CHAPTER IV

Nonassociative algebras derived from matrices

4.1. Introduction.

We develop further the concept of a matrix product, also introducing Lie and Kac-Moody algebras. Let A and B be hyperintricate matrices. The hyperintricate methodology is firstly redefined by the introduction for a permutation P of the hyperintricate interlayer operator $\veebar_P$, which permutes layers on the right in the matrix product A $\veebar_P$ B.

Next we introduce operations in the intricate case which in combination allow each element of A and B in these products to be expressed uniquely in the given variables as bilinear expressions. These operations, the diamond operator ◊, the right roll operator $°_S$ and the left roll operator $_S°$, are in general nonassociative. Their algebras are expressed in the intricate formalism, and this is extended to the hyperintricate case by the $\veebar_P$ operator.

We develop the interrelationship between these ideas, by discussing combined and other operations, and discuss split products which reduce these operations to more primitive ones.

4.2. The hyperintricate interlayer operator, $\veebar_P$.

Let P be a permutation on n objects, which can be represented in permutation cycle notation, as in the example

(1 2 3)(4 5)(6 7 8 ... n).

Let $Я_n$ and $Ч_n$ be two n-hyperintricate numbers. The usual matrix product for layer t, where $1 \leq t \leq n$ corresponds to multiplying for each t the intricate number r_t, which is at layer t for $Я_n$, and h_t which is also at intricate layer t for $Ч_n$.

For $Я_n$ and $Ч_n$, define the operation $Я_n \veebar_P Ч_n$ as the matrix operation which corresponds on layer t to the intricate product of r_t with $h_{P(t)}$. Then for two permutations P1 and P2 involving a third hyperintricate number $И_n$

$$Я_n \veebar_{P1} (Ч_n \veebar_{P2} И_n) = (Я_n \veebar_{P1} Ч_n) \veebar_{P1\,P2} И_n.$$

The interlayer operator is distributive.

$$Я_n \veebar_P (Ч_n + И_n) = (Я_n \veebar_P Ч_n) + (Я_n \veebar_P И_n).$$
$$(Я_n + Ч_n) \veebar_P И_n = (Я_n \veebar_P И_n) + (Ч_n \veebar_P И_n). \ \square$$

4.3. The intricate diamond operator, ◊.

We explain the diamond product firstly in terms of 2 × 2 matrices

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \lozenge \begin{Vmatrix} e & f \\ g & h \end{Vmatrix} = \begin{Vmatrix} ae + bg & af + bh \\ -ec - dg & cf + dh \end{Vmatrix}.$$

The diamond operation is not expressible in any way by the normal matrix product.

Under the transformation c → -c, d → -d the right hand side matrix becomes

$$\begin{Vmatrix} ae + bg & af + bh \\ ec + dg & -cf - dh \end{Vmatrix}$$

A reversal of sign between bottom left and top right elements may be found under the transformation $a \to -a$, $e \to -e$, $b \to -b$ and $g \to -g$.

We have the following relations for intricate basis elements 1, i, α, ϕ.

$$1 \Diamond 1 = 1,\ i \Diamond i = -1,\ \alpha \Diamond \alpha = 1,\ \phi \Diamond \phi = 1,$$
$$1 \Diamond i = \phi = i \Diamond 1,\ 1 \Diamond \alpha = \alpha = \alpha \Diamond 1,\ 1 \Diamond \phi = i = \phi \Diamond 1,$$
$$i \Diamond \alpha = -i = -\alpha \Diamond i,\ i \Diamond \phi = \alpha = -\phi \Diamond i,\ \alpha \Diamond \phi = \phi = -\phi \Diamond \alpha.$$

The equations above show that 1, i, α and ϕ have inverses under $\Diamond$, with the left inverse equal to the right inverse. □

For an intricate number $M = m1 + ni + p\alpha + q\phi$, define the conjugate $M^{\Diamond}* = M*$ by

$$M^{\Diamond}* = m1 - ni - p\alpha - q\phi.$$

We obtain

$$M^{\Diamond}* \Diamond M = m^2 + n^2 - p^2 - q^2 = M \Diamond M^{\Diamond}*$$

and thus

$$M^{-1} = M^{\Diamond}*/(m^2 + n^2 - p^2 - q^2),$$

when not divided by zero. □

The diamond operator is distributive. For intricate A, B and C

$$(A + B) \Diamond C = (A \Diamond C) + (B \Diamond C)$$
$$A \Diamond (B + C) = (A \Diamond B) + (A \Diamond C).$$

The diamond operation is not associative in general, for example

$$(1 \Diamond i) \Diamond \phi = 1 \neq 1 \Diamond (i \Diamond \phi) = \alpha.$$

Operations of multiplication on the left or right by i, α and ϕ under the usual matrix product interchange whole rows and columns, multiplying each row or column everywhere by 1 or -1. The diamond operation multiplies just one element by -1, or under transformation of signs of elements an odd number by -1. It follows that two such operations by i, α or ϕ can sometimes convert to usual matrix multiplication. For instance

$$(A \Diamond 1) \Diamond 1 = A = 1 \Diamond (1 \Diamond A).$$ □

4.4. The intricate left and right roll operators, $_S{\circ}$ and $\circ_S$.

The right roll operator $\circ_1$ rotates clockwise the entries of the intricate matrix on the right, and then the usual matrix product is formed. Thus

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \circ_1 \begin{Vmatrix} e & f \\ g & h \end{Vmatrix} = \begin{Vmatrix} ag + bh & ae + bf \\ cg + dh & ce + df \end{Vmatrix}$$

This is not expressible in terms of the usual matrix product, nor the diamond product, nor a combination of these.

