CHAPTER II

Hyperintricate numbers

2.1. Introduction.

We develop a representation of $2^n \times 2^n$ matrices by analogy with the intricate case, which is a regular extension of this idea. These are called *hyperintricate numbers*, and we investigate some of the properties of this representation.

2.2. Construction and properties of hyperintricate numbers.

The sum of two $m \times m$ matrices A and B, with elements for A given by a_{ij}, where i is the ith row and j is the jth column, and for B by b_{ij}, is the matrix C where

$$C = c_{ij} = a_{ij} + b_{ij}.$$

The corresponding product D is

$$D = d_{ik} = AB = \sum_j a_{ij} b_{jk},$$

where $\sum$ indicates summation, in this case over the variable j. This is the generalisation of a matrix product already given in 1.5 for 2×2 matrices. We seek to develop this idea within an extended framework already given for these intricate numbers.

We can define n-*hyperintricate* numbers recursively, by building up starting from intricate ones. Consider a $2^n \times 2^n$ matrix. Let "+" be a chosen $2^{n-1} \times 2^{n-1}$ matrix which is a hyperintricate basis element of lower dimension, for example an intricate basis element 1, i, α or ϕ. Let "–" be the corresponding matrix with all negative entries from "+". Consider the set of $2^n \times 2^n$ hyperintricate basis elements, where an intricate number has "+" = 1, "–" = -1

$$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}, \begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}, \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}.$$

Any $2^n \times 2^n$ matrix can be represented uniquely by a linear combination of these.

A $j \times j$ matrix may be extended both right and below with zero entries to give a larger $2^n \times 2^n$ matrix, or main diagonal entries of 1 may be substituted here to keep determinants non-zero. Determinants are described from section 2.9 in this chapter. By this means matrix theorems may be expressed hyperintricately.

I now introduce some notation. I will do this by giving examples of 4×4 matrices. Write

$$1_1 = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{bmatrix}, \qquad \alpha_i = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & 0 \\ & 0 & 0 & -1 \\ & & 1 & 0 \end{bmatrix},$$

$$i_1 = \begin{bmatrix} & 0 & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & 0 \\ 0 & -1 & & \end{bmatrix}, \quad \phi_i = \begin{bmatrix} & 0 & 0 & 1 \\ & & -1 & 0 \\ 0 & 1 & & 0 \\ -1 & 0 & & \end{bmatrix}.$$

So "+" corresponds with the subscript, which will be described as an example of a layer, for example in α_i. A memory aid is '*subscripts are the little part*'.

If in general each of the 16 real 4 × 4 matrices are represented by e.g. $\alpha_i = A_B$, then

$(A_B) + (A_C) = A_{(B + C)}$,
$(A_B) + (C_B) = (A + C)_B$,
$(A_B)(C_D) = (AC)_{BD}$,
$A_{-B} = -(A_B) = (-A)_B$.

For further nesting of matrices, consider instead of stepping down a further layer, introducing (possibly) a comma, thus: $A_{B,C}$, so that matrix multiplication becomes

$(AB)_{CD,EF} = (A_{C,E})(B_{D,F})$.

The *layers* of a basis element $m_{n\ ...\ p}$, are the vectors m, n, ... p, and its *layer dimension* is the number of layers.

We define an n-*hyperimaginary* number to be an n-hyperintricate number with each layer restricted to the set {1, i}. We can also define *hyperactual* numbers, containing elements of {1, α} in all layers and *hyperphantom* numbers for which every layer $\in$ {1, ϕ}. Hyperactual and hyperphantom number are not members of a field. This arises because (1 + α) and (1 + ϕ) have determinant zero, and so have no inverse and $(a1_1 + bi_i)$ has inverse $(a1_1 - bi_i)/(a^2 - b^2)$, which does not exist for a = b. Another way of looking at this is that complex numbers are the only algebra of the three which is conformal (see *Number, space and logic* [Ad18]).

Intricate and hyperintricate numbers appear in four ways – as scalars, satisfying a non-commutative algebra, as vectors (or eigenvectors described later) with a linearly independent basis, as matrices – where the first instance is intricate numbers, and in the hyperintricate case, say as the object similar to a tensor, $m_{n,p}$, where m, n and p are vectors.

Layers may be *permuted*, and except for interior coefficient algebras, and compression and expansion studied later in this chapter, uniformly applied the resulting algebra under addition and multiplication is the same.

2.3. Deriving hyperintricate basis element coefficients from a matrix.

The *trace* (the sum of diagonal entries) of an intricate 1 is 2, and of α is zero. If we take consecutively the diagonal entries of α, which are [1, -1], and subtract the second entry, -1, we get 2 for α, and applied to 1 the same process now gives 0. We will denote the trace of the real component by tr1 and this modified trace of the actual component by trα. So for trα

$\text{tr}\alpha(1) = [1 - 1] = 0$,
$\text{tr}\alpha(\alpha) = [1 - (-1)] = 2$.

For each diagonal matrix, the trace component of say 1_α, represented by [1, -1, 1, -1] is not zero over its average only on multiplying, for negative values, by -1, and this procedure for trα applied to 1_α, α_1 and α_α gives the value 4. This idea can be generalised to non-diagonal hyperintricate values to obtain general coefficients from the defining matrix.

