

CHAPTER V

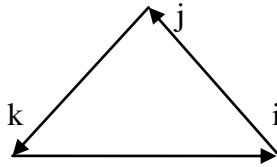
Novanions

5.1. Introduction.

Using homology theory J. F. Adams proved in 1960 that the only possible division algebras with a finite basis are at maximum those of the octonions, which are nonassociative, where quaternions and complex numbers are subalgebras of these. However, I introduce the explicit result that there exist nonassociative algebras, the n-novanions, with nonzero division, but when just the scalar part is zero two nonzero novanions may have a zero product. I obtain here the proof for $n = 10$, and extend it for n greater than 10. Novanions have a dimension of $n = 1 + 3^f \prod_{i \in \mathbb{N}} (3^{g_i} - 2)_i$, for which $f \in \mathbb{N}_{\cup 0}$, $g_i \in \mathbb{N}$. As a consequence, for the n-novanions I have opened a new type of quasi division algebra existence question, with novel results.

5.2. The quaternions and exquaternions.

The quaternions may be represented by a triangle diagram



where the nodes i, j and k satisfy

$$ij = k, jk = i, ki = j, \tag{1}$$

in other words, for a positive sign in the above relations, we are following the arrows. When we are going in a direction opposite to the arrows, we have a negative sign:

$$ji = -k, kj = -i, ik = -j. \tag{2}$$

We have here that 1 commutes with all elements, and also

$$1^2 = 1, i^2 = j^2 = k^2 = -1. \tag{3}$$

Some matrix representations of the quaternions are given in chapter III, section 3.7. We now introduce the nonassociative exquaternions (which thus cannot be represented by a matrix).

The linearly independent intricate basis elements of chapter I, section 1.6 satisfy

$$\begin{aligned} 1^2 &= 1, i^2 = -1, \alpha^2 = 1, \phi^2 = 1, \\ 1i &= i = i1, 1\alpha = \alpha = \alpha 1, 1\phi = \phi = \phi 1, \\ i\alpha &= -\phi = -\alpha i, i\phi = \alpha = -\phi i \text{ and } \alpha\phi = i = -\phi\alpha. \end{aligned}$$

This algebra may be modified so that, for instance, all relations are maintained except for one case of the sign, which is altered to

$$i\alpha = \phi = -\alpha i.$$

If the resulting algebra were associative, then

$$\alpha = i\phi = i(i\alpha) = (i^2)\alpha = -\alpha$$

and

$$-i = \phi\alpha = (i\alpha)\alpha = i\alpha^2 = i,$$

so this *extricate* (in contradistinction to *intricate*) algebra is not associative, and thus is not represented by a matrix.

Indeed, we may adopt the relations

$$i(i\alpha) = -(i^2)\alpha$$

and

$$(i\alpha)\alpha = -i(\alpha^2).$$

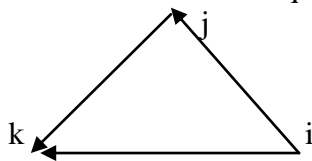
If we apply the quaternion representation of chapter III to the exquaternions, in the sense of an algebra rather than a matrix, then a representation is

$$e_1 = i_1, e_2 = \alpha_i, e_3 = \phi_i,$$

and the exquaternion multiplication table on elements becomes

| \times | I | e_1 | e_2 | e_3 |
|----------|-------|--------|--------|-------|
| I | 1 | e_1 | e_2 | e_3 |
| e_1 | e_1 | -1 | e_3 | e_2 |
| e_2 | e_2 | $-e_3$ | -1 | e_1 |
| e_3 | e_3 | $-e_2$ | $-e_1$ | -1 |

A representative instance of the exquaternions can be depicted in the diagram



where the direction of the arrows indicates the sign in products. For this particular example

$$(j - k)(1 - i) = 0, \tag{4}$$

so that the exquaternions do not form a division algebra. \square

We can combine the intricate and extricate algebras to form a *duplicate* algebra. For this purpose, it is convenient under the extricate algebra to introduce basis elements written as $i_{\#}$ (or $i^{\#}$ in layers) for i , $\alpha_{\#}$ (or $\alpha^{\#}$) for α , and $\phi_{\#}$ or $\phi^{\#}$ for ϕ . The general duplicate number is now written as

$$a1 + bi + c\alpha + d\phi + b'i_{\#} + c'\alpha_{\#} + d'\phi_{\#}.$$

We need a multiplication operation for such numbers. If A and B are intricate numbers, $A_{\#}$ and $B_{\#}$ extricate numbers, define

$$(A + A_{\#}) \times (B + B_{\#}) = (AB) + (A_{\#}B) + (A \times_{\#} B_{\#}) + (A_{\#} \times_{\#} B_{\#}),$$

where AB is expressed under intricate multiplication as an intricate number, $A_{\#} \times_{\#} B_{\#}$ under extricate multiplication $\times_{\#}$ as an extricate number, whereas the $A_{\#}B$ term is taken as an intricate multiplication with the $A_{\#}$ variables transformed to intricate ones, and $A \times_{\#} B_{\#}$ as extricate multiplication, with the A variables transformed to extricate ones.

We may also form the following general nonassociative algebra, for the set

$$\Lambda = \{\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\},$$

where $\Lambda_0, \Lambda_1, \Lambda_2$ and Λ_3 are scalars and

$$(A + A_{\#}) \times_{\Lambda} (B + B_{\#}) = \Lambda_0(AB) + \Lambda_1(A_{\#}B) + \Lambda_2(A \times_{\#} B_{\#}) + \Lambda_3(A_{\#} \times_{\#} B_{\#}).$$

The hyperduplicate algebra is obtained by the same process that a hyperintricate algebra is obtained from an intricate one. In describing nonassociative operations the hyperduplicate algebra needs extension to represent a general multiplication table.