If two rolls take place, we will denote the operation by $\circ_2$. In general when S rolls take place and $S' = S \pmod 4$ then $\circ_{S'} = \circ_S$.

Because

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \circ_1 \begin{Vmatrix} e & f \\ g & h \end{Vmatrix}$$

amounts to one roll, the iteration

$$1 \circ_1 (1 \circ_1 A) = 1 \circ_2 A$$

leads to the result that $\circ_S$ can be derived from $\circ_1$ operations. For notational convenience on occasion we may drop the S suffix in $\circ_1$ to form $\circ$.

In terms of the intricate operation, $\circ_1$ has the following algebra

$1 \circ 1 = \phi$, $i \circ i = \phi$, $\alpha \circ \alpha = \phi$, $\phi \circ \phi = \phi$,
$1 \circ i = -\alpha = -i \circ 1$,
$1 \circ \alpha = i = \alpha \circ 1$,
$1 \circ \phi = 1 = \phi \circ 1$,
$i \circ \alpha = -1 = \alpha \circ i$,
$i \circ \phi = i = \phi \circ i$,
$\alpha \circ \phi = \alpha = -\phi \circ \alpha$.

From these relations it can be seen that the $\circ_S$ algebra has inverses. The left and right inverse are identical. □

The conjugate $M^{\circ *}$ of $M = m1 + ni + p\alpha + q\phi$ is

$$M^{\circ *} = m\phi - n\alpha + pi - q1,$$

so that

$$M \circ M^{\circ *} = m^2 + n^2 - p^2 - q^2 = M^{\circ *} \circ M.$$ □

For intricate A, B and C the $\circ_S$ algebra is distributive

$(A + B) \circ_S C = (A \circ_S C) + (B \circ_S C)$
$A \circ_S (B + C) = (A \circ_S B) + (A \circ_S C)$.

The left roll operator ${}_S\circ$ rotates clockwise the entries of the intricate matrix on the left, then forming the usual matrix product. The intricate algebra for ${}_S\circ$ maps bijectively

$$A \ {}_S\circ\ B \leftrightarrow B \circ_S A,$$

the order in the product in terms of usual matrix multiplication now being reversed in the mapping. For the case of a simultaneous ${}_S\circ$ and $\circ_{S'}$ operation, we denote this by ${}_S\circ_{S'}$. □

4.5. Combined and other operations.

The usual, diamond and roll products can be applied in different combinations on each layer of a hyperintricate number, possibly including the hyperintricate layer operator. On occasion we may denote the layers in a column, with the layer operators for each layer matched with the column. For example

$1 \ \lozenge \ \phi = i = -i_1$.
$\alpha \circ_1 i \quad -1$

There are a number of other operations. The tilde operator, ~, converts the hyperintricate diagonal $1 \rightarrow 1$, and all other hyperintricate numbers over all layers are multiplied collectively by minus one. Thus if $A = 1 + ai_i$, then $A^{\sim} = 1 - ai_i$.

Since the usual product is not commutative, in general

$A(B^{\sim}) \neq B(A^{\sim})$.

We will denote $A(B^{\sim})$ under usual multiplication by A ~ B. Because of multiplicative associativity of minus signs, but not additive associativity, these features carry over to the ~ operation.

The conjugate * acts on individual layers, but otherwise acts like ~.

The transpose T acts on individual layers, and converts $i \to -i$. We have seen that for hyperintricate numbers the transpose on all layers corresponds to the matrix transpose as commonly understood.

Under usual matrix multiplication T is contravariant

$(AB)^T = B^T A^T$.

We may combine the usual covariant product AB = AB with the T operator to form the right transpose operation

$A\ _{TR}\ B = A(B^T)$.

The left transpose operator LT satisfies

$A\ _{LT}\ B = (A^T)B$.

Normally, the transpose operation is not associative, nor is $A\ _{LT}\ B$ or $A\ _{TR}\ B$.

We may introduce transposes (as $i \to -i$) on a layer with diamond and roll operators. For instance

$1 \Diamond_{TR} \phi = 1 \Diamond (\phi^T) = 1 \Diamond \phi = i$. □

4.6. Split products.

Usual matrix multiplication satisfies

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} . \begin{Vmatrix} e & f \\ g & h \end{Vmatrix} = \begin{Vmatrix} ae + bg & af + bh \\ ec + dg & cf + dh \end{Vmatrix} = \begin{Vmatrix} bg & bh \\ dg & dh \end{Vmatrix} + \begin{Vmatrix} ae & af \\ ec & cf \end{Vmatrix}$$

where we will write this as

A.B = C + D.

The matrix C is the left split product and we also write

A|.B = C.

The right split product is

A.|B = D.

These operations can be extended to the diamond and roll products, where we write

$A|\Diamond B$,

$A\Diamond|B$,

$A|\circ_1 B$, etc.

We define the split transpose of A.B by

$(A|.B)^T = B^T|.A^T$.

The split products may be carried over to layers, with different operations on each layer. □

4.7. Modifier functions and products.

A function h may be applied to a matrix $A = a_{ij}$ to form the matrix $h_{ij} = h(A)$. A special kind is localised to independent functions, so h_{ij} is a function of the a_{ij} locally of the form $h(a_{ij})$. The identity function occurs when $h(a_{ij}) = 1a_{ij}$.

These functions may be combined with products of the form already discussed, to form composite products.