For example, let $Я_2$ be the 2-hyperintricate number

$$\begin{aligned} Я_2 = {} & a_{11}1_1 + a_{1i}1_i + a_{1\alpha}1_\alpha + a_{1\phi}1_\phi \\ & + b_{i1}i_1 + b_{ii}i_i + b_{i\alpha}i_\alpha + b_{i\phi}i_\phi \\ & + c_{\alpha 1}\alpha_1 + c_{\alpha i}\alpha_i + c_{\alpha\alpha}\alpha_\alpha + c_{\alpha\phi}\alpha_\phi \\ & + d_{\phi 1}\phi_1 + d_{\phi i}\phi_i + d_{\phi\alpha}\phi_\alpha + d_{\phi\phi}\phi_\phi. \end{aligned}$$

For $Я_2 = r_{jk}$ as elements of a matrix, if we consider trace members r_{11}, r_{22}, r_{33} and r_{44} in sequence

$$\begin{bmatrix} r_{11} & & & \\ & r_{22} & & \\ & & r_{33} & \\ & & & r_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & & & \\ & a_{11} & & \\ & & a_{11} & \\ & & & a_{11} \end{bmatrix} + \begin{bmatrix} c_{\alpha 1} & & & \\ & c_{\alpha 1} & & \\ & & -c_{\alpha 1} & \\ & & & -c_{\alpha 1} \end{bmatrix}$$

$$+ \begin{bmatrix} a_{1\alpha} & & & \\ & -a_{1\alpha} & & \\ & & a_{1\alpha} & \\ & & & -a_{1\alpha} \end{bmatrix} + \begin{bmatrix} c_{\alpha\alpha} & & & \\ & -c_{\alpha\alpha} & & \\ & & -c_{\alpha\alpha} & \\ & & & c_{\alpha\alpha} \end{bmatrix}.$$

then

$r_{11} = [a_{11} + c_{\alpha 1} + a_{1\alpha} + c_{\alpha\alpha}]$
$r_{22} = [a_{11} + c_{\alpha 1} - a_{1\alpha} - c_{\alpha\alpha}]$
$r_{33} = [a_{11} - c_{\alpha 1} + a_{1\alpha} - c_{\alpha\alpha}]$

and

$$r_{44} = [a_{11} - c_{\alpha 1} - a_{1\alpha} + c_{\alpha\alpha}]. \quad (1)$$

Substituting in sequence r_{12}, r_{21}, r_{34} and r_{43} for the above expressions, we obtain equivalent results, by substituting in the second subscript on the right above the transformation from diagonal to antidiagonal terms $1 \to \phi$ and $\alpha \to i$.

Likewise for r_{13}, r_{24}, r_{31} and r_{42}, equivalent results on the right are found from $a \to d$, $c \to b$, and for the first subscript $1 \to \phi$ and $\alpha \to i$.

For r_{14}, r_{23}, r_{32} and r_{41} in sequence, the substitutions are $a \to d$, $c \to b$, and for both the first and second subscripts $1 \to \phi$ and $\alpha \to i$.

To obtain a_{11}, $c_{\alpha 1}$, $a_{1\alpha}$ and $c_{\alpha\alpha}$ respectively, for respective terms of r_{11}, r_{22}, r_{33} and r_{44} in (1), there is an inverse type of relationship maintaining the signs:

$a_{11} = [r_{11} + r_{22} + r_{33} + r_{44}]/4$
$c_{\alpha 1} = [r_{11} + r_{22} - r_{33} - r_{44}]/4$
$a_{1\alpha} = [r_{11} - r_{22} + r_{33} - r_{44}]/4$

and

$c_{\alpha\alpha} = [r_{11} - r_{22} - r_{33} + r_{44}]/4$

extendable to the other cases.

For a general n-hyperintricate matrix, each term is divided by the number of rows, 2^n. □

2.4. Exterior, interior and relative coefficient algebras.

A real number, r, multiplied by a hyperintricate basis element A_B multiplies each element of the matrix by r. Then

$$rA_B = (aA)_{(bB)} \quad (1)$$

where ab = r.

Hyperintricate basis elements may have coefficients acting on the left or right (or both) which are themselves hyperintricate. These coefficients may be considered as a sum of terms of real values multiplied by hyperintricate basis elements.

Generally speaking, there is more than one type of algebra in which the coefficients are multiplied by hyperintricate basis elements. In all cases real components of the coefficient bases are treated as in (1).

An *exterior coefficient algebra* takes the layer dimension, n, of the coefficients and to the m-hyperintricate basis element to which it is attached, appends a basis element of the coefficient to the (say) trailing 1 layers of the m-hyperintricate basis element. The exterior coefficient algebra is commutative with respect to a coefficient and an m-hyperintricate basis element, for example, in an intricate number converted to a 2-hyperintricate with exterior coefficients

$(a1_\alpha)1_1 + (b1_\alpha)i_1 + (c1_\phi)\alpha_1 + (d1_\phi)\phi_1$.

The *interior coefficient algebra* extracts layers from the basis element of the coefficient, permutes them in a uniform way and multiplies corresponding layers to those in the m-hyperintricate basis to which it is attached. For identity permutations and coefficients with layer dimension equal to that of basis elements, this corresponds to normal matrix multiplication. The interior coefficient algebra is not commutative in general.

The exterior coefficient algebra may be considered as a special case of the interior coefficient algebra, in which the coefficient is multiplied by trailing layers of 1 in the m-hyperintricate basis element.

The *relative coefficient algebra* operates on all layers rather than selective ones and treats r, a and b in (1) as intricate or hyperintricate numbers. Consider the example $A_B = 1_1$, r = i, a = α, b = ϕ, and real numbers t = uv. We would have

$tiA_B = (u\alpha A)_{(v\phi B)} = t\alpha_\phi$

$= -(u\phi A)_{(v\alpha B)} = -t\phi_\alpha$,

which is not the case. A solution not directly involving equivalence classes is to treat r, a and b as conforming to the scalar algebra given in (1.6.1) and to perform operations relative to the basis A_B. We write to indicate this

$tiA_B = (uaA)_{(vbB)}$ (rel A_B),

then we have for example

$ti1_1 = (u\alpha 1)_{(v\phi 1)}$ (rel 1_1),

$ti1_1 = -(u\phi 1)_{(v\alpha 1)}$ (rel 1_1).

The relative coefficient algebra is not commutative in general. We emphasise that only one basis element, say A_B, is used throughout compound expressions. The consistency of relative coefficient algebras is proved by compressing A_B (say on the right) to the product AB and then evaluating all coefficients, consistently applied under the same type of compression, in separate expressions on the left. □

2.5. Symmetric, antisymmetric and upper triangular matrices.

The hyperintricate representation is unique, arising from the linear independence of distinct basis elements; if two hyperintricate numbers are given by say, $a1_1 + bi_\alpha = a'1_1 + b'i_\alpha$, then since $(a - a')1_1 + (b - b')i_\alpha = 0$, a = a′ and b = b′.