We will see later that there exist nonassociative algebras, which cannot be represented by matrices and are partial division algebras, the novanions. \square

To demonstrate that the product of two quaternions with real coefficients

$$(a1 + bi + cj + dk)(p1 + qi + rj + tk) \quad (5)$$

cannot be zero unless $a = b = c = d = 0$ or $p = q = r = t = 0$, we will adopt a proof ‘the long way round’, which we will simplify and extend later to the nonassociative 10-novonians.

Equating by hypothesis the real and quaternionic parts of (5) to zero, we obtain the following 4 equations in 8 unknowns.

real part:

$$ap - bq - cr - dt = 0 \quad (6)$$

i part:

$$pb + aq + ct - dr = 0 \quad (7)$$

j part:

$$pc + ar - bt + qd = 0 \quad (8)$$

k part:

$$pd + at + br - cq = 0. \quad (9)$$

If $a = 0$, then equations (6), (7) (8) and (9) imply

$$(b^2 + c^2 + d^2)q = 0, \quad (10)$$

$$(b^2 + c^2 + d^2)r = 0 \quad (11)$$

and

$$(b^2 + c^2 + d^2)t = 0, \quad (12)$$

so either $b = c = d = 0$, which we exclude, or q, r and $t = 0$, which implies (5) cannot be satisfied except for $b = c = d = 0$.

On eliminating p , provided $a \neq 0$ we obtain

$$(a^2 + b^2)q + (-ad + cb)r + (ac + db)t = 0, \quad (13)$$

$$(ad + cb)q + (a^2 + c^2)r + (-ab + cd)t = 0 \quad (14)$$

and

$$(-ac + bd)q + (ab + cd)r + (a^2 + d^2)t = 0. \quad (15)$$

On eliminating q ,

$$[(a^2 + c^2)(a^2 + b^2) + (ad + cb)(ad - cb)]r + [(-ab + cd)(a^2 + b^2) - (ad + cb)(ac + db)]t = 0 \quad (16)$$

and

$$[(ab + cd)(a^2 + b^2) - (-ac + db)(-ad + cb)]r + [(a^2 + d^2)(a^2 + b^2) - (-ac + db)(ac + db)]t = 0. \quad (17)$$

Then on eliminating r , whilst putting

$$A = a^2, B = b^2, C = c^2, D = d^2, \quad (18)$$

we obtain provided $p = q = r = t = 0$ does not hold

$$\begin{aligned} & [(A + C)(A + B) + (AD - CB)][(A + D)(A + B) - (-AC + DB)] \\ & - [(A + B)^2(-AB + CD) + (-AD + CB)(-AC + DB)] \\ & + (-ab + cd)(A + B)(-ac + db)(-ad + cb) + (ab + cd)(A + B)(ad + cb)(ac + db) \\ & = 0. \end{aligned} \quad (19)$$

The last line with terms in (19) is

$$2(A + B)[BCD + ACD + ABD + ABC], \quad (20)$$

so we have

$$\begin{aligned} & (A + B)^2A[A + B + C + D] \\ & + (A + B)[A^2D + AD^2 + A^2C + AC^2 + ABC + ACD + ABD] = 0 \end{aligned} \quad (21)$$

which is impossible, since $A > 0$. \square

5.3. The octonions and exoctonions.

The algebra of the nonassociative octonions, is given by the Fano plane in the figure below.

Typical cyclic identities are

$$\begin{aligned} e_7 e_5 &= e_2 \\ e_5 e_6 &= e_1 \end{aligned} \tag{1}$$

and

$$e_4 e_2 = e_6.$$

Note what we have said here. The inner triple $e_2 e_4 e_6$ acts like a quaternion, but the outer triple $e_1 e_3 e_5$ does not. Nevertheless, we will need to allocate later a list ordered as right triple = $e_2 e_4 e_6$ + central triple = $e_1 e_3 e_5$ + one = e_7 . Octonions form a division algebra, in particular

$$e_c^2 = -1, \tag{2}$$

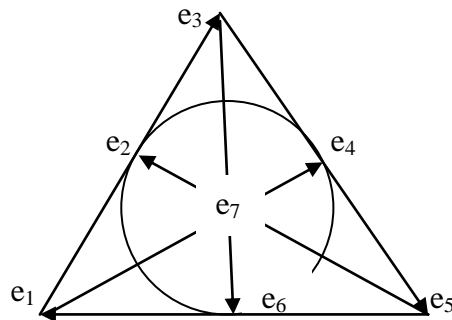
$$e_a e_b = -e_b e_a \quad (a \neq b),$$

and the inverse of

$$a1 + \sum_{n=1}^7 b_n e_n$$

is

$$(a1 - \sum_{n=1}^7 b_n e_n) / (a^2 + \sum_{n=1}^7 b_n^2). \tag{3}$$



The octonions, \mathbb{O} , are also generated by the Cayley-Dickson construction [Ba01]. This builds up algebras from the complex numbers, to the quaternions, to the octonions, to the sixteen dimensional sedenions, etc.