For example, we may consider the dimension of a matrix to be n + b, where n is a natural number and b ∊ the semi-closed, semi-open interval [0, 1[. Then $h_{ij} = 1$ for the array elements indexed by 1 to n, and b otherwise. For matrices of negative dimension, consider $h_{ij} = -1$ in all elements.

4.8. Lie algebras of type $sl_n(\mathbb{C})$.

The best introduction to the theory of Lie algebras and root systems is probably Roger Carter's in *Lectures on Lie groups and Lie algebras* [CSM95]. Our purpose in following closely the development there is so that it is embedded in this work, but we do not take its development as far as discussing groups of Lie type. To generalise $sl_n(\mathbb{C})$ to $gl_n(\mathbb{C})$ see [1Ca72], [1Ca05] and for Lie groups [1Ca93].

A *Lie algebra* is a vector space g over a field $\mathbb{F}$ in which is defined a multiplication

$g \times g \to g,$

$x, y \to [x, y],$

where [x, y] is called the *Lie bracket*, satisfying the axioms

(i) [x, y] is linear in x and y.

(ii) [x, x] = 0 for all x ∊ g.

(iii) The *Jacobi identity* [x, [y, z]] + [y, [z, x]] + [z, [x, y]] holds.

The set of n × n matrices, A, B, ... with entries in a field $\mathbb{F}$ can be made into a Lie algebra with Lie bracket [A, B] by setting

$$[A, B] = AB – BA,$$

and this is denoted by $gl_n(\mathbb{F})$, the general linear Lie algebra over the field $\mathbb{F}$.

It follows for these matrices that

$$[A + A', B + B'] = [A, B] + [A, B'] + [A', B] + [A', B']$$
$$[A, A] = 0$$
$$[A, B] = -[B, A],$$

and the Jacobi identity holds

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0, \qquad (1)$$

since this is

$$\begin{aligned} &ABC – BAC – CAB + CBA \\ &+ BCA – CBA – ABC + ACB \\ &+ CAB – ACB – BCA + BAC \\ &\quad = 0, \end{aligned}$$

but the Lie bracket is not generally associative

$$[[A, B], C] \neq [A, [B, C]].$$

Let $sl_n(\mathbb{C})$ be the set of n × n matrices with complex entries, a zero sum of diagonal entries, so the trace tr is zero, and with Lie brackets.

Example 4.8.1. The Lie algebra of a 2 × 2 intricate matrix has a $sl_2(\mathbb{C})$ basis of i, α, ϕ, since

$$[i, \alpha] = -2\phi, [i, \phi] = 2\alpha \text{ and } [\alpha, \phi] = 2i,$$

and all of i, α, ϕ are matrices with zero trace.

The trace tr, of the matrix Lie bracket is zero. For instance, since

$$tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} B_{ji} A_{ij} = tr(BA),$$

if A and B ∊ $gl_n(\mathbb{C})$ then

$$tr[A, B] = tr(AB - BA) = tr(AB) - tr(BA) = 0, \quad (2)$$

for any two n × n matrices. As we have seen in chapter II, the diagonal trace of an intricate actual number α and the hyperintricate traces involving αs are all zero, so for $2^n \times 2^n$ matrices, equation (2) reduces to the case where the real hyperintricate trace is zero,

$$tr1[A, B] = tr1(AB - BA) = tr1(AB) - tr1(BA) = 0, \quad (3)$$

where 1 is interpreted as a hyperintricate diagonal 1.

Thus $gl_n(\mathbb{C})$ contains $sl_n(\mathbb{C})$, which is a nontrivial proper subspace when n > 1. By a subspace we mean the set of all [A, B].

If h is the set of diagonal matrices of $sl_n(\mathbb{C})$ where tr is zero, this is a subalgebra of rank n – 1. Further, [h, h] = 0, so h is commutative.

For finitely many summands, direct sums are the same as direct products, where for sets this is the Cartesian product. For infinite subspaces, for direct sums all but a finite number of coordinates must be zero, whilst for direct products all but a finite number of multiples must be 1. Let E_{ij} be an elementary matrix, the unit diagonal matrix with just rows i and j swapped. Then

$$sl_n(\mathbb{C}) = h \oplus \sum_{i \neq j} \mathbb{C}E_{ij}, \quad (4)$$

where ⊕ is a direct sum of subspaces. This is because row i = 1 can be written as a linear combination of j – 1 terms in E_{ij} where i < j, down to row n – 1 as a linear combination with 1 term, and similarly for j < i, which supplies sufficient degrees of freedom to write complex elements of (4) uniquely.

Given a vector space **V** over a field $\mathbb{F}$, the span of a set **S** of vectors (not necessarily finite) is defined as the intersection **W** of all subspaces of **V** that contain **S**. **W** is referred to as the subspace spanned by **S** or by the vectors in **S**. Conversely we say **S** spans **W**.

By a dual space of a linear vector space **V**, if we interpret **V** as the space of *columns* of n real (Eudoxus) numbers, $\mathbb{U}$, or complex numbers, $\mathbb{C}$, its dual space $\mathbf{V}^*$ is typically written as the space of *rows* of n Eudoxus or complex numbers. If **V** consists of the linearly independent space of geometrical Eudoxus vectors in the plane, then the level curves of an element of $\mathbf{V}^*$ form a family of parallel lines in **V**. So an element of $\mathbf{V}^*$ can be intuitively thought of as a particular family of parallel lines covering the plane. For a linearly independent space **V** of dimension n, the elements of **V*** are parallel hyperplanes of dimension n. In the finite case the space **V*** has the same dimension as **V** and the dual of the dual is isomorphic to the original space: **V**** = **V**. If the vectors of **V** are linearly dependent, so the rank r is less than the dimension n of **V**, then the rank of **V*** is r, being the number of linearly independent basis elements of **V***.