A matrix U is *symmetric* when its elements satisfy $u_{jk} = u_{kj}$, and the elements of a matrix V are *antisymmetric* when $v_{jk} = -v_{kj}$. Any matrix W may be represented uniquely as W = U + V.

The $2^n \times 2^n$ matrix V is antisymmetric when all of its hyperintricate components are antisymmetric. The square of a symmetric basis element $\sum_j u_{ij}u_{jk} = \sum_j u_{ij}u_{kj}$ is 1 and of an antisymmetric basis element $\sum_j v_{ij}v_{jk} = -\sum_j v_{ij}v_{kj}$ is -1. So this antisymmetry happens when a search of the layers of a basis element finds an odd number of i's, which has a square -1, otherwise the basis element is symmetric, with square 1. □

The *transpose* W^T of a matrix $W = w_{jk}$, swapping rows and columns, is obtained under the map $i \to -i$ for all i layers of its component basis elements. No other basis element than i changes sign under the map $w_{jk} \to w_{kj}$. □

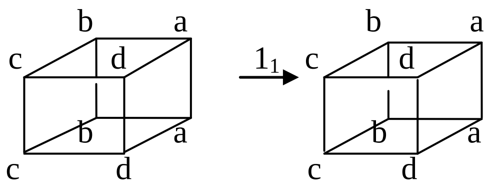
When j = k we say w_{jk} is along the (main) diagonal. A $2^n \times 2^n$ *upper triangular* matrix has all zero entries below the main diagonal. This diagonal is represented entirely by hyperactual components, because 1 and α are the only diagonal intricate matrices.

A necessary condition for the remainder outside the diagonal to be upper triangular is that each antisymmetric component where a ϕ or i layer is ranked earliest is summed equally with the symmetric component in which this ϕ or i is interchanged, which zeroises the lower part.

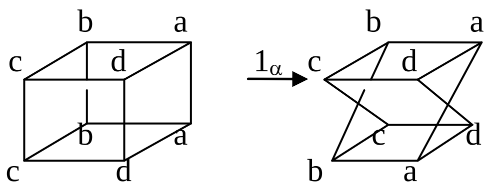
This condition is sufficient. Main diagonal symmetries, including antisymmetries, are symmetries with the largest such scope, only determined by the leading ϕ or i layer. The lower triangular region will then only be zero if a leading ϕ or i layer for each component is summed with an interchanged i or ϕ layer of equal value. □

2.6. Hyperintricate multiplication from the symmetries of hyperobjects.

In the hyperintricate representation we can describe a basis element component in a layer by a symmetry transformation of a square. A geometrical interpretation in cohomology theory is that each separate vertex of each square may be thought of as connected with a corresponding vertex for all squares in other layers, forming a possibly twisted hypercube.



a representation of the hyperintricate number 1_1



a representation of the hyperintricate number 1_α

Algebraically, the twisted intricate transformation $\mathbb{W}$, where $\mathbb{W}^2 = 1$ can be represented by α or ϕ, where $\alpha^2 = \phi^2 = 1$, and twisted intricate numbers can be represented for example by

$$a1_1 + bi_1 + c\alpha_1 + d\phi_1 + a'1_\alpha + b'i_\alpha + c'\alpha_\alpha + d'\phi_\alpha.$$

To extend this idea from chapter I, there is also a formulation in which $\mathbb{W}^2 = -1$, which maps on to i as the second layer component. □

2.7. Vector spaces.

A vector space is a collection of objects, **V**, called *vectors*, denoted in bold type, which can be added together and multiplied by numbers called *scalars*, given in ordinary letters. Scalars can be real numbers, which we later describe as Eudoxus numbers, but there can also be scalar multiplication by complex numbers, rational numbers or generally any field (described in the next chapter). The operations of vector addition and scalar multiplication satisfy the rules, called axioms

Axiom	*Meaning*
(1) Associativity of addition	**u** + (**v** + **w**) = (**u** + **v**) + **w**
(2) Commutativity of addition	**u** + **v** = **v** + **u**
(3) Identity element for addition	There exists an element **0** belonging to **V**, called the *zero vector*, so that **v** + **0** = **v** for all **v** that belong to **V**.
(4) Inverse elements for addition	For every **v** that belongs to **V**, there exists an element –**v** that belongs to **V** called the additive inverse of **v**, with the property **v** + (-**v**) = **0**.
(5) Compatibility of scalar multiplication with field multiplication	a(b**v**) = (ab)**v**.
(6) Scalar multiplication identity element	There is a scalar 1 satisfying 1**v** = **v** for all **v**.
(7) Distributivity of scalar multiplication with respect to vector addition	a(**u** + **v**) = a**u** + a**v**.
(8) Distributivity of scalar multiplication with respect to field addition	(a + b)**v** = a**v** + b**v**.

2.8. Change of basis.

The representations discussed in this section are not diagonal or antidiagonal, but classical representations by row and column. Change of basis is used in chapter IX to describe the properties of matrix polynomials.

A nonsquare matrix A with u rows and v columns can be multiplied by another nonsquare matrix X with v rows (the same as the number of columns in A) and w columns. The element in the jth row and kth column in the result is found by multiplying each element from the jth row with the corresponding element from the kth column, adding the result together, just as for $n \times n$ matrices.

For example, if u = v = 2 and w = 1, with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix},$$

where **x** with one column is called a *column vector*.

If we start off with a set of independent linear equations, for example

$$a_{11}x_1 + a_{12}x_2 = y_1$$
$$a_{21}x_1 + a_{22}x_2 = y_2,$$

these can be represented by the matrix equation

$A\mathbf{x} = \mathbf{y}$, (1)

where $\mathbf{x}$ and $\mathbf{y}$ are column vectors.