Define a $*$ -algebra to be an algebra equipped with conjugation, a linear map $*$ satisfying

$$a^{**} = a, \tag{4}$$

$$(ab)^* = b^* a^*. \tag{5}$$

Starting from any $*$ -algebra, the Cayley-Dickson construction gives a new algebra

$$(a, b)(c, d) = (ac - db^*, ad^* + cb), \tag{6}$$

with conjugation defined by

$$(a, b)^* = (a^*, -b). \tag{7}$$

This generates the basis element multiplication table $\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$ of the octonions

| \times | 1 | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|----------|-------|--------|--------|--------|--------|--------|--------|--------|
| 1 | 1 | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
| e_1 | e_1 | -1 | e_3 | $-e_2$ | e_5 | $-e_4$ | $-e_7$ | e_6 |
| e_2 | e_2 | $-e_3$ | -1 | e_1 | e_6 | e_7 | $-e_4$ | $-e_5$ |
| e_3 | e_3 | e_2 | $-e_1$ | -1 | e_7 | $-e_6$ | e_5 | $-e_4$ |
| e_4 | e_4 | $-e_5$ | $-e_6$ | $-e_7$ | -1 | e_1 | e_2 | e_3 |
| e_5 | e_5 | e_4 | $-e_7$ | e_6 | $-e_1$ | -1 | $-e_3$ | e_2 |
| e_6 | e_6 | e_7 | e_4 | $-e_5$ | $-e_2$ | e_3 | -1 | $-e_1$ |
| e_7 | e_7 | $-e_6$ | e_5 | e_4 | $-e_3$ | $-e_2$ | e_1 | -1 |

We can generate for each

$$\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}: e_i \times e_j \rightarrow e_k$$

a Cayley-Dickson construction of

$$e_i \times (-e_j) \rightarrow -e_k,$$

so that each of the 7 non-scalar basis elements in a row of the table can be multiplied by -1 to provide $2^7 = 128$ possible Cayley-Dickson constructions.

The Fano plane has 7 non-scalar basis elements. The number of non-scalar quaternionic triplets is 7, each of which, since exquaternions are excluded, operates under a forward or a reversed orientation – again 2^7 possibilities. The following Fano triplets map bijectively to the standard Cayley-Dickson construction for the octonions

$$(e_1, e_2, e_3), (e_3, e_4, e_5), (e_1, e_4, e_6), (e_4, e_6, e_2), (e_1, e_7, e_6), (e_4, e_7, e_3), (e_5, e_7, e_2).$$

Thus distinct possibilities for the Fano plane are isomorphic to distinct instances of the Cayley-Dickson construction. \square

A Fano plane can be devised to form the exoctonions. Representing the exquaternions of section 2 by the 4×4 block A, the exoctonion multiplication table may be represented by the blocks

A B

C D

where A or its transpose A^T may be chosen and D or its transpose D^T . An example of the complex numbers, exquaternions and exoctonions is represented in the following nested table for an instance of the exoctonions.

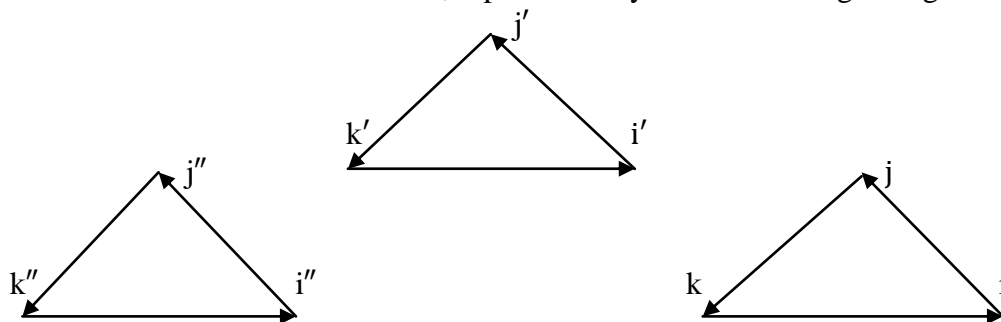
| \times | I | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|----------|-------|--------|--------|--------|--------|--------|--------|--------|
| I | 1 | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
| e_1 | e_1 | -1 | e_3 | e_2 | e_5 | $-e_4$ | $-e_7$ | e_6 |
| e_2 | e_2 | $-e_3$ | -1 | e_1 | e_6 | e_7 | $-e_4$ | $-e_5$ |
| e_3 | e_3 | $-e_2$ | $-e_1$ | -1 | e_7 | $-e_6$ | e_5 | $-e_4$ |
| e_4 | e_4 | $-e_5$ | $-e_6$ | $-e_7$ | -1 | e_1 | e_2 | e_3 |
| e_5 | e_5 | e_4 | $-e_7$ | e_6 | $-e_1$ | -1 | e_3 | e_2 |
| e_6 | e_6 | e_7 | e_4 | $-e_5$ | $-e_2$ | $-e_3$ | -1 | e_1 |
| e_7 | e_7 | $-e_6$ | e_5 | e_4 | $-e_3$ | $-e_2$ | $-e_1$ | -1 |

The exoctonions do not form a division algebra, since they contain the exquaternions as a subalgebra, but the inverse of $a_01 + a_1e_1 + a_2e_2 + \dots$ etc. exists and is

$$(a_01 - a_1e_1 - a_2e_2 - \dots \text{ etc.}) / (a_0^2 + a_1^2 + a_2^2 + \dots \text{ etc.}). \square$$

5.4. The 10-novonions.

We now introduce the 10-novonions, represented by the set of triangle diagrams



where in general each triangle is a quaternion without 1.