Now choose a representative x of h

$$x = \begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

with $\lambda_1 + ... + \lambda_n = 0$, so these are linearly dependent of rank n – 1, then let

$$[xE_{ij}] = (\lambda_i - \lambda_j)E_{ij},$$

so we have a mapping

$$x \to (\lambda_i - \lambda_j).$$

Note that we have n(n – 1) 1-dimensional representations of h arising in this way, being the number of combinations of $(\lambda_i - \lambda_j)$ with $i \neq j$. These are called the *roots* of $sl_n(\mathbb{C})$ with respect to h. Let Φ be this set of roots. We will show how it lies in the dual space h*.

If $\beta \in \Phi$ then $-\beta \in \Phi$ also, since the map $x \to (\lambda_j - \lambda_i)$ is the negative of the map $x \to (\lambda_i - \lambda_j)$. Thus the roots are not independent. The roots do however span h*. For define $\beta_i \in \Phi$ by

$$\beta_i(x) = \lambda_i - \lambda_{i+1}.$$

Then $\beta_1, \beta_2, ... \beta_{n-1}$ are linearly independent and form a basis of h*. Let $\Pi = \{\beta_1, \beta_2, ... \beta_{n-1}\}$. Π is called the set of fundamental roots, or simple roots. We consider the way the roots are expressed as linear combinations of the fundamental roots. The root $x \to (\lambda_i - \lambda_j)$ is equal to

$$\beta_1 + \beta_2 + ... + \beta_{n-1} \quad \text{if } i < j$$
$$-(\beta_1 + \beta_2 + ... + \beta_{n-1}) \quad \text{if } j < i.$$

Thus each root in Φ is a linear combination of fundamental roots with coefficients in ℤ, and these integers can all be partitioned into nonnegative combinations Φ^+ of Π and nonpositive combinations Φ^- so that

$$\Phi = \Phi^+ \cup \Phi^- \quad \text{and} \quad \Phi^+ = -\Phi^-.$$

Given an element x of a Lie algebra g, we define the *adjoint* action of x on g as the map

$$ad_x: g \to g$$

where for all z in g

$$ad_x(z) = [x, z].$$

Let g be a Lie algebra over a field 𝔽. Then the linear mapping $x \to ad_x$ is a representation of the Lie algebra and is called the adjoint representation of the algebra. The Lie bracket is, by definition, obtained from two operators:

$$[ad_x, ad_y] = ad_x \circ ad_y - ad_y \circ ad_x$$

where ○ denotes composition of linear maps. By the Jacobi identity we have

$$ad_x \circ ad_y(z) - ad_y \circ ad_x(z) = [x, [y, z]] - [y, [x, z]] = [[x, y], z]$$
$$= ad_{[x, y]}(z).$$

Now, supposing g is of finite dimension, the trace of the composition of two such maps defines a bilinear form

$$\langle x, y \rangle = tr(ad_x \circ ad_y),$$

where <x, y> is the *Killing form* on g. The name Killing form first appeared in a paper of Armand Borel in 1951. Since tr(AB) = tr(BA), the Killing form is symmetric

$$\langle x, y \rangle = \langle y, x \rangle.$$

The Killing form is an invariant form, in the sense that it is associative

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle,$$

since

$$\langle [x, y], z \rangle = tr([x, y]z) = tr(xy\, z - yx\, z) = tr(xy\, z) - tr(y\, xz).$$

Similarly,

$<x, [y, z]> = tr(x\,[y, z]) = tr(x\,yz) - tr(x\,zy)$.

Finally, $tr(y(x\,z)) = tr((x\,z)y)$.

The Killing form on g is nondegenerate in that

$<x, y> = 0$ for all $y \in g$ implies $x = 0$.

We may restrict the Killing form on g to h, to give a map $h \times h \to \mathbb{C}$. It can be shown that this map is nondegenerate on h, so that

$x \in h$ and $<x, y> = 0$ for all $y \in g$ implies $x = 0$.

We may thus define a map $h \to h^*$ given by $x \to f_x$ where

$f_x(y) = <x, y>$ for $x, y \in h$,

which is a linear map $h \to h^*$. Thus each element of h^* has a form f_x for just one $x \in h$, so we can define a map $h^* \times h^* \to \mathbb{C}$ by

$<f_x, f_y> = <x, y>$ for $x, y \in h$.

We may restrict this linear form to the real (Eudoxus) vector space $h^*_{\mathbb{U}}$. It can be shown from the representation of i as an integer matrix that its values lie in $\mathbb{U}$. Thus we have a map

$h^*_{\mathbb{U}} \times h^*_{\mathbb{U}} \to \mathbb{U}$.

This map has the property that

$<\lambda, \lambda> \geq 0$ for all $\lambda \in h^*_{\mathbb{U}}$.

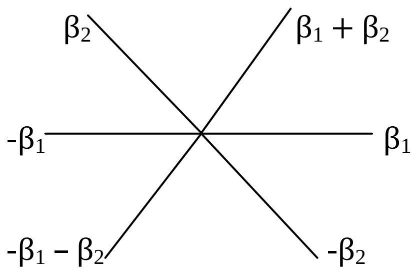
Furthermore $<\lambda, \lambda> = 0$ implies $\lambda = 0$, so that the scalar product on $h^*_{\mathbb{U}}$ is positive definite. $h^*_{\mathbb{U}}$ is therefore a Euclidean space.

