the end of the proof the student may be none the wiser as to the process by which the end result was derived from the assumptions at the beginning, even though she or he may be able to memorise the proof, in effect without understanding it.

It has been my point of view that the reader needs to be helped with insight as to how a result is obtained. This may mean a diversion explaining the history of a result (in an appendix), and why it has seemed reasonable. Nevertheless it is totally impractical to document the many thousands of pages of computation, with its many errors, detours and blind alleys, which have gone into putting this work together. The student is therefore given a brief look at the ideas behind the proofs, so to compute these results himself should not be too difficult.

I have inserted exercises to work through the text, and provided answers at the end of the eBook. One of my minor sins is that I do not solve exercises involving a mechanical study of a text, even though I attempt to solve problems which I think and decide are interesting, either to find a new route to what is already known, or to explore unknown territory. However, I now realise that exercises as normally presented are useful when describing mathematics, so that if the reader wishes, copying ways of working can be successful. This work introduces the reader to the creative process in mathematics and is not intended as a text to develop mathematical performance, so all exercises are optional.

The path of my research interests is unusual and self-directed, so that as well as consulting some of today's mathematicians, most of whom have insights which cannot be found from books, I have the whole body of mathematical literature available to me at the University of Sussex, so that I have found myself in the strange situation of being in a sort of internal dialogue with accomplished mathematicians from the past. In effect my teachers are Euler, Lagrange, Gauss, Galois, Dirichlet, Jacobi, Eisenstein, Riemann, Cayley, Dickson, Burnside, together with many illustrious writers from the twentieth century, some of the present day. Seldom could there be a better mathematical education than this.

I acknowledge the influence of the physicist David Bohm, who in the early 1980's via the physics student Ebrihim Baravi became interested in my investigations in variants of complex exponentiation. It is not generally known that David considered the implications of this for the Riemann hypothesis, and suggested a modification of the complex binomial theorem, which was the same as my own, but took me three decades to return to as the $\mathbb{D}$w exponential algebras. In this work, Bohmian determinism is transferred from physics to mathematics.

I would like to thank Graham Ennis for his ability to listen and his support, Paul Hammond and Raoof Mirzaei for their comments, Roger Fenn for pointing out the history of intricate numbers and a mapping to the symmetries of a square, above all Doly García whose analysis and questioning has led to major improvements in the text, Tim Gibbs, whose stimulating interest has resulted in many significant questions which has made me bold in trying to answer them, Roger Goodwin for sharing his encyclopaedic knowledge of the mathematical literature, and to acknowledge an idea of Daniel Hajas and a calculation of Doly García, which have given rise to significant understandings in the theory of novanions. Thanks are due to Jim Hamilton and Jenny Venton who were effective in correcting some calculations on polynomial comparison methods, causing the temporary removal of statements I was making about them. The contents of this work are my own, as is the responsibility for any errors.

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Brighton & Hove

January 2017

Released on The Assayer free eBook website: 27th October 2015.

Completion of Volume I: 17th January 2016.

Completion of Volume II: 31st December 2016.

Completion of Volume III: 17th January 2017.

No further updates: 31st December 2017.

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Mathematical terms

The following terms, symbols, ideas and definitions are used in the text. The arrangement is by ideas rather than alphabetic. This may be scanned as a more technical alternative to the details of the contents in the Foreword, or as a further introduction to the contents which follow in the eBook.

1. *Sets* S, T (chapter III).
∅ (*the empty set*). The set with no members.
⊙ (*the void set*). The set satisfying a false condition.
∈ (*belongs to*). If x is a member of a set S then x ∈ S.
⊂ (*properly included in*). If a set S is included in a set T and S does not equal T.
⊆ (*included in*). Inclusion, when S = T is possible.
$\complement S_T$ (*complement of* S *in* T). Those x not belonging to S but that belong to T, and S ⊂ T.
⋃ (*union of sets*). If x belongs to S or x belongs to T then x belongs to S ⋃ T.
⋂ (*intersection of sets*). If x ∈ S and x ∈ T then x ∈ S ⋂ T.