We can change the basis from $\mathbf{x}$ to the linearly independent basis $\mathbf{x}'$ so that

$x_1' = p_{11}x_1 + p_{12}x_2$

$x_2' = p_{21}x_1 + p_{22}x_2$,

so we can define a new matrix P in which

$\mathbf{x}' = P\mathbf{x}$. (2)

If we apply the same transformation to $\mathbf{y}$, we change to a basis in which $\mathbf{y}'$ is represented as

$\mathbf{y}' = P\mathbf{y}$. (3)

From (1) in (3)

$\mathbf{y}' = PA\mathbf{x}$,

then from (2)

$\mathbf{y}' = PAP^{-1}\mathbf{x}'$,

and this corresponds to a transformation of A under change of basis to PAP^{-1}. □

By definition, square matrices B and A are *similar*, when there exists a non-singular matrix P where

$B = PAP^{-1}$.

Our discussion shows that two $n \times n$ matrices A and B are similar if and only if they represent the same linear transformation, in particular a linear transformation represented by a matrix A relative to a basis x_i is represented by a matrix PAP^{-1} in a new basis x'_i where P is non-singular. The algebra of matrices applies to the case of diagonal matrices.

2.9. Classical rules for determinants.

Formulas organised by rows and columns for the solution of simultaneous linear equations lead naturally to determinants. For instance consider the solution of three linear equations

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1$,

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2$,

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = y_3$ (1)

for which by definition the denominator of each solution x_k turns out to be the determinant, in this case

$a_{11}a_{22}a_{33} + a_{12}a_{22}a_{31} + a_{13}a_{31}a_{32}$

$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$.

This consists of six products. Each involves one factor a_{1j} from the first row, one from the second row and one from the third. The same situation occurs for the columns. If we look, say, at the columns with their subscripts 1, 2 and 3, the positive terms occur for even permutations of (1 2 3), that is, by doing an even number of swaps of two elements, and the odd terms occur for odd permutations, with an odd number of swaps.

In general, the determinant of a matrix A satisfies

$\det A = \sum_\sigma (-1)^{N_\sigma} \prod_{j=1}^{n} A(j, \sigma(j))$, (2)

where the sum is taken over all N! permutations σ, $(\sigma(1), \ldots \sigma(n))$ of the sequence of column indices 1, ... n, and where $N\sigma$ is the minimal number of pairwise transpositions needed to transform $\sigma(1), \ldots \sigma(n)$ to 1, ... n.

Theorem 2.9.1. *If* A^T *is the transpose of the matrix* A, *then* $\det A^T = \det A$.

Proof. Swap rows and columns, then (2) may be considered as permutations over rows rather than columns. Even permutations $A(j, \sigma(j))$ on columns map to even permutations $A(\sigma(j), j)$, on rows, and likewise for odd permutations, so the determinant remains invariant on forming the transpose. ☐

Theorem 2.9.2. *If* A *has two rows* (*or columns*) *one a multiple of the other*, *then* $\det A = 0$.

Proof. Make the determinant have two equal rows by dividing the multiple, c, throughout the row and then multiplying det A by c. Swapping the two rows changes the sign of det in (2), but this keeps the determinant the same, so it must be zero. ☐

Theorem 2.9.3. *Adding a constant* c *times row* j *to row* k *leaves* det A *unchanged*.

Proof. Adding one row at j multiplied by c to another at k creates the entry

$$A(k, c\sigma(j) + \sigma(k)) = cA(k, \sigma(j)) + A(k, \sigma(k)) \qquad (3)$$

where $A(j, \sigma(j))$ and $cA(k, \sigma(j))$ have rows differing by a multiple, so by theorem 2.9.2 and definition (2)

$$\det A = \sum_\sigma (-1)^{N_\sigma} \prod_{j\neq k=1}^{n} A(j, \sigma(j)) \prod_{j=k} (cA(k, \sigma(j)) + A(k, \sigma(k))) \qquad (4)$$

where k is held fixed; the result is 0 + det A. ☐

The operation of adding a constant c times row j to row k of theorem 2.9.3 is known as an *elementary operation*. Thus elementary operations leave the determinant unchanged.

Theorem 2.9.4. *If the entries on the main diagonal of an* $n \times n$ *matrix are all non-zero, then elementary operations can always reduce the matrix to main diagonal form*.

Proof. We will choose the typical example of a 4×4 matrix.

$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ r_1 & r_2 & r_3 & r_4 \\ q_1 & q_2 & d_3 & q_4 \\ p_1 & p_2 & p_3 & d_4 \end{bmatrix}$$

Start on the bottom row, one to the left of the rightmost column. Let this element have value p_3. Let the corresponding element on the row above p_3 have value d_3, so d_3 is along the main diagonal of the matrix. To start the inductive proof, multiply the row above p_3 by (p_3/d_3) and subtract that row from the bottom row, so that the bottom row has 0 rather than p_3 in the p_3 position.

Now remove the element q_4 one place above the bottom row at the last column in a similar way.

By the induction hypothesis assume all zeros off the main diagonal from the bottom row up to but not including row j have been zeroised. Then we can repeat the above process for row j, and similarly for the columns to the right of column j

$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ r_1 & r_2 & 0 & 0 \\ q_1 & 0 & d_3 & 0 \\ p_1 & 0 & 0 & d_4 \end{bmatrix}. \ ☐$$

Theorem 2.9.5. *The determinant is multiplicative*:

$$(\det A)(\det A') = \det(AA').$$

Proof. Let the entries on the main diagonal of an n × n matrix be all non-zero. By elementary operations this can be reduced to the main diagonal, with all other elements zero. Then in this case by formula (2) the determinant is multiplicative.

Now assume some entries along the main diagonal are zero.

$$\begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & 0 & r_3 & r_4 \\ 0 & q_2 & d_3 & 0 \\ 0 & p_2 & 0 & d_4 \end{bmatrix}$$

Start with the largest row and column, j, where the diagonal entry is zero. If this is the final entry, swap with another row and change the sign of the determinant. In the typical example above r_3 can be set non-zero if not already the case by adding the row containing d_3, so that positions to the left of r_3 can be zeroised. Then d_3 and d_4 can be used to zeroise firstly q_2 and p_2, then r_3 and r_4 respectively. Then the matrix is diagonal. If the row containing position j is still zero, this is the same for a product with A′, and by formula (2) the determinants of these are zero, so

$$(\det A)(\det A') = \det(AA') = 0. \ \square$$

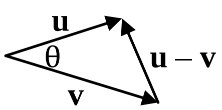
2.10. The determinant as a hypervolume in orthogonal space.

