CHAPTER I

Intricate numbers

1.1. Introduction.

We introduce some basic ideas in mathematics, including a sketch of linear algebra and complex numbers. Complex numbers may be represented by matrices. We develop a full representation of 2 × 2 matrices, which have 4 components rather than the two of complex numbers. These are called *intricate numbers* (otherwise known as split-quaternions or coquaternions), which contain the complex numbers as a subalgebra. We explore the properties of this representation.

1.2. The concrete and the abstract.

When we begin to use numbers, they describe objects we can identify,

1 chick + 3 chicks = 4 chicks

and we learn that the properties of numbers remain the same no matter which objects they are describing. The numbers with the chicks above are a concrete representation of numbers.

When we discard the objects to which numbers apply, this is an abstract approach. What we assume in this eBook is that the reader can cope with the abstraction, which becomes rather general. We need sometimes to plug in numbers to check our calculations, but essentially we will assume that the reader understands that we are using letters for numbers, and does not feel discomforted by it. This way of looking at mathematics is called algebra.

The name *algebra* derives from a term in the book *Kitab al-jabr wa'l muqabala* [K1831], by the ninth-century Uzbek scientist at the court of al-Ma'mum in Baghdad, Mohammed ibn Mûsâ al-Khowârizmi, where he discusses methods for the positive solution of equations.

The laws of *addition* we first come across satisfy the following examples

$$2 + 5 = 7 = 5 + 2$$

and

$$5 + (4 + 3) = 5 + 7 = 12$$
$$= (5 + 4) + 3 = 9 + 3 = 12.$$

Later, we are taught that the same sort of rules apply to any number, and so we are led to represent numbers by letters, and the rules for the letters are called algebra.

So in terms of algebra, the previous rules for addition can be written as

$$a + b = b + a, \qquad (1)$$
$$c + (d + e) = (c + d) + e. \qquad (2)$$

These rules hold if any number is put in place of a, b, c, d or e. So for equation (1) we have previously used a = 2 and b = 5.

The rule (1) for addition is called the *abelian* law, and the rule (2) the *associative* law.

A question we can now ask is, can these rules be derived from simpler ones? In fact there can be nonstandard nonassociative addition, and this is described in chapter III, section 3.

From the laws of addition, we can introduce *multiplication*. So from

$$a + a + ... + a \text{ (for m terms)}$$

we can form the multiple $m \times a$, sometimes called the product $m \times a$, which is often written without the '$\times$' sign as ma.

We have put this abstractly using algebra. Should it be surprising now that multiplication satisfies rules like (1) and (2)?

$$ab = ba, \quad (3)$$

$$a(bc) = (ab)c. \quad (4)$$

For instance

$$2 \times 3 = 6 = 3 \times 2$$

$$5 \times (4 \times 3) = 5 \times 12 = 60$$

$$= (5 \times 4) \times 3 = 20 \times 3 = 60.$$

For multiplication, rule (3) is often called the *commutative* law, and rule (4) is still called the associative law.

There are even rules connecting addition and multiplication:

$$a(b + c) = ab + ac. \quad (5)$$

This rule is called the *distributive* rule. Try an example.

We can also prove a theorem using (3) and (5). From (5)

$$a(b + c) = ab + ac.$$

Then from (3) this is

$$ba + ca,$$

so we are justified in calling this

$$(b + c) a,$$

which also works with examples. Another way of looking at this is (b + c) is another number, called d, and we know by (3)

$$ad = da.$$

It is an interesting fact that the rules (3) and (4) do not always hold. That is, we can construct number systems which are neither commutative nor associative. In this chapter, we will introduce some types of number, called intricate numbers, where the commutative rule (3) fails. But they still satisfy rules (1), (2), (4) and the distributive rule (5).

For addition there is a special element called *zero*, and denoted by 0, satisfying

$$a + 0 = a, \quad (6)$$

and we can see from rule (1) this also satisfies

$$0 + a = a + 0 = a.$$

We can now introduce negative numbers, satisfying

$$a + (-a) = 0 = (-a) + a. \quad (7)$$

For multiplication, there is something similar. There is a special number, called *one*, denoted by 1, satisfying

$$a \times 1 = a = 1 \times a, \quad (8)$$

and we can introduce *divisors*, by the rule

$$a \times \left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right) \times a. \qquad (9)$$

Remarkably there exist number systems with 1 = 0 but 0a = a, and these are not trivial; not every number k in this system is 0. For prime p, n is transformed to k = n/p, so p/p = 1.