This Euclidean space $h^*_{\mathbb{U}}$ contains the set of roots Φ. The properties of the configuration formed by the roots in $h^*_{\mathbb{U}}$ is important in the classification of the simple Lie algebras g.

Example 4.8.2. Let $g = sl_2(\mathbb{C})$. Then dim h = 1. Let $\Pi = \{\beta_1\}$. Then $\Phi = \{\beta_1, -\beta_1\}$. The configuration formed by Φ in the 1-dimensional space $h^*_{\mathbb{U}}$ is

$-\beta_1$ ——— 0 ——— β_1

For $g = sl_3(\mathbb{C})$, dim h = 2, and $\Pi = \{\beta_1, \beta_2\}$, so $\Phi = \{\beta_1, \beta_2, \beta_1 + \beta_2, -\beta_1, -\beta_2, -\beta_1 - \beta_2\}$. The configuration formed by Φ in the 2-dimensional Euclidean space $h^*_{\mathbb{U}}$ is



The configuration formed by the root system Φ is best understood by introducing a certain group of non-singular linear transformations of $h^*_{\mathbb{U}}$ called the *Weyl group*. For each $\beta \in \Phi$ let $s_\beta: h^*_{\mathbb{U}} \to h^*_{\mathbb{U}}$ be the map defined by

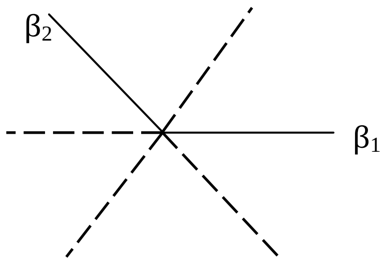
$$s_\beta(\lambda) = \lambda - 2\frac{\langle\beta, \lambda\rangle}{\langle\beta, \beta\rangle}\beta.$$

Note that $s_\beta(\beta) = -\beta$ and $s_\beta(\lambda) = \lambda$ whenever $\langle\beta, \lambda\rangle = 0$. Thus s_β is the reflection in the hyperplane orthogonal to β. Let W be the group generated by the maps s_β for all $\beta \in \Phi$. W is called the Weyl group.

W has favourable properties. Firstly it permutes the roots, that is, $w(\beta) \in \Phi$ for every $\beta \in \Phi$ and $w \in W$. Consequently W is finite, because there are only a finite number of permutations of Φ, in which Φ spans the linear transformations $h^*_{\mathbb{U}}$, where each of these permutations comes from just one such linear transformation. We also have $\Phi = W(\Pi)$, which means given any $\beta \in \Phi$, there exists a $\beta' \in \Pi$ and $w \in W$ such that $\beta = w(\beta')$. Furthermore, W is generated by the $s_{\beta'}$ for $\beta' \in \Pi$.

The importance of the Weyl group is that it enables us to reconstruct the full root system Φ given only the set Π of fundamental roots. For given Π the Weyl group is determined, being the group generated by the reflections $s_{\beta'}$ for $\beta' \in \Pi$. The root system Φ is then determined, since $\Phi = W(\Pi)$. Thus, given Π, the root system Φ is obtained by successive reflections $s_{\beta'}$ until no further vectors can be obtained.

An example when $g = sl_3(\mathbb{C})$ is shown below



Given β_1 and β_2 the remaining roots are obtained by reflecting successively by s_{β_1} and s_{β_2}.

We note that

$$s_{\beta_i}(\beta_j) = \beta_j - 2\frac{\langle\beta_i, \beta_j\rangle}{\langle\beta_i, \beta_i\rangle}\beta_i.$$

If $\beta_i, \beta_j \in \Pi$ with $i \neq j$, then $s_{\beta_i}(\beta_j)$ is a root, and so is a $\mathbb{Z}$-combination of β_i and β_j. Since the coefficient of β_j is 1, the coefficient of β_i must be a nonnegative integer, because the given root lies in Φ^+. It follows that

$$2\frac{\langle\beta_i, \beta_j\rangle}{\langle\beta_i, \beta_i\rangle} \in \mathbb{Z} \text{ and is } \leq 0.$$

We define $A_{ij} = 2\frac{\langle\beta_i, \beta_j\rangle}{\langle\beta_i, \beta_i\rangle}$,
where the elements of A_{ij} are called *Cartan numbers*, and the matrix they form the *Cartan matrix*. We have $A_{ij} \in \mathbb{Z}$, where $A_{ii} = 2$ and A_{ij} for $i \neq j$ is ≤ 0.

Let θ_{ij} be the angle between β_i and β_j. This angle can be found from the cosine formula

$$< \beta_i, \beta_j > = < \beta_i, \beta_i >^{1/2} < \beta_j, \beta_j >^{1/2} \theta_{ij}.$$

Therefore

$$4\cos^2\theta_{ij} = 2\frac{\langle\beta_i, \beta_j\rangle}{\langle\beta_i, \beta_i\rangle}.2\frac{\langle\beta_j, \beta_i\rangle}{\langle\beta_j, \beta_j\rangle},$$

giving

$$4\cos^2\theta_{ij} = A_{ij}A_{ji}.$$

We will write $n_{ij} = A_{ij}A_{ji}$. Then $n_{ij} \in \mathbb{Z}$ and $n_{ij} \geq 0$. Furthermore, because

$$-1 \leq \cos \theta_{ij} \leq 1$$

we get

$$0 \leq 4\cos^2 \theta_{ij} \leq 4,$$

and since when $i \neq j$, $\theta_{ij} \neq 0$, we obtain

$$0 \leq 4\cos^2 \theta_{ij} < 4.$$

Thus the only possible values of n_{ij} are 0, 1, 2 and 3.