We can multiply a matrix with one row, which is called a row vector, by a matrix with one column, a column vector, for example

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 = \mathbf{u}.\mathbf{v}, \qquad (1)$$

where we call **u**.**v** the *scalar product* of the two vectors **u** and **v**, and we can consider a column vector as the transpose of a row vector, and vice versa.

From looking at the vector diagram



we can construct the scalar product

$$(\mathbf{u} - \mathbf{v}).(\mathbf{u} - \mathbf{v}) = (\mathbf{u}.\mathbf{u}) - 2(\mathbf{u}.\mathbf{v}) + (\mathbf{v}.\mathbf{v}), \qquad (2)$$

where **u**.**u** is the length ||**u**|| squared, given by and restricted to the orthogonal space formula, considered here in 3-dimensional space, but which can be extended to n dimensions

$$\mathbf{u}.\mathbf{u} = u_1^2 + u_2^2 + u_3^2 = \|\mathbf{u}\|^2, \qquad (3)$$

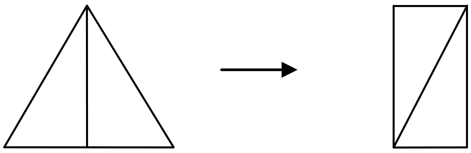
and equation (2) can be compared with the trigonometric cosine formula

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\cos\theta\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2, \qquad (4)$$

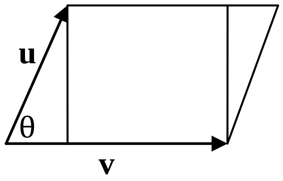
so that we can take as a definition

$$\cos\theta = \frac{(\mathbf{u}.\mathbf{v})}{\|\mathbf{u}\|\|\mathbf{v}\|}. \qquad (5)$$

An isosceles triangle has two equal sides, which is the only case we need. For such a triangle the area shown next is $\frac{1}{2}$base × height:



giving the area of a parallelogram as



$$\text{base} \times \text{height} = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta, \quad (6)$$

so that the square of this area is

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2(1 - \cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}.\mathbf{v})^2, \quad (7)$$
$$= (u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1v_1 + u_2v_2)^2$$
$$= u_1^2v_2^2 + u_2^2v_1^2 - 2u_1u_2v_1v_2, \quad (8)$$

in which (7) is in fact the determinant of the matrix AA^T, where

$$A = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}, \quad (9)$$

but we have proved

$$\det AA^T = (\det A)(\det A^T) = (\det A)^2, \quad (10)$$

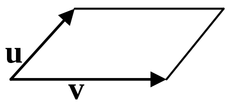
so in two dimensions the determinant is $\pm$ the area of the parallelogram spanned by **u** and **v**.

For more than two dimensions the induction process establishes a corresponding bijection between the determinant of an n-dimensional matrix and an n-dimensional parallelepiped, which is the n-dimensional analogue of a parallelogram. In this case the height is described by a vector at right angles in the induction procedure to a k-dimensional parallelepiped, and the theorem is established for (k + 1). □

2.11. Rank and nullity.

There is a visual description of linear dependence and independence. When two nonzero vectors lie along the same line they are linearly dependent, when a**u** + b**v** = 0 is possible, and when forming a parallelogram they are linearly independent, when a**u** + b**v** = 0 is impossible in both cases for a, b ≠ 0.

linearly dependent



linearly independent

In general when k vectors are linearly independent they form a k-dimensional parallelepiped, but when they are linearly dependent at least one vector lies in a continuation of the hypervolume of a parallelepiped defined by some or all of the other vectors. This means when k vectors are linearly dependent they define a parallelepiped of dimension less than k.

The *rank*, r, of n vectors $\mathbf{v}_1$, $\mathbf{v}_2$ ... $\mathbf{v}_n$ is the maximum number of linearly independent vectors. The *nullity* we say is (n – r), so

rank + nullity = dimension.

If we take (n – r) vectors defining a nullity then they themselves will be linearly dependent or independent, so in this case we have

rank + (rank of nullity) + (nullity of nullity) = dimension.

In general we have

$r_1 + r_2 + ... + r_j$ = dimension,

where r_1 is the rank of the vector space, r_2 is the rank of the nullity, etc.

For a n × n matrix each row defines a row vector. Thus the matrix defines n vectors with a row rank and a row nullity. Thus each of r_1, r_2, ... r_j defines the dimension of a descending sequence of parallelepipeds, where the parallelepipeds of say k linearly independent vectors define a nonzero hypervolume. □

2.12. Eigenvectors and eigenvalues.

Eigenvector and eigenvalue theorems will also be discussed and used in chapters V and VI. A vector **x** with finite basis is an *eigenvector* and λ is an *eigenvalue* of the n × n matrix A if and only if for all linearly independent **x**

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Thus for the unique diagonal matrix I

$$IA\mathbf{x} = A\mathbf{x} = \lambda I\mathbf{x}$$

and

$$(A - \lambda I)\mathbf{x} = 0$$

has dependent rows and therefore by theorems 2.8.3 and 2.8.2

$$\det(A - \lambda I) = 0.$$ □

Similar matrices have the same eigenvalues, since

$$\begin{aligned}\det(A - \lambda I) &= \det(PP^{-1}(A - \lambda I))\\ &= \det(P)\det(A - \lambda I)\det(P^{-1})\\ &= \det(PAP^{-1} - \lambda PIP^{-1})\\ &= \det(PAP^{-1} - \lambda I).\end{aligned}$$ □

By the fundamental theorem of algebra proved in chapter VII section 6 with introductory discussion in sections 4 and 5 of that chapter, if λ is complex but not otherwise intricate, the independent solutions for λ of the equation

$$\det(A - \lambda I) = 0,$$

in number at most the dimension, n, of A, are unique. □

2.13. Compression and expansion.

The *compression* of a v-hyperintricate number from $2^v \times 2^v$ matrix basis elements to $2^w \times 2^w$ basis elements, where we are compressing v – w + 1 vectors, consists in multiplying together in order the vectors to be compressed in the v-hyperintricate algebra.