2. ℕ is the set {1, 2, 3 ...} of *positive whole numbers*, otherwise called *natural numbers*. If this set contains the element 0, we denote it in this eBook by $\mathbb{N}_{\cup 0}$. If we wish to emphasise that it does not contain zero, we use $\mathbb{N}_{\neq 0}$.
ℤ (from the German Zahl for number) is the set {..., -3, -2, -1, 0, 1, 2, ...} of negative, zero or positive *integers*.
𝕄 is the set of *transfinite natural numbers*, containing whole numbers, the infinite set of which is uncountably infinite.
ℚ is the set of *rational numbers* m/n, where m ∈ ℤ and n ∈ ℕ, for example ½.
ℝ is the set of *unbounded real numbers* of the form ±m/n, where m ∈ $\mathbb{M}_{\cup 0}$ and n ∈ $\mathbb{M}_{\neq 0}$.
𝔸 in this eBook is the set of *algebraic numbers*, sums and differences of *radicals* of the form p^q, where p, q ∈ ℚ, but p and q together are not both zero, for example $1 + 2\sqrt[3]{\frac{-1}{5}}$.
3. *Congruence arithmetic* (*mod n*). Finite or 'clock' arithmetic where whole numbers come back to themselves, so its set is {0, 1, ... (n – 1)} and n = 0.
Prime number. A whole number which only when divided by 1 and itself gives a whole number. Example: 7.
Totient (φ(s)). For a natural number s as a product of primes p, q, ... r to powers j, k, ... m, if $s = (p^j)(q^k)...(r^m)$ then $\varphi(s) = s\left[\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)...\left(1-\frac{1}{r}\right)\right]$. Example: φ(20) = 8.

4. *Abelian*. Occurs for a set with a general operation + (not necessarily addition as usually understood) when a + b = b + a always.
Commutative. Abelian, but generally written for × rather than +.
Associative. Satisfying a + (b + c) = (a + b) + c, or a(bc) = (ab)c, etc.

5. *Eudoxus numbers*, 𝕌. (chapter VII or *Discussion on ladder numbers and zero algebras* in the eBook Innovation in mathematics, [Ad14]). For u ∈ $\mathbb{U}_{\neq 0}$, for every s ∈ $\mathbb{N}_{\neq 0}$, there exist t, v ∈ $\mathbb{N}_{\neq 0}$ so that s > ut > v.

Complex numbers, $\mathbb{C}$. Numbers of the form a + bi, where a, b $\in \mathbb{R}$ and i = $\sqrt{-1}$.
Gaussian integers. Complex numbers where a and b above are integers.
$\mathbb{F}$ is a *field* (chapter III). It contains axioms (rules) for addition and multiplication. Examples could be the real numbers $\mathbb{R}$, Eudoxus numbers $\mathbb{U}$ and complex numbers $\mathbb{C}$.
$\mathbb{Y}$ is a *zero algebra* (chapter III). This is similar to a field except for the existence of multizeros.
Exponential algebra, (chapters XV and XVI). Contains axioms for exponentiation.
$\mathbb{D}$w *exponential algebra*, (chapter XVI). A nonstandard exponential algebra.
Superexponentiation (chapter XVII). An operation, of which the first three are addition, multiplication and exponentiation, where the nth is obtained by repeating the (n – 1)th.
Superexponential algebra. Contains axioms connecting superexponential nth operations for various n.

6. *Implies*. (exercise, chapter XII). For statements A and B, A implies B is only false when A is true and B is false.
Sufficient. A is sufficient for B means A implies B.
Necessary. A is necessary for B means A is implied by B (the same as B implies A).

7. *Function* (chapter III). A set of pairs, {x, f(x)}, all x of which have a value f(x).
Injection. A mapping from all the sets {a, b} to {f(a), f(b)}, where f(a) $\neq$ f(b) if a $\neq$ b.
Surjection. A mapping where every f(x) in the set {f(x)} has a value from an x.
Bijection. A mapping which is simultaneously injective and surjective.