The primed variables (), (') and (") act as holders of information concerning an algebra for them. When the variables all contain a common instance, for example (k), (k') and (k"), then the algebra is that of the quaternions, in which we have a cyclic algebra

$$kk' = k'' = -k'k. \quad (1)$$

When the variables contain different instances, such as k and i', then the product contains the primed variable that does not belong to the first two elements, but the primed part commutes.

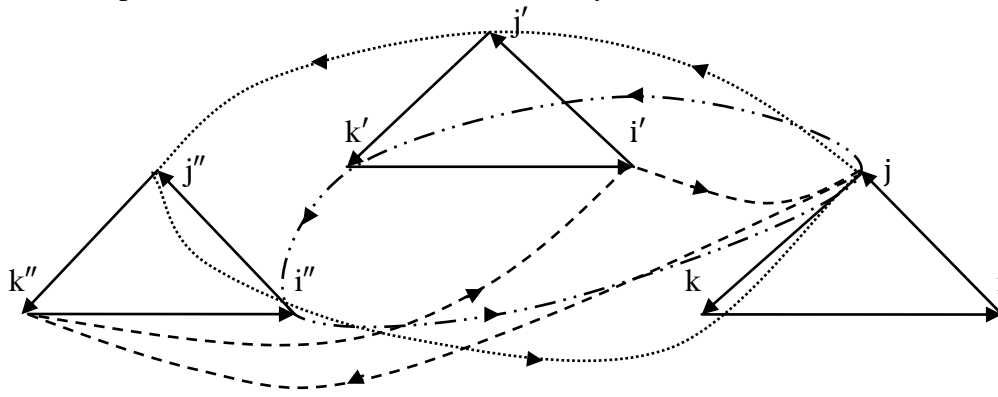
On top of this is the fact that the variables, say k and i, satisfy a quaternion algebra, so say

$$ki = j = -ik \quad (2)$$

and consequently

$$ki' = j'' = -i'k. \quad (3)$$

In order to picture the 10-novanions more closely, we will show the connections from node j



Our claim is that the inverse of

$$a1 + \sum_{n=1}^3 \sum_{\text{primed } m=1}^3 b_n^m e_n$$

is

$$(a1 - (\sum_{n=1}^3 \sum_{\text{primed } m=1}^3 b_n^m e_n)) / (a^2 + (\sum_{n=1}^3 \sum_{\text{primed } m=1}^3 (b_n^m)^2)), \quad (4)$$

and this constitutes a type of division algebra with no divisors of zero provided $a1 \neq 0$ – the 10 dimensional 10-novanions.

We see that the n-novanions are nonassociative, since they have more than 4 basis elements; more explicitly

$$(j''k')j' = ij' = k'' \neq j''(k'j') = -j''i' = k.$$

We wish to enquire under what conditions there exist two 10-novanion numbers multiplied together giving zero:

$$(a1 + bi + cj + dk + b'i' + c'j' + d'k' + b''i'' + c''j'' + d''k'') \times (p1 + qi + rj + tk + q'i' + r'j' + t'k' + q''i'' + r''j'' + t''k'') = 0. \quad (5)$$

Their product is

real part:

$$ap - bq - cr - dt - b'q' - c'r' - d't' - b''q'' - c''r'' - d''t'' = 0, \quad (6)$$

i part:

$$bp + aq - dr + ct - b''q' - d''r' + c''t' + b'q'' - d'r'' + c't'' = 0, \quad (7)$$

j part:

$$cp + dq + ar - bt + d''q' - c''r' - b''t' + d'q'' + c'r'' - b't'' = 0, \quad (8)$$

k part:

$$dp - cq + br + at - c''q' + b''r' - d''t' - c'q'' + b'r'' + d't'' = 0, \quad (9)$$

i' part:

$$b'p + b''q - d''r + c''t + aq' - d'r' + c't' - bq'' - dr'' + ct'' = 0, \quad (10)$$

j' part:

$$c'p + d''q + c''r - b''t + d'q' + ar' - b't' + dq'' - cr'' - bt'' = 0, \quad (11)$$

k' part:

$$d'p - c''q + b''r + d''t - c'q' + b'r' + at' - cq'' + br'' - dt'' = 0, \quad (12)$$

i'' part:

$$b''p - b'q - d'r + c't + bq' - dr' + ct' + aq'' - d''r'' + c''t'' = 0, \quad (13)$$

j'' part:

$$c''p + d'q - c'r - b't + dq' + cr' - bt' + d''q'' + ar'' - b''t'' = 0, \quad (14)$$

k'' part:

$$d''p - c'q + b'r - d't - cq' + br' + dt' - c''q'' + b''r'' + at'' = 0. \quad (15)$$

D is a possibly nonassociative division algebra if for any element a in D and any nonzero element b in D there exists just one element x in D with a = bx and only one element y in D with a = yb.

If a = 0, the 10-novanions contain possibilities for two nonzero 10-novanions giving a product which is zero. We give an example due to Doly García, showing that the 10-novanions satisfying a = 0 do not form a division algebra of standard type

$$(i + i' + i'')(j + j' - 2j'') = 0. \quad (16)$$

Thus the 10-novanions are not a division algebra, since for an arbitrary real number g

$$(i + i' + i'')[j + j' - 2j'']g = 0. \quad (17)$$

From now on we will assume a ≠ 0. By a symmetrical argument applied also to the theorem which follows, we need to assume with this that p ≠ 0.