The compression operation, κ, with abelian addition and non-commutative multiplication, satisfies for basis elements, and correspondingly for composites (we may use here real numbers r and s, although we can incorporate r and s as intricate numbers via an interior or relative coefficient algebra)

$\kappa(rA_B) = r^2AB$,

$\kappa(rA_B + sC_D) = (r^2AB) + (s^2CD)$,

as may be verified using basis element universal objects.

Where B or C are 1 or B = C, we connect compression with matrix multiplication via

$$\kappa(rA_BsC_D) = \kappa(rA_B)\kappa(sC_D) \qquad (1)$$

otherwise for distinct non-real B and C, by non-commutation of basis elements we obtain

$\kappa(rA_BsC_D) = -\kappa(rA_B)\kappa(sC_D)$,

for example

$\kappa[(\alpha_\phi)^2] = -\kappa(\alpha_\phi)\kappa(\alpha_\phi) = -i^2 = 1$.

The zero matrix is compressed to a zero matrix, and the unit matrix to a unit matrix. However $\kappa(rA_0) = 0$, and A_0 is 0, but $\kappa(\alpha_\alpha) = 1$ and $\alpha \neq 1$. Compression is an additive epimorphism (this is a surjective mapping, described in 3.5) from the v-hyperintricate algebra to the w-hyperintricate algebra.

The hyperimaginary, hyperactual and hyperphantom algebras commute, so for these numbers κ is commutative, and (1) always holds.

The equation

$\kappa(A_BC_D) = \kappa(A_C)\kappa(B_D)$,

which is not multiplicative in the usual sense for homomorphisms, and the definition

$\kappa(A_B + C_D) = \kappa(A_B) + \kappa(C_D)$

define a type of ring structure we call a compression ring. There is a unit:

$\kappa(1_1C_D) = 1_1\kappa(1_C)\kappa(1_D) = 1_1\kappa(C_1)\kappa(1_D)$

and the algebra is distributive:

$\kappa[U_V(W_X + Y_Z)] = \kappa[U_VW_X] + \kappa[U_VY_Z]$.

There is an opposite operation, κ^{op}, called *expansion*, so that for expansion

$\kappa^{op}(r^2AB) = rA_B$,

$\kappa^{op}[(r^2AB) + (s^2CD)] = (rA_B) + (sC_D)$.

Let B be an intricate number of the form

$B = b + fJ_1$,

where $J_1^2 = 0$ or ±1, and C be of the form

$C = c + gJ_2$,

where

$J_1J_2 = d – J_2J_1$.

Then

$BC = bc + fcJ_1 + bgJ_2 + fgJ_1J_2$

$CB = bc + fcJ_1 + bgJ_2 + fg(d – J_1J_2)$

so that for some v, w and pure intricate J_3

$\kappa(A_BC_D) = \kappa(A_CB_D) + v\kappa(A_D) + w\kappa[(AJ_3)_D]$.

We have proved that for intricate A, B, C and D there exists a v and intricate X such that

$$\kappa(A_BC_D) = \kappa(A_CB_D) + vX, \qquad (2)$$

conversely that for given A, D a selection of B and C can be made so that for arbitrary v and X, (2) holds. □

We can extend this type of notion of compression *for intricates down to reals* by taking the determinant of the matrix basis elements. First note that the number 1 we have been using is in fact a diagonal 2×2 matrix. To distinguish this from its real value elements, denote the latter occasionally by 1~.

We can now compress intricate basis elements down to ±1~. We have the mappings

$$1 \to 1\sim, i \to 1\sim, \alpha \to -1\sim \text{ and } \phi \to -1\sim,$$

where we denote this compression mapping by λ, so that the determinant

$$\lambda(r1 + s\alpha + t\phi + ui) = [(r + s)(r - s) - (t + u)(t - u)1\sim$$
$$= (r^2 - s^2 - t^2 + u^2)1\sim,$$

and therefore we have proved

$$\lambda(r1 + s\alpha + t\phi + ui) = \lambda(r1) + \lambda(s\alpha) + \lambda(t\phi) + \lambda(ui).$$

However, for the real component, and except possibly for sign similarly for i, α and ϕ

$$\lambda[(r_1 + r_2)1] = \lambda(r_11) + \lambda(r_21) + 2r_1r_21\sim = (r_1 + r_2)^2 1\sim.$$

The expansion λ^{op} is defined in like manner to κ^{op}. ☐

2.14. The hyperintricate trace, determinant and layer algebra.

Let 1, T = a1 + bi + cα + dϕ and U, V, W be intricate matrices. The determinant, det, or hypervolume, is additive for intricate numbers when every basis element for T is linearly independent of those for U: det(T + U) = det(T) + det(U), but not otherwise. This follows from equation 1.6.3 for the intricate determinant. The trace operators, tr1 = 2a, tri = 2bi, trα = 2cα and trϕ = 2dϕ, and det satisfy

det(a = 1, b = 0, c = 0, d = 0) = 1, det(a = 0, b = 1, c = 0, d = 0) = 1,
det(a = 0, b = 0, c = 1, d = 0) = -1 and det(a = 0, b = 0, c = 0, d = 1) = -1
tr1(a = 1, b = b, c = c, d = d) = 2, tri(a = a, b = 1, c = c, d = d) = 2i,
trα(a = a, b = b, c = 1, d = d) = 2α and trϕ(a = a, b = b, c = c, d = 1) = 2ϕ
det(TU) = det(T)det(U)
tr1(T + U) = tr1(T) + tr1(U), etc.
det(tr1(T)) = tr1(T), det(tri(T)) = -tri(T)i,
det(trα(T)) = -trα(T)α, det(trϕ(T)) = -trϕ(T)ϕ
tr1(det(T)) = det(T), tri(det(T)) = 0
trα(det(T)) = 0 and trϕ(det(T)) = 0. ☐