8. *Magma*, M (chapter III). A set with one binary operation, with no other properties specified.
Polymagma (chapter XVII). Maps a number of copies of a set to itself.
Group, G (chapter III). This satisfies the multiplicative axioms for a field, except that multiplication may be noncommutative: ab $\neq$ ba.
Schur multiplier. A multiplicative group on identifying 1 and -1 as one element.
Subgroup. A set of elements in a group which satisfies within itself all the properties of the containing group.
Order of a group. The number of elements (or members) in a group.
Homomorphism of groups is a surjective map h: G $\to$ G′ of groups with h(ab) = h(a)h(b).
Isomorphism of groups. A bijective homomorphism.
Automorphism of a group is an isomorphism of a group to itself.
Inner automorphism of a group is an automorphism of the form x $\leftrightarrow$ a^{-1}xa.
Outer automorphism of a group. An automorphism which is not inner.
Normal subgroup is invariant under all inner automorphisms of the containing group G.
Simple group has no normal subgroups other than itself and 1.

9. *Ring*, A. Satisfies the additive and multiplicative axioms of a field, except there is no general division and multiplication may be noncommutative. Example: matrices.
Unital ring. A ring with a multiplicative identity, 1. We assume rings are unital.
Automorphism of a ring. (chapter X). A bijective map of a ring A, H: A $\leftrightarrow$ A, where H(ab) = H(a)H(b) and H(a + b) = H(a) + H(b).

10. *Matrix* (*plural matrices*). An array of numbers B = b_{jk}, where the element b_{ij} exists in the i^{th} row and j^{th} column. (chapters I and II).
Symmetric matrix. U = $u_{jk} = u_{kj}$.
Antisymmetric matrix. V = $v_{jk} = -v_{kj}$.
Matrix transpose. If W = w_{jk} , then the transpose $W^T = w_{kj}$.
Unit diagonal matrix. Denoted by I = b_{jk}, where $b_{jk} = 1$ when j = k, otherwise $b_{jk} = 0$.
Trace of a matrix. The sum of all (main) diagonal entries b_{jk}, where j = k.
Determinant (*or hypervolume*) *of a matrix*, (*det* B). (chapters I and II).
Singular matrix, D. Satisfies det D = 0.

11. *Intricate number*. A representation of 2 × 2 matrices, that is, with two rows and two columns, given by a1 + bi + cα + dϕ. (chapter I).
Intricate basis element. One of the vectors 1, i, α or ϕ above.
Real basis element. The number 1 in its intricate representation.
Imaginary basis element. The number i in its intricate representation.
Actual basis element. The number α in its intricate representation.
Phantom basis element. The number ϕ in its intricate representation.
Intricate conjugate. The number a1 – bi – cα – dϕ.
J. J = bi + cα + dϕ in which $J^2 = 0$ or ±1.
JAF. A changed basis for i, α and ϕ.

12. *Hyperintricate number*. A representation of $2^n \times 2^n$ matrices. (chapter II).
Layer. For example, a hyperintricate number with a component in 3 layers is $A_{B,C}$ where A, B and C are intricate numbers, possibly intricate basis elements.
n-hyperintricate number. A hyperintricate number representable by sums of components in n layers. Sometimes denoted by Я$_n$.
n-hyperintricate conjugate, Я*$_n$. Satisfies Я*$_n$Я$_n$ = det Я$_n$.
J-abelian hyperintricate number. A number giving the example $A_B + ... + D_E$, where A = p1 + qJ, B = p′1 + q′J′, ..., D = t1 + uJ, E = t′1 + u′J′. Two such hyperintricate numbers with identical J and J′ commute.
Compression. The map κ: $A_B \rightarrow AB$.
Expansion. A map κ^{op}: $AB \rightarrow A_B$.