Equations (6) to (15) form a matrix E + aI, where E is an antisymmetric matrix and I is the unit diagonal, multiplied on the right by the eigenvector (p, q, r, t, q', r', t', q'', r'', t''). Below we give a proof that the eigenvalues of a real antisymmetric matrix are entirely imaginary, provided in chapter 11 of [Uh01], so these correspond to -a, which is real, whereas we are now excluding the only possibility for this, a = 0. □

Within the field of complex numbers ℂ, the complex conjugate of c = a + ib is c* = a - ib. For the corresponding matrix C with entries c_{jk} = a_{jk} + ib_{jk}, the conjugate C* = a_{jk} - ib_{jk}. The transpose of a matrix C is denoted by C^T and has entries c_{kj}. The transpose is a contravariant (order reversing) operation:

$$(CD)^T = D^T C^T.$$

A matrix is defined as antisymmetric if C^T = -C.

Theorem: All eigenvalues of a real antisymmetric matrix E = -E^T are pure imaginary.

Proof. Consider the case of the eigenvalue λ and possibly complex eigenvector x ≠ 0. According to the Fundamental Theorem of Algebra λ ∈ ℂ. Hence

$$Ex = λx. \quad (18)$$

If we take the complex conjugate of both sides of the eigenvalue-eigenvector equation (18), we obtain

$$(Ex)^* = (λx)^* = λ^*x^*.$$

Transposing yields

$$(\mathbf{E}\mathbf{x})^{*\text{T}} = \mathbf{x}^{*\text{T}}\mathbf{E}^{*\text{T}} = \lambda^*\mathbf{x}^{*\text{T}}.$$

Define the norm

$$\|\mathbf{E}\mathbf{x}\|^2 = (\lambda^*)\lambda(\mathbf{x}^{*\text{T}})\mathbf{x}, \quad (19)$$

where

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^{\text{T}}\mathbf{x}} \in \mathbb{R}.$$

Since $\mathbf{E}^{\text{T}} = -\mathbf{E}$ and $\mathbf{E}^{*\text{T}} = \mathbf{E}^{\text{T}}$ for a real antisymmetric matrix \mathbf{E} , we can write (19) as

$$\begin{aligned} \|\mathbf{E}\mathbf{x}\|^2 &= \mathbf{x}^{*\text{T}}\mathbf{E}^{*\text{T}}\mathbf{E}^{\text{T}}\mathbf{x}, \\ &= \mathbf{x}^{\text{T}}\mathbf{E}^{\text{T}}\mathbf{E}\mathbf{x} \\ &= -\mathbf{x}^{\text{T}}\mathbf{E}^2\mathbf{x} \\ &= -\mathbf{x}^{\text{T}}\lambda^2\mathbf{x}, \end{aligned}$$

because $\mathbf{E}^2\mathbf{x} = \mathbf{E}(\mathbf{E}\mathbf{x}) = \mathbf{E}(\lambda\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, so

$$\|\mathbf{E}\mathbf{x}\|^2 = -\lambda^2\mathbf{x}^{\text{T}}\mathbf{x}. \quad (20)$$

Now $\mathbf{x}^{\text{T}}\mathbf{x} \neq 0$, and thus $\lambda^*\lambda = -\lambda^2$, by comparing (19) and (20). Thus a real antisymmetric matrix $\mathbf{E} = -\mathbf{E}^{\text{T}}$ can only have imaginary eigenvalues λ . \square

5.5. n-novaniums.

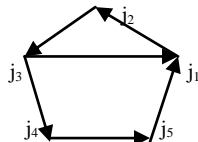
The octonions given by a Fano plane under suitable orientations may be given three copies in primed variables $()$, $()'$ and $()''$, and by an analogous procedure this constitutes a $1 + (3 \times 7) = 22$ dimensional division algebra. Further details will be given in sections 5.6 and 5.7.

Extending these ideas further to multiple occurrences of the three or seven primed variables, say $()$, $()'$ and $()''$, we obtain in general an $n = 1 + 3^{f7^g}$ dimensional novanium algebra, the n-novaniums, in which if a common variable, k , is employed, then the lowest value within brackets of say (k) , $(k)'$ and $(k)''$ is evaluated. \square

Note that, just as we introduced the exquaternions, in a perfectly analogous fashion we can assemble exnovaniums. \square

5.6. The search for other novanium algebras.

Are there other novanium algebras of a type not already covered? This question has been stimulated by a first-year Sussex University student's identification of novaniums with strings in physics, [Ad17], for which we wish to investigate the bosonic allocation $1 + 5^2$. In the pentagonal diagram shown next, an initial attempt depicts only one out of five subtriangles.



The pentagon can be enumerated cyclically, so that

$$j_1j_2 = j_3, j_2j_3 = j_4, j_3j_4 = j_5, j_4j_5 = j_1, j_5j_1 = j_2, \quad (1)$$

and jumping a vertex we evaluate the closest triangle

$$j_3j_1 = j_2, j_4j_2 = j_3, j_5j_3 = j_4, j_1j_4 = j_5, j_2j_5 = j_1, \quad (2)$$

where on inverting the orientation, we get a minus sign.