The tensor product may be thought of as a type of product of vectors. The hyperintricate layer operator $\veebar$ behaves as a tensor product with extra structure, where $T_U = T\veebar U$ satisfies [JL09]

$$tr1(T\veebar U) = tr1(T)tr1(U),$$
$$tri(T\veebar U) = -tri(T)tri(U)i, \text{ etc.}$$
$$(T\veebar 1)(1\veebar U) = T\veebar U$$
$$\det(T\veebar 1) = \det(T)$$
$$\det(1\veebar U) = \det(U),$$

so

$$\det(T\veebar U) = \det(T)\det(U).$$

The relation

$$(T + U) \veebar (V + W) = (T\veebar V) + (T\veebar W) + (U\veebar V) + (U\veebar W)$$

can be used with the determinant of the above for linearly independent basis elements between T and U, and also between V and W respectively, or to extend this further, if the basis elements 1, $\mathcal{J}$, $\mathcal{A}$, $\mathcal{F}$ of chapter I, section 8, between T and U and similarly 1, $\mathcal{J}'$, $\mathcal{A}'$, $\mathcal{F}'$ for V and W are linearly independent. Then

$$\det[(T + U) \veebar (V + W)] = [\det(T) + \det(U)][\det(V) + \det(W)],$$

the above generalised accordingly for k layers. But if T = V = 1 and U = W = i, then the determinant, which has dependent rows, is not additive on hyperintricate components for

$$0 = \det[T\veebar V + U\veebar W] \neq \det(T)\det(V) + \det(U)\det(W) = 2. \;\square$$

Since hyperintricate matrices have even dimension, say

$$A = aI = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix},$$

the determinant of a coefficient times a hyperintricate basis element that is not intricate is an even power of real numbers, and so is always positive. This is because beyond the intricate, the 4 × 4 matrix basis elements are in diagonal or antidiagonal 2 × 2 blocks, and in the notation we gave in 2.2, the "+" block could be something like ϕ, the "–" is then $-\phi$, the determinant is -1 in both cases, and we have seen

$$\det(\phi\veebar-\phi) = \det(\phi)\det(-\phi) = 1.$$

The general result for arbitrary hyperintricate numbers follows from a proof by induction.

The determinant of the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

has, by formula 2.9.(4), the value -1. B is the hyperintricate number

$$B = 1_{(1 + \alpha)/2} + \alpha_{(1 - \alpha)/2},$$

or

$$B = ½(1_1 + 1_\alpha + \alpha_1 - \alpha_\alpha).$$

Looking at B as the sum of the matrices C and D

$$B = C + D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

in which opposite alternate rows in C and D are zero, gives a different representation than the hyperintricate one which uniquely defines a matrix. □

2.15. J-abelian hyperintricate determinants and inverses.

Bourbaki writes [Bo73] in the historical note in *Algebra* I, "Toeplitz ... makes the fundamental observation that the theory of determinants is not needed to prove the principal theorems of linear algebra". Positing a greater significance to determinants, as hypervolumes spanned by vectors, we introduce the J-layered approach to the hyperintricate representation, developing this to accommodate the structure of linear algebra in the J-abelian case, leaving a general discussion of determinants and inverses to section 16.

An n-*hyperintricate number is* <u>*J-abelian*</u> *if* U, V, ... W *are intricate numbers for the layers of the* n-*hyperintricate number* $\Sigma U_{V...W}$, *where for each layer the value of* J *is constant* (*but* J *can vary over different layers*), J *is not real and* $J^2 = 0$ *or* ±1.

The n-hyperintricate representation has 4^n independent components, but the number of independent components in a J-abelian n-hyperintricate number $U_{V...W}$ is 4n, and this is maximally incremented for n > 2 by forming the sum $\Sigma U_{V...W}$, there being c = 3n of the J components plus one scalar, plus $\sum_{r=0}^{n-1} \frac{n!}{r!(n-r)!}$ independent mixed J components.

In the tensor form of a 2-hyperintricate number, with $2^2 \times 2^2 = 16$ independent components, this can be represented by the sum of independent J- and J′-abelian tensors

$$m_p + m'_{p'},$$

where both m_p and $m'_{p'}$ separately have $2^2 + 2^2 = 8$ independent components.

For intricate numbers A, B, ... H, the determinant of A = a1 + bi + cα + dϕ satisfies

$$\det A = AA^*,$$

where A* is the intricate conjugate

$$A^* = a1 - bi - c\alpha - d\phi.$$

Since the 2-hyperintricate

$$A_B C_D = (AC)_{(BD)},$$

the compression of these satisfies

$$\kappa(A_B C_D) = ACBD.$$

Thus the determinant of the above satisfies

$$\det[\kappa(A_B C_D)] = \det A \det C \det B \det D$$
$$= \det[\kappa(A_B)]\det[\kappa(C_D)].$$

Compression also satisfies

$$\kappa(E_F + G_H) = \kappa(E_F) + \kappa(G_H)$$
$$= EF + GH,$$

which may be thought of as a definition, so that κ defines a ring epimorphism.

Determinants are not additive and also in general

$$\det(E_F + G_H) \neq \det[\kappa(E_F) + \kappa(G_H)]. \ \square$$

Since

$$A_B = (A_1)(1_B),$$

it follows in this case

$$\det(A_B) = \det[\kappa(A_B)] = \det A \det B.$$

For such numbers

$$(A_B)(A^*_{B^*}) = (AA^*)(BB^*)1_1,$$

which implies

$$(A_B)^{-1} = (A^*_{B^*})/[(AA^*)(BB^*)]. \ \square$$

A second idea is to represent each J-abelian n-hyperintricate number as a sum of $\underline{\vee}$ composite layers, where each $\underline{\vee}$ layer of the composite, $1 \leq k \leq n$, is given by the intricate number $(a_{rk}1 + J_k L_{rk})$, where $J_k = b_k i + c_k \alpha + d_k \phi$ and $J_k^2 = 0$ or ±1.