13. *Vector*, **v** (in bold). A matrix as one row (a row vector), or as one column (a column vector). Example: the row vector (x, y, z).
Scalar product of two vectors. The matrix product of multiplying each element of a row vector in turn with the corresponding elements of a column vector. Example: $x^2 + y^2 + z^2$.
Eigenvector. A vector **x** satisfying for matrix B, B**x** = λ**x**.
Eigenvalue. A value λ for the eigenvector **x** above. Example: λ is a complex root value.
Vector space. Contains vectors with magnitude and direction, which can be added together and multiplied by scalars in a field.
Module. A module over a ring is a generalisation of a vector space over a field, being an additive abelian group like a vector space, where the scalars are the elements of a ring.

14. *Division algebra*. (chapters III and V). A ring with division where multiplication might be nonassociative. Two elements of a division algebra cannot be multiplied giving zero unless one of them is zero.
Quaternions. (chapters III and V). A type of associative division algebra.
Exquaternions. (chapter V). A type of algebra obtained from the quaternions, neither associative nor with complete division.
Octonions, $\mathbb{O}$. (chapter V). A nonassociative division algebra.
Exoctonions. (chapter V) A type of algebra derived from the octonions, neither associative nor with complete division.
n-novanions. (chapter V). An n dimensional nonassociative division algebra, but not when both the real parts in a multiplication are zero.

15. *Norm*. Applied to complex numbers a + bi, the norm is $\sqrt{(a^2 + b^2)}$. For intricate numbers $a1 + bi + c\alpha + d\phi$ the norm squared is $a^2 + b^2 - c^2 - d^2$. Applied to a $n \times n$ matrix B, the norm is the positive nth root of det B. Applied to n-novanions $a1 + bi + c\alpha + d\phi + b'i' + c'\alpha' + d'\phi' + \ldots$, the norm is $\sqrt{(a^2 + b^2 + c^2 + d^2 + b'^2 + c'^2 + d'^2 + \ldots)}$.
Interlayer operator $\veebar_P$. (chapter IV).
Diamond operator $\Diamond$. (chapter IV).
Left roll operator s°. (chapter IV).
Right roll operator $^\circ s$. (chapter IV).
Split product. (chapter IV).

16. *Standard protocol*. (chapter VII). The ordinal infinity $\mathbf{\Omega}_{\mathbb{N}} = \sum_{\text{all } \mathbb{N}} 1$. This is not a natural number, and is treated as being irreducible.
Ladder number. A superexponential expression in $\mathbf{\Omega}_{\mathbb{N}}$, with Eudoxus coefficients.
Strict transfer principle. The axioms for variables in a superexponential algebra also hold for the variable $\mathbf{\Omega}_{\mathbb{N}}$.
Winding number. The number of times a loop winds round a point.

17. *Additive format of a polynomial equation*. The form $ax^n + bx^{n-1} + \ldots + d = 0$. (chapters VII and VIII).
Monic polynomial. Example in the case of a polynomial equation: when a above = 1.
Fundamental theorem of algebra. The complex polynomial in additive format given by $ax^n + bx^{n-1} + \ldots + d$ always has some values which are zero.
Multiplicative format of a polynomial equation. The form $(x - p)(x - q) \ldots (x - t) = 0$.
Zero of a polynomial. A value of a polynomial $f(x) = ax^n + bx^{n-1} + \ldots + d$ so that $f(x) = 0$.
Root of a polynomial equation. The roots of a polynomial equation $f(x) = 0$ are the values of x satisfying this.
Degree of a polynomial. The value of n for f(x).
Duplicate root. A root of the equation $(x + a)^2 = 0$.
Antiduplicate root. A root of the equation $(x + a)(x - a) = 0$.
Independent roots. Occur when no known dependency relation is used in the solution of a polynomial equation.
Dependent roots. Occur when a known dependency relation is used in the solution of a polynomial equation.

Polynomial entity. A polynomial equation with dependent roots.
Multipolynomial. A polynomial in a number of variables.
Variety (from the French variété for manifold). A polynomial equation in a number of variables. Example: $3x^2y + xyz + 4x^2z^2 = 0$.