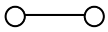
We will now encode these results about the system Π of fundamental roots in terms of a graph.

The *Dynkin diagram* Δ of g is the graph with nodes labelled 1, ..., m mapping bijectively to the set Π of fundamental roots, so that the nodes i, j with $i \neq j$ are joined by n_{ij} bonds.

Example 4.8.3. Let $g = sl_3(\mathbb{C})$. Then $\Pi = \{\beta_1, \beta_2\}$ and

$$s_{\beta_1}(\beta_2) = \beta_1 + \beta_2, s_{\beta_2}(\beta_1) = \beta_1 + \beta_2,$$

thus $A_{12} = A_{21} = -1$, and $n_{12} = 1$, giving the graph Δ



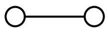
1 2

The Dynkin diagram is uniquely determined by g. The choice of fundamental system Π does not matter, since it can be shown that two fundamental systems Π_1 and Π_2 have the property that $\Pi_1 = w(\Pi_2)$ for some $w \in W$.

The Dynkin diagram of g has the following properties. Δ is a connected graph provided g is a nontrivial simple Lie algebra. Any two nodes are joined by at most 3 bonds. Let $Q(x_1, ..., x_m)$ be the quadratic form

$$Q(x_1, \cdots, x_m) = 2\sum_{i=1}^{m} x_i^2 - \sum_{i,j,i \neq j}^{m} \sqrt{n_{ij}}\, x_i x_j.$$

This quadratic form is determined by the Dynkin diagram. For example if Δ is



1 2

then we have

$$Q(x_1, x_2) = 2x_1^2 + 2x_2^2 - 2x_1 x_2.$$

The quadratic form $Q(x_1, ..., x_m)$ is positive definite because it contains the scalar product in

$$Q(x_1, \cdots, x_m) = 2\left\langle \frac{\sum_{i=1}^{m} x_i \beta_i}{\sqrt{\beta_i, \beta_i}}, \frac{\sum_{i=1}^{m} x_i \beta_i}{\sqrt{\beta_i, \beta_i}} \right\rangle.$$

We will consider the problem of determining all graphs Δ with the above properties.

Theorem 4.8.4. *Consider graphs* Δ *with the following properties*:

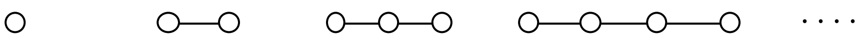
(i) Δ *is connected*

(ii) *The number of nodes joining any two bonds is* 0, 1, 2 *or* 3

(iii) *The quadratic form* Q *determined by* Δ *is positive definite*.

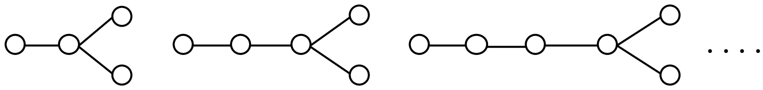
Then Δ *must be one of the graphs in the following list*:

A_1 A_2 A_3 A_4

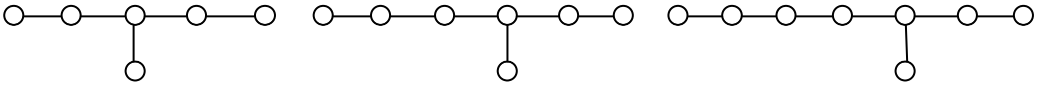


B_2 B_3 B_4

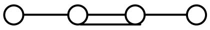
D$_4$ D$_5$ D$_6$



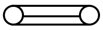
E$_6$ E$_7$ E$_8$



F$_4$



G$_2$ list 1



We first consider to what extent the Dynkin diagram determines the matrix of Cartan integers. Recall that

$$n_{ij} = A_{ij}A_{ji} \qquad i \neq j,$$

and that the A_{ij} are integers ≤ 0. Furthermore $A_{ij} = 0$ if and only if $A_{ji} = 0$.

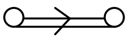
If $n_{ij} = 0$ then $A_{ij} = A_{ji} = 0$. If $n_{ij} = 1$ we must have $A_{ij} = -1$ and $A_{ji} = -1$. However, if $n_{ij} = 2$ we have two possibilities: either $A_{ij} = -1$ and $A_{ji} = -2$ or $A_{ij} = -2$ and $A_{ji} = -1$, and because

$$A_{ij} = 2\frac{\langle \beta_i, \ \beta_j \rangle}{\langle \beta_i, \ \beta_i \rangle},$$

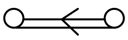
we obtain

$$\frac{A_{ij}}{A_{ji}} = \frac{\langle \beta_j, \ \beta_j \rangle}{\langle \beta_i, \ \beta_i \rangle}.$$

In the first case, we have $\langle \beta_i, \ \beta_i \rangle > \langle \beta_j, \ \beta_j \rangle$, and in the second case $\langle \beta_i, \ \beta_i \rangle < \langle \beta_j, \ \beta_j \rangle$, which we notate by putting an arrow on the Dynkin diagram pointing towards the long root, where the arrow can be thought of as an inequality between the root lengths. In the first case we have the diagram

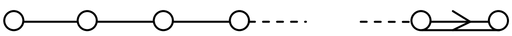


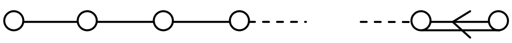
and for the second case



Likewise if $n_{ij} = 3$ there are two possible factorisations of $n_{ij} = A_{ij}A_{ji}$ which can be distinguished by putting similar directional arrows on the triple bond.