We will represent this n-hyperintricate number by

$$Я_n = \sum_{r=1}^{\lceil c/4n \rceil} [\underline{\vee}\ (k = 1 \text{ to } n)(a_{rk}1 + J_kL_{rk})],$$

where we are using the ceiling function, $\lceil c/4n \rceil$, the whole number $\geq$ c/4n, and the iterated composite layer operator $\underline{\vee}$.

For each layer we select the value of $J_k \in \{i, \alpha, \phi\}$, or its corresponding ***JAF*** format $J_k \in \{\mathcal{J}, \mathcal{A}, \mathcal{F}\}$, and with $J_k \neq \pm 1$ for any layer, where the J_k are identical over r and independent over k.

We are now able to introduce, for an intricate number X = $(a1 + bJ_k) + (c1 + dJ_k)$ a type of conjugate, $X^{k\sim}$, so that

$$X^{k\sim} = (a1 - bJ_k) + (c1 - dJ_k),$$

which implies that $XX^{k\sim}$ is real. On expanding out $Я_n$, the kth layer is selected so that

$$Я_n^{k\sim} = \sum_{r=1}^{\lceil c/4n \rceil} [$$

$$\underline{\vee}\ (m = 1 \text{ to } n)(a_{rk}1 + J_kL_{rk})(\text{except } m = k \text{ with})(a_{rk}1 - J_kL_{rk})].$$

For each k, $Я_nЯ_n^{k\sim}$ is real in layer k.

Let $Y^1 = Я_nЯ_n^{1\sim}$ and $Y^k = Y^{k-1}(Y^{k-1})^{k\sim}$. We are able to form a real value from the product

$$Y^n = Я_nЯ_n^*,$$

where we have introduced the n-hyperintricate conjugate $Я_n^*$.

The mth power Δ^m of the determinant Δ of $Я_n$, is a multiplicative function. We will allocate

$$Я_nЯ_n^* = \Delta g,$$

where g is a factor.

When $Я_n$ is a real number times a hyperintricate basis element, or a ***JAF*** format extension of this, the value of this factor is 1. If this factor is a constant it is always 1, as can be shown since Δg is a multiplicative function. But if g is not a constant, then the degree of the powers of compressed hyperintricate components in $Я_nЯ_n^*$, 2^n, is not equal to the degree of Δ in its hyperintricate matrix representation, which is not the case. Thus the inverse of $Я_n$ when it exists is $Я_n^*/\Delta$. □

2.16. The general hyperintricate inverse.

If a block diagonal of a 2-hyperintricate number is $1_{Я1} + \alpha_{Я2}$, where $Я_1$ and $Я_2$ are intricate numbers, then we have represented by the matrix

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

that $A = Я_1 + Я_2$ and $D = Я_1 - Я_2$, similarly an antidiagonal $\phi_{Я3} + i_{Я4}$ for the matrix

$$\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$$

gives $B = Я_3 + Я_4$ and $C = Я_3 - Я_4$.

Let A, B, C, D, X, Y and Z be square matrix sub-blocks of the same arbitrary size, and 1 be the unit diagonal matrix. Then since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B \\ C & D \end{bmatrix}, \quad (1)$$

where we can also write

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & A^{-1}BD^{-1} \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}, \quad (2)$$

we obtain from the definition of the column expansion of a determinant

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det (1 - CA^{-1}BD^{-1}))(\det D). \quad (3)$$

$D - CA^{-1}B$ is known as the Schur complement of A.

Equation (2) implies when D and A can be inverted

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 1 & A^{-1}BD^{-1} \\ C & 1 \end{bmatrix}^{-1} \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

We can obtain the block inverse (multiply by its non inverse to check)

$$\begin{bmatrix} 1 & X \\ Y & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -X \\ -Y & 1 \end{bmatrix} \begin{bmatrix} (1 - XY)^{-1} & 0 \\ 0 & (1 - YX)^{-1} \end{bmatrix},$$

which does not exist when X is the inverse of Y, so putting $X = A^{-1}BD^{-1}$ and $Y = C$, by this algorithm of Boltz-Banachiewicz [2Be09] the inverse of the matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

given by

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

when invertible in this way satisfies in terms of A^{-1} and the inverse Schur complement

$$E = (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1},$$

where we have used with $Z = BD^{-1}C$,

$$(A - Z)^{-1} = A^{-1}(1 + Z(A - Z)^{-1}),$$

the remaining entries being

$$F = -(DB^{-1}A - C)^{-1} = -A^{-1}B(D - CA^{-1}B)^{-1},$$
$$G = -(AC^{-1}D - B)^{-1} = -(D - CA^{-1}B)^{-1}CA^{-1},$$
$$H = (D - CA^{-1}B)^{-1}.$$

There exist other solutions by similar methods, to be found in [2Be09] pages 117-118, not directly obtainable by the previous formulas, for instance when det A = 0, with other non-singular combinations involving A, B, C and D.

Thus for n-hyperintricate numbers this operation can be defined recursively. □

We can define the hyperintricate conjugate X* of a hyperintricate number X by the formula

$$XX^* = \det X,$$

and this works for an equivalence class of X* when X is singular, otherwise

$$X^* = X^{-1} \det X. \ □$$

2.17. Exercises.

(A) Obtain the inverse of the J-abelian number

$$(a1 + b\alpha + c\phi)_{(b1 + di + c\alpha)},$$

using intricate conjugates.

(B) Using example (A), what is its determinant by this method? Check that it corresponds with the determinant of section 16, equation (3).

(C) A 4 × 4 *companion matrix* is defined as

$$C = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix}.$$

Show that the determinant

$$\det(C - xI) = \det\begin{bmatrix} -x & 0 & 0 & -a_0 \\ 1 & -x & 0 & -a_1 \\ 0 & 1 & -x & -a_2 \\ 0 & 0 & 1 & -x - a_3 \end{bmatrix}$$

is given by

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + x^4.$$