This latter fact implies we have an inbuilt norm and inverse; the inverse of

$$a_1 + \sum_{n=1}^5 b_n j_n$$

is

$$a1 - \sum_{n=1}^5 b_n j_n / (a^2 + \sum_{n=1}^5 b_n^2), \quad (3)$$

which is nonassociative, as is demonstrated by

$$(j_3 j_1) j_4 = -j_3 \neq j_3 (j_1 j_4) = -j_4.$$

The question arises as to whether this constitutes a novanion algebra, which would now be extended from previous considerations to include the dimensions

$$n = 1 + 3^f 5^g 7^h.$$

The possibility of the existence of the division algebra violating equation

$$(a1 + b j_1 + c j_2 + d j_3 + e j_4 + f j_5) \times (p1 + q j_1 + r j_2 + t j_3 + u j_4 + v j_5) = 0 \quad (4)$$

will now be investigated. Under the constraints (1) and (2) we obtain the set of equations

real part:

$$ap - bq - cr - dt - eu - fv = 0, \quad (5)$$

j₁ part:

$$bp + aq - fr + 0 - fu + (c + e)v = 0, \quad (6)$$

j₂ part:

$$cp + (f + d)q + ar - bt + 0 - bv = 0, \quad (7)$$

j₃ part:

$$dp - cq + (b + e)r + at - cu + 0 = 0, \quad (8)$$

j₄ part:

$$ep + 0 - dr + (c + f)t + au - dv = 0, \quad (9)$$

j₅ part:

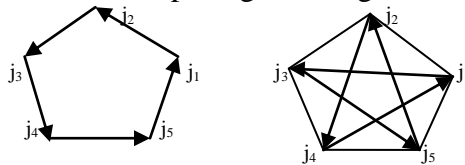
$$fp - eq + 0 - et + (d + b)u + av = 0, \quad (10)$$

from which it follows that the E type matrix is not antisymmetric, but it may be represented as the sum of two matrices F and G, where F has all pure imaginary eigenvalues:

$$F = \begin{bmatrix} 0 & -b & -c & -d & -e & -f \\ b & 0 & -f & c & -f & e \\ c & f & 0 & -b & d & -b \\ d & -c & b & 0 & -c & e \\ e & f & -d & c & 0 & -d \\ f & -e & b & -e & d & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & c \\ 0 & d & 0 & 0 & -d & 0 \\ 0 & 0 & e & 0 & 0 & -e \\ 0 & -f & 0 & f & 0 & 0 \\ 0 & 0 & -b & 0 & b & 0 \end{bmatrix},$$

and we do not have pure imaginary eigenvalues for $F + G - \lambda I$. \square

We find a similar type of situation for the pentagonal diagrams



and we have failed by introducing modifications of equations (5) to (10) to come up with other solutions, where for pentagonal diagrams an exhaustive analysis shows these do not exist. \square

However, the identification relates to the number $1 + 25$, and we now wish to probe the allocation $25 = 7 + 9 + 9$, where 7 is the number of non-real basis elements of the octonions, and 9 is the number for the 10-novanions. There is an analogy here. The octonion non-real basis elements of 7 may be represented as $1 + 3 + 3$, where 3 is the number of such basis

elements for the quaternions, and 1 for the complex numbers. We are forced for a number of reasons to decompose such an allocation into triplets, basically to retain the cyclic algebra for the quaternions.

The allocation will be as follows, where we subscript 3 and 1 to distinguish them

$$3_a, 3_b, 3_c \tag{i}$$

$$3_d, 3_e, 3_f \tag{ii}$$

$$3_g, 3_h, 1_u \tag{iii}$$

where allocations (i) and (ii) are internally similar to 10-novonions, and allocation (iii) is internally an octonion. We will explain why we use the word ‘similar’ later.

There are a number of possible configurations.

We want an algebra linking between (i), (ii) and (iii). Vertical allocations are present. We will choose next from straight lines going from left to right, for example the diagonal going upwards from $3_g, 3_e$ to 3_c . This is similar to a 10-novonion algebra. The descending line from $3_a, 3_e$ to 1_u is an octonion algebra. We then incorporate the algebra taking for example $3_d, 3_b$ to 1_u , an octonion algebra, or $3_g, 3_b$ to 3_f , this is similar to a 10-novonion algebra.

We have used the words ‘similar to a 10-novonion algebra’, and we now explain why. If we look at allocation (iii), this is part of the $3_g 3_h 1_u$ octonion, where 3_h and 1_u are linked. Although 3_g is indeed a quaternion, we have already mentioned that 3_h is not. Therefore the vertical allocation given by $3_a 3_d 3_g$ is a 10-novonion, since it is made of genuine quaternions, but the vertical allocation $3_b 3_e 3_h$ is not. $3_b, 3_e$ and 3_h occur in octonion representations. If we were to state that the central triple $3_b, 3_e$ and 3_h algebras were quaternions, we would have an inconsistency. Therefore for these allocations as part of a ‘similar to 10-novonion’ structure, we decide that the octonion structure overrides the 10-novonion one. Since there is only one special 1_u part for the octonions, this part of the allocation is unique. The similar 10-novonion structure is now not a closed algebra within the 10-novonions; part of it belongs to the octonions. The corresponding situation just for 10-novonions with no octonionic overlap but with novonionic overlap is described by the octonionic allocation already discussed.

Since there is no other mixing of allocations, the result is as consistent as the 10-novonions and the octonions. This can be checked with equations like 5.4.(6) to (15), for which it is clear eigenvalues are pure imaginary. Finally a calculation like 5.4.(16) shows that this is a novonion algebra. \square

The existence of 10-, 26- and 80-novonions (the latter obtained by an array cube of items like (i) to (iii) – all configurations lie in planes of the cube, and an m -cube gives rise to a $(3^m \pm 1)$ -novonion) implies that results derived for division algebras have a different extension for n -novonions. \square

5.7. Sedenions and 64-novonions.

The 16-dimensional sedenions are formed by the Cayley-Dickson construction [Ba01]. Since they are not alternative, they do not form a division algebra. That is, we do not have

$$x(xy) = (xx)y$$

and

$$(yx)x = y(xx)$$

for all x and y in the algebra, the proof using basis elements. Every associative algebra is alternative, but so too are some strictly non-associative algebras such as the octonions.