In the cases where $\Delta = B_2$, F_4 and G_2 in list 1, the diagrams are symmetric, so there is no difference, but when $\Delta = B_m$ for $m > 2$ we can find two different types of graph, which we denote by B_m and C_m in the diagrams below.

B$_m$ 

C$_m$ 

Thus for B_m the last fundamental root is shorter than the others, and for C_m it is longer. □

Written explicitly the Dynkin diagrams correspond to the matrices of Cartan integers

$$A_m = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & 0 & \\ & & -1 & & & & & \\ & & & & & -1 & & \\ & 0 & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}$$

$$B_m = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & 0 & \\ & & -1 & & & & & \\ & & & & & -1 & & \\ & 0 & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -2 & 2 \end{bmatrix}$$

$$C_m = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & 0 & \\ & & -1 & & & & & \\ & & & & & -1 & & \\ & 0 & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -2 \\ & & & & & & -1 & 2 \end{bmatrix}$$

$$D_m = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & 0 & \\ & & -1 & & & & & \\ & & & & & -1 & & \\ & 0 & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & 0 \\ & & & & & & 0 & 2 \end{bmatrix}$$

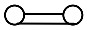
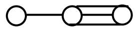
$$G_2 = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_6 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_7 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

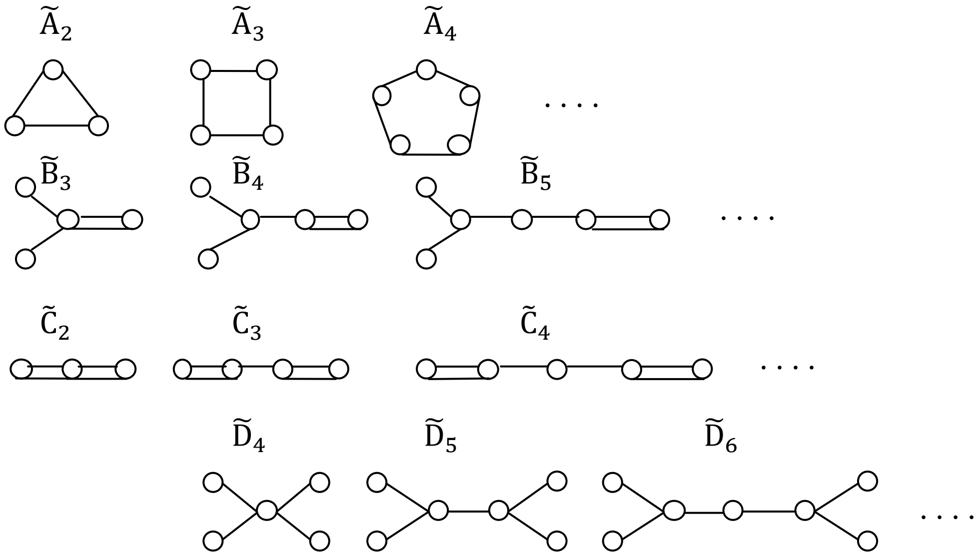
$$E_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

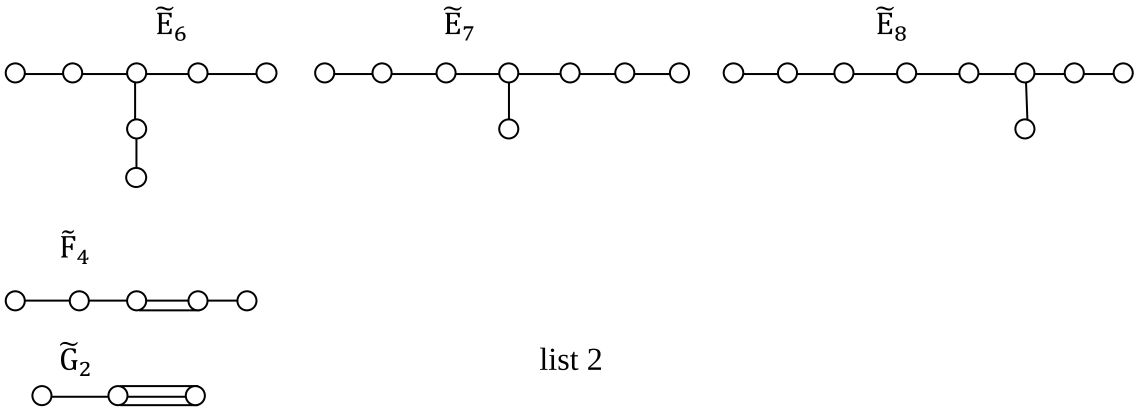
Proof of theorem 4.8.4. A *subgraph* of a graph Δ is obtained from Δ by removing certain nodes or decreasing certain bond lengths or both (so is a subgraph of ). The list of graphs given in the theorem will be called the standard list. We note that each subgraph of a graph on the standard list is also on the standard list. It is not difficult to show that if the quadratic form of a graph Δ is positive definite, then the quadratic form of any subgraph of Δ is positive definite also.

Now the quadratic form of a graph Δ is represented by a symmetric matrix M.