18. *Equivalence relation* ≡ *in a set S*. Satisfies a ≡ a (*reflexive*), if a ≡ b then b ≡ a (*symmetric*) and if a ≡ b and b ≡ c then a ≡ c (*transitive*), for a, b, c ∈ S.
Equivalence class. A partition of a set where an equivalence relation between elements defines membership of the partition.
Partial order $\leq$ *of a set S*. Satisfies a $\leq$ a, if a $\leq$ b and b $\leq$ a then a = b (*antisymmetric*) and if a $\leq$ b and b $\leq$ c then a $\leq$ c, for a, b, c ∈ S.
Total order $\leq$ *of a set S* is a partial order existing for all a, b, c ∈ S.
Well-ordering $\leq$ *of a set S*. A total order where every nonempty subset has a least element.

19. *Left* (*or right*) *coset* of a subgroup S of G is the set of elements aS (or respectively Sa), with s ∈ S and a ∈ G.
Quotient group G/S *of* G mod S. The family of left cosets of the group G with subgroup S, sG, s ∈ S.

20. *Ideal*, C. (chapters III and XI). A subset of a ring, A. {c, d} ∈ C and a ∈ A implies that (c – d) ∈ C and both ac and ca ∈ C.
Principal ideal, (a). The ideal generated by one element, a, of the ring A. For every r ∈ A, (a) is ra. Example: for a ≠ 0 belonging to the integers $\mathbb{Z}$, (3a) ⊂ (a) ⊂ $\mathbb{Z}$.
Prime ideal, P. If a and b are two elements of A such that their product ab is an element of P, then a or b is in P, and P is not equal to the whole ring A. Example: integers containing all the multiples of a given prime number, together with zero. Example: the zero ideal (0).
Maximal ideal, M. In any ring A, this is an ideal M contained in just two ideals of A, M itself and the entire ring A. Every maximal ideal is prime. Nonexistence: the zero ideal (0) is not a maximal ideal of $\mathbb{Z}$ because (0) ⊂ (2) ⊂ $\mathbb{Z}$, nor is the ideal (6), since (6) ⊂ (2) ⊂ $\mathbb{Z}$.
Nilradical, N(A). The intersection of all prime ideals of a ring.
Jacobson radical, J(A). The intersection of all maximal ideals of a ring.

21. *Monomial* (chapter XI) in a variety is a term without the coefficient. Example: x^4y^2z.
Monomial order of monomials in a variety. For example: $1 < x^4y^2z$ and if $x^4y^2z < x^4y^3z^2$, then $(x^4y^2z)(x^ay^bz^c) < (x^4y^3z^2)(x^ay^bz^c)$.
Lexicographic order $<_{lex}$ of monomials in a variety. The monomial order where the first powers g, h that differ satisfy g < h. Example: $x^4y^gz < x^4y^hz^2$, with g < h.
Degree lexicographic order $<_{deglex}$ of monomials in a variety. For example: $x^ay^bz^c <_{deglex} x^dy^ez^f$ if a + b + c < d + e + f, and if they are equal, then revert to lexicographic order.
Degree reverse lexicographic order $<_{degrevlex}$ of monomials in a variety is defined by $x^ay^bz^c <_{degrevlex} x^dy^ez^f$ if a + b + c < d + e + f, and if they are equal, then revert to right to left lexicographic order with the degrees interchanged. Example: $x^2y^2z^3 <_{degrevlex} x^4yz^2$ so the total degrees are equal, but then order with 3 and 2 in z interchanged.
Gröbner basis. A set consisting of multipolynomial divisors of a multipolynomial with unique remainder.

22. *Open set*. Example: the interval $a < x < b$ with the end points a and b removed.
Closed set. Example: the interval $a \leq x \leq b$ with the end points a and b present.
Topology. A theory of space using open and closed sets.
Homotopy. A theory of paths through a topological space.

23. *Exact sequence*. (chapter III).
Homology. A theory of holes. The dimension of the nth homology is the number of holes in a space for dimension n.

24. *Explanation*. A theory or theorem using matrices or other combinatorial means.
Charade. The image of an explanation in homological algebra.
Deconstruction. The mathematical refutation of a generally accepted result.