For the example table that follows

$$(e_1 + e_{10})(e_{15} - e_4) = 0. \quad \square$$

The standard complex numbers, quaternions, octonions and sedenions can be nested by inclusion and an instance is represented in the following table for the sedenions

| \times | 1 | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 | e_8 | e_9 | e_{10} | e_{11} | e_{12} | e_{13} | e_{14} | e_{15} |
|----------|----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1 | 1 | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 | e_8 | e_9 | e_{10} | e_{11} | e_{12} | e_{13} | e_{14} | e_{15} |
| e_1 | e_1 | -1 | e_3 | $-e_2$ | e_5 | $-e_4$ | $-e_7$ | e_6 | e_9 | $-e_8$ | $-e_{11}$ | e_{10} | $-e_{13}$ | e_{12} | e_{15} | $-e_{14}$ |
| e_2 | e_2 | $-e_3$ | -1 | e_1 | e_6 | e_7 | $-e_4$ | $-e_5$ | e_{10} | e_{11} | $-e_8$ | $-e_9$ | $-e_{14}$ | $-e_{15}$ | e_{12} | e_{13} |
| e_3 | e_3 | e_2 | $-e_1$ | -1 | e_7 | $-e_6$ | e_5 | $-e_4$ | e_{11} | $-e_{10}$ | e_9 | $-e_8$ | $-e_{15}$ | e_{14} | $-e_{13}$ | e_{12} |
| e_4 | e_4 | $-e_5$ | $-e_6$ | $-e_7$ | -1 | e_1 | e_2 | e_3 | e_{12} | e_{13} | e_{14} | e_{15} | $-e_8$ | $-e_9$ | $-e_{10}$ | $-e_{11}$ |
| e_5 | e_5 | e_4 | $-e_7$ | e_6 | $-e_1$ | -1 | $-e_3$ | e_2 | e_{13} | $-e_{12}$ | e_{15} | $-e_{14}$ | e_9 | $-e_8$ | e_{11} | $-e_{10}$ |
| e_6 | e_6 | e_7 | e_4 | $-e_5$ | $-e_2$ | e_3 | -1 | $-e_1$ | e_{14} | $-e_{15}$ | $-e_{12}$ | e_{13} | e_{10} | $-e_{11}$ | $-e_8$ | e_9 |
| e_7 | e_7 | $-e_6$ | e_5 | e_4 | $-e_3$ | $-e_2$ | e_1 | -1 | e_{15} | e_{14} | $-e_{13}$ | $-e_{12}$ | e_{11} | e_{10} | $-e_9$ | $-e_8$ |
| e_8 | e_8 | $-e_9$ | $-e_{10}$ | $-e_{11}$ | $-e_{12}$ | $-e_{13}$ | $-e_{14}$ | $-e_{15}$ | -1 | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
| e_9 | e_9 | e_8 | $-e_{11}$ | e_{10} | $-e_{13}$ | e_{12} | e_{15} | $-e_{14}$ | $-e_1$ | -1 | $-e_3$ | e_2 | $-e_5$ | e_4 | e_7 | $-e_6$ |
| e_{10} | e_{10} | e_{11} | e_8 | $-e_9$ | $-e_{14}$ | $-e_{15}$ | e_{12} | e_{13} | $-e_2$ | e_3 | -1 | $-e_1$ | $-e_6$ | $-e_7$ | e_4 | e_5 |
| e_{11} | e_{11} | $-e_{10}$ | e_9 | e_8 | $-e_{15}$ | e_{14} | $-e_{13}$ | e_{12} | $-e_3$ | $-e_2$ | e_1 | -1 | $-e_7$ | e_6 | $-e_5$ | e_4 |
| e_{12} | e_{12} | e_{13} | e_{14} | e_{15} | e_8 | $-e_9$ | $-e_{10}$ | $-e_{11}$ | $-e_4$ | e_5 | e_6 | e_7 | -1 | $-e_1$ | $-e_2$ | $-e_3$ |
| e_{13} | e_{13} | $-e_{12}$ | e_{15} | $-e_{14}$ | e_9 | e_8 | e_{11} | $-e_{10}$ | $-e_5$ | $-e_4$ | e_7 | $-e_6$ | e_1 | -1 | e_3 | $-e_2$ |
| e_{14} | e_{14} | $-e_{15}$ | $-e_{12}$ | e_{13} | e_{10} | $-e_{11}$ | e_8 | e_9 | $-e_6$ | $-e_7$ | $-e_4$ | e_5 | e_2 | $-e_3$ | -1 | e_1 |
| e_{15} | e_{15} | e_{14} | $-e_{13}$ | $-e_{12}$ | e_{11} | e_{10} | $-e_9$ | e_8 | $-e_7$ | e_6 | $-e_5$ | $-e_4$ | e_3 | e_2 | $-e_1$ | -1 |

Whether there exist other 16-dimensional algebras not generated by the Cayley-Dickson construction was answered in the negative in 1960. If we introduced the following diagram

$$\begin{array}{c} 3_a, 3_b, 1_s \\ 1_t \\ 3_c, 3_d, 1_u \end{array}$$

where octonions $3_a, 3_b$ give $\pm 1_t$ and $3_c, 3_d$ also gives $\pm 1_t$, allocation $1_s, 1_t, 1_u$ is a quaternion, then we have for octonions $3_a, 3_c$ gives $\pm 1_s$; $3_b, 3_d$ gives $\pm 1_s$, then 3_c and 3_b is matched with $\pm 1_u$ and 3_a and 3_d with $\pm 1_u$, then this configuration is inconsistent in all configurations.