We recall from linear algebra that $Q(x_1, ..., x_m)$ is positive definite if and only if all the leading minors of M have positive determinant. However the leading minors of M are simply matrices M corresponding to certain subgraphs of Δ. In order to show that $Q(x_1, ..., x_m)$ is positive definite it is therefore sufficient to check that det $M > 0$ for each graph Δ on the standard list. This is readily verified.

We now wish to prove conversely that the graphs on the standard list are the only ones satisfying the given conditions. In order to do this we introduce a second list.





list 2

It may be readily checked that each graph Δ on list 2 has a quadratic form $Q(x_1, ..., x_m)$ with symmetric matrix M satisfying det M = 0. Thus $Q(x_1, ..., x_m)$ is not positive definite. Hence any graph Δ satisfying our given conditions can contain no subgraph on list 2.

Let Δ be a graph satisfying our conditions (i) (ii) and (iii). Then Δ has no cycles, otherwise Δ would contain a subgraph of type $\tilde{A}_m$ Δ has at most one multiple bond, otherwise Δ would contain a subgraph of type $\tilde{C}_m$. Δ cannot have both a multiple bond and a branch point, otherwise it would contain a subgraph $\tilde{B}_m$. Also Δ cannot have more than one branch point, otherwise Δ would contain a subgraph $\tilde{D}_m$.

Suppose Δ has a triple bond. Then Δ must be G_2, as otherwise Δ would contain a subgraph $\tilde{G}_2$. We may therefore assume that Δ contains no other triple bond than for G_2.

Suppose Δ has a double bond. Then Δ contains no branch point, so is a chain. If the double bond is at one end of the chain then Δ = B_m, if not then Δ must be F_4, otherwise Δ would contain a subgraph $\tilde{F}_4$.

So we may assume that Δ contains only single bonds. If Δ has no branch points then Δ = A_m. So posit Δ contains a branch point, which must have only three branches because it cannot contain a subgraph $\tilde{D}_4$. Let the length of the branches be m_1, m_2 and m_3, with $m = m_1 + m_2 + m_3 + 1$ and $m_1 \geq m_2 \geq m_3$. Then $m_3 = 1$, otherwise Δ would contain a subgraph $\tilde{E}_6$. Also $m_2 \leq 2$ otherwise Δ would contain a subgraph $\tilde{E}_7$. If $m_2 = 1$ then Δ = D_m. So suppose $m_2 = 2$. Then $m_1 \leq 4$, otherwise Δ would contain a subgraph $\tilde{E}_8$. If $m_1 = 2$ then Δ = E_6. If $m_1 = 3$ then Δ = E_7. Lastly if $m_1 = 4$ then Δ = E_8.

Therefore Δ must be one of the graphs on list 1, the standard list. □

The classification of the simple Lie algebras was achieved by W. Killing in a series of papers in Mathematische Annalen between 1888 and 1890, and independently by Eli Cartan in his Paris thesis of 1894.

4.9. Kac-Moody algebras. [Ka90]

There are 26 sporadic groups, of which the largest is the Monster. 20 of them are subgroups or subquotients of the Monster.

It is interesting to consider to what extent the Monster is related to the Lie theory. It is known that the Monster is the automorphism group of an infinite dimensional algebra called a vertex operator algebra. Vertex operators appear in the representation theory of the infinite dimensional Lie algebras known as affine Kac-Moody algebras. These are Lie algebras corresponding to the extended Dynkin diagrams in list 2 of section 8. Thus the Monster can be related to the theory of Kac-Moody algebras.

A generalised Kac-Moody algebra is a Lie algebra that is similar to a Kac-Moody algebra, except that it is allowed to have imaginary simple roots. Generalised Kac-Moody algebras are also sometimes called GKM algebras, Borcherds-Kac-Moody algebras, BKM algebras, or Borcherds algebras. The best known example is the Monster Lie algebra. A question that arises from the allocation of a real matrix $\mathrm{i} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, is whether the existence of imaginary roots is bogus.

The reason for the existence of infinite Kac-Moody algebras can be expressed in nonstandard set theory. The property of being a finite Lie algebra holds for all finite $m \in \mathbb{N}$. For standard set theory infinite $m \in \mathbb{N}$ are void. A Kac-Moody algebra is nowhere a finite Lie algebra, so infinite Kac-Moody algebras are at least admissible in logical reasoning. For nonstandard set theory there exist infinite m, so the set of these is nonvoid. Thus in nonstandard set theory infinite Kac-Moody algebras exist and the property of being a finite Lie algebra and an infinite Kac-Moody algebra can both hold.

4.10. Exercises.

(A) Let X and Y be symmetric matrices and A and B be antisymmetric matrices. Show

$$(X + A)(Y + B) + (X - A)(Y + B) + (X + A)(X - B) + (X - A)(X - B) = 4XY,$$
$$(X + A)(Y + B) + (X - A)(Y + B) - (X + A)(X - B) - (X - A)(X - B) = 4XB,$$
$$(X + A)(Y + B) - (X - A)(Y + B) + (X + A)(X - B) - (X - A)(X - B) = 4AY,$$
$$(X + A)(Y + B) - (X - A)(Y + B) - (X + A)(X - B) + (X - A)(X - B) = 4AB.$$

Express the above relations in terms of the transposes $(A + Y)^T$ and $(Y + B)^T$. Reformulate these in terms of the matrix product, the left matrix product and transposes.

(B) Using a 2 × 2 matrix in its intricate representation or otherwise, show that the product of two symmetric matrices is not necessarily symmetric.