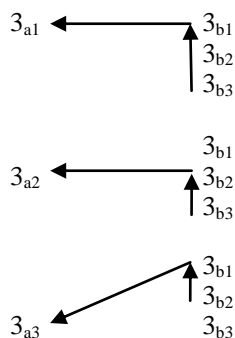
Irrespective of this result, the Cayley-Dickson construction generating a 64-dimensional algebra shows that this is not a division algebra, since in particular this contains the sedenions as a subalgebra. However, a 64-novonion has $63 = 3^2 \times 7$ non-scalar basis elements, and we will see that novonion algebras of this type are consistent. A 64-novonion is given by the cube with slices

$$\begin{array}{ccc} 3_a, 3_b, 1_p & 3'_a, 3'_b, 1'_p & 3''_a, 3''_b, 1''_p \\ 3_c, 3_d, 1_q & 3'_c, 3'_d, 1'_q & 3''_c, 3''_d, 1''_q \\ 3_e, 3_f, 1_r & 3'_e, 3'_f, 1'_r & 3''_e, 3''_f, 1''_r \end{array}$$

To evaluate a typical slice algebra, if we take the leftmost array above, we know that 3_b is not a quaternion, so we will build an override structure for the composition of two elements in 3_b . Within this slice 3_b belongs to three octonionic arrangements, those given by $3_a, 3_b, 1_p$, or $3_c, 3_b, 1_r$, or $3_e, 3_b, 1_q$, so we need to select an override on the nonquaternion 3_b , so that when two elements are multiplied within it, just one allocation to an octonionic structure is selected.

We will need to look at this typical example in detail, so denote the three elements of 3_b by $3_{b1}, 3_{b2}$ and 3_{b3} . We will display the 3 elements of 3_b combining in pairs to form arrows with the following typical structures. We will choose at first, arbitrarily, a link to the $3_a, 3_b, 1_p$

octonionic structure. Of course, two arrows shown below combine to give an oriented quaternion triple, for which reversal of arrows leads to a minus value.



The central triples 3_d and 3_f have similar structures, mapping to separate values in 3_c and 3_e respectively. We have stated the 1_p , 1_q and 1_r elements combined with 3_b give on composition with one element of 3_b the octonion structures $(3_a, 3_b, 1_p)$, $(3_c, 3_b, 1_r)$ and $(3_e, 3_b, 1_q)$. Because 3_{b1} links to 3_{a1} , we have to ensure that the link 1_p to 3_{b1} does not also link to 3_{a1} , but this can be arranged.

Alternative structures can be considered. For example, if 3_{b3} , 3_{b1} links to 3_{a1} as before, we could also have 3_{b3} , 3_{b2} linking to 3_{c1} and 3_{b2} , 3_{b1} linking to 3_{e1} . \square

5.8. Further investigations.

This conclusion is related to questions similar to those about division algebras in homotopy theory, and we will investigate connections with the following circumstances

- (i) The homotopy group for a sphere.
- (ii) Steenrod square operations.
- (iii) Parallelisation of vector fields on spheres.
- (iv) Fiber bundle structures.
- (v) $K(X)$ as a universal description for objects described by semigroups.
- (vi) Bott periodicity.
- (vii) Universal Quillen homotopy.

An investigation of issues (i) to (vii) above will be given in *Number, space and logic* [Ad18].

The exceptional Lie algebras G_2 , F_4 , E_6 , E_7 and E_8 are related to the existence of division algebras limited in number to those embedded within the octonions [Wi09], [CSM95].

From section 3, for the octonions the only nonquaternionic triple is $e_1e_3e_5$ and

$$\begin{aligned} [[e_1 e_3] e_5] + [[e_3 e_5] e_1] + [[e_5 e_1] e_3] &= 2[e_2 e_5] + 2[e_4 e_1] + 2[e_6 e_3] \\ &= -12e_7. \end{aligned}$$

However, if we do not assume, say $[e_1 e_3] = e_1e_3 - e_3e_1 = 2e_2$, and we redefine

$$\begin{aligned} [e_1 e_3] &= ae_2, \\ [e_3 e_5] &= be_4, \\ [e_5 e_1] &= ce_6, \end{aligned}$$

then

$$a + b + c = 0$$

means the Jacobi identity is satisfied.

The 10-novations we have considered contain the quaternions as a subalgebra. Even when an override structure is imposed, its three elements constitute a quaternion. For the octonion structures we have considered, these also contain a quaternion, and this still applies if a nonstandard override is applied. Thus the n-novations all contain quaternion subalgebras.

The implications of the existence of algebras of novation type for the classification of Lie algebras and of groups could be addressed.

5.9. Exercises.

(A) This exercise is computationally exhausting on paper. An equation-solver might be more efficient. In equations (6) to (15) of section 4, put $a = p = 0$, $b = b$, $c = d = b' = 1$, $c' = d' = b'' = c'' = d'' = 0$. Including the value of b , what is the solution of equation (5)?