From the rules for multiplication, we can repeat the process to obtain *exponentiation*, in the same sort of way that from addition we can repeat the process to form multiplication. So

$$a \times a \times ... \times a \text{ (for m terms)} = a^m.$$

Sometimes in what follows we will be nonstandard and write

$$a^m = a\uparrow m.$$

then in general (using ≠ for *does not equal*)

$$a\uparrow m \neq m\uparrow a,$$

for example

$$2^3 = 8 \neq 3^2 = 9,$$

so exponentiation is not commutative. Also

$$(2\uparrow 3)\uparrow 2 = 8\uparrow 2 = 64$$
$$\neq 2\uparrow(3\uparrow 2) = 2\uparrow 9 = 512,$$

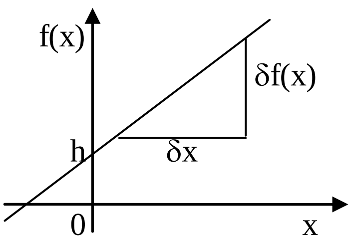
so exponentiation is not associative.

We round off this section by mentioning that it is an objective of this work to describe *superexponential* operations, the first type being addition, the second multiplication obtained from repeated addition, the third exponentiation obtained from repeated multiplication, and to generalise, an (n + 1)th superexponential operation obtained from an nth superexponential operation.

1.3. Linear algebra.

In mathematics the word *linear* comes from the word line, but it is used to describe only *straight* lines.

In the graph below the slope of the straight line



is defined as a change of f(x), δf(x), divided by a change of x, δx, and for a straight line this is always unique and constant, say g

$$\frac{\delta f(x)}{\delta x} = g. \qquad (1)$$

At x = 0 the value of f(x) = f(0) = h. So the equation of the straight line is

$$f(x) = gx + h. \qquad (2)$$

A transformation, or mapping, or function from x to f(x) is called linear if it is in the form (2). Not all functions f(x) are linear. Your lecturer is not entirely made out of straight lines either!

We can have a linear function in more than one variable, for example

$$f(x, y) = g_1x + g_2y + h. \qquad (3)$$

This example describes a flat surface. We can be very general, and describe n-dimensional flat surfaces by increasing the number of variables to n. The equations

$$g_{11}x_1 + g_{12}x_2 + g_{13}x_3 = h_1$$
$$g_{21}x_1 + g_{22}x_2 + g_{23}x_3 = h_2$$
$$g_{31}x_1 + g_{32}x_2 + g_{33}x_3 = h_3 \qquad (4)$$

are an example of *simultaneous* linear equations, and it is the objective of linear algebra to describe their solution.

The mathematician of classical Greek antiquity Archimedes gives a 'cattle problem', which reduced to modern day mathematical notation can be represented by linear equations

W = *number of white bulls*

B = *number of black bulls*

Y = *number of yellow bulls*

D = *number of dappled bulls*

w = *number of white cows*

b = *number of black cows*

y = *number of yellow cows*

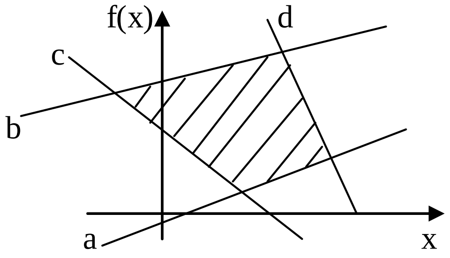
d = *number of dappled cows*

Then the first part of the problem can be stated as the following seven equations in eight unknowns:

(i)	$W = (1/2 + 1/3)B + Y$	(the white bulls were equal to a half and a third of the black bulls together with the whole of the yellow bulls)
(ii)	$B = (1/4 + 1/5)D + Y$	(the black bulls were equal to the fourth part of the dappled bulls and a fifth, together with, once more, the whole of the yellow bulls)
(iii)	$D = (1/6 + 1/7)W + Y$	(the remaining bulls, the dappled, were equal to a sixth part of the white bulls and a seventh, together with all of the yellow bulls)
(iv)	$w = (1/3 + 1/4)(B + b)$	(The white cows were precisely equal to the third part and a fourth of the whole herd of the black)
(v)	$b = (1/4 + 1/5)(D + d)$	(the black cows were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together)
(vi)	$d = (1/5 + 1/6)(Y + y)$	(the dappled cows in four parts, that is in totality, were equal in number to a fifth part and a sixth of the yellow herd)
(vii)	$y = (1/6 + 1/7)(W + w)$	(the yellow cows were in number equal to a sixth part and a seventh of the white herd)

We will see in chapter II that equations (4) and the cattle problem can be described using arrays of numbers called matrices, and columns of numbers called column vectors.

In the graph below a, b, c and d are straight lines.



It is the objective of *linear programming* to find a value that is, say, greater than a, less than b, greater than c and less than d. This set of values is given inside the shaded region.

1.4. Complex numbers.

Complex numbers were first introduced to describe solutions of cubic polynomial equations, that is, to describe those values of x for numbers p and q satisfying the equation

$$x^3 + px + q = 0. \qquad (1)$$


Niccolò Fontana Tartaglia

This equation was solved by mathematician Scipione del Ferro (1465 – 1526). Del Ferro kept his achievement secret until just before his death, when he told his student Antonio Fiore about it, and in 1530 Niccolò Tartaglia (1500–1557) also found the solution. The Italian mathematicians who discovered this solution did not have negative numbers at their disposal, never mind complex ones [Wikipedia]. The solutions of the cubic equation given in chapter VIII are more simply stated using complex numbers.

The idea is this. A positive number squared is positive, and so is a negative number squared. So there appears to be no number whose square is negative. What we do is to introduce it. We put

$$i = \sqrt{-1}, \qquad (2)$$

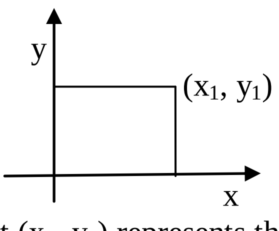
so that

$$i^2 = -1, \qquad (3)$$

and then we can work with complex numbers (a + ib).

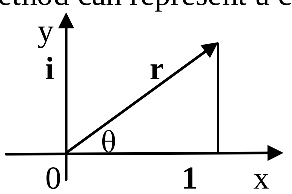
A question raised by early mathematicians was: what does this mean? Rather than answer this question, we reformulate it by describing three methods by which we can represent complex numbers. We will now demonstrate both a geometric and an algebraic representation of these numbers. The matrix approach is given in section 1.6.

We will combine geometry in a plane with an algebra of sets of pairs of numbers in a method introduced by René Descartes, shown below.



The point (x_1, y_1) represents the projection to x_1 on the x axis and y_1 on the y axis. The x and y axes are at right angles, and the projections hit these axes at right angles.

This method can represent a complex number by a vector (in bold)



$$\mathbf{r} = r(\cos\theta)\mathbf{1} + r(\sin\theta)\mathbf{i}, \qquad (4)$$

where **1** and **i** are unit vectors.

The functions cos θ and sin θ are related by the Pythagoras theorem, so writing as is usual $(\cos\theta)^2 = \cos^2\theta$ and $(\sin\theta)^2 = \sin^2\theta$,

$$\cos^2\theta + \sin^2\theta = 1. \tag{5}$$

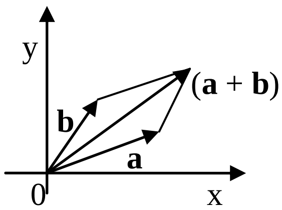
We can represent cos θ and sin θ algebraically, as is done in chapter XV, using the Euler relations

$$\cos\theta = \frac{e^{-i\theta} + e^{i\theta}}{2}, \qquad \sin\theta = i\frac{e^{-i\theta} - e^{i\theta}}{2} \tag{6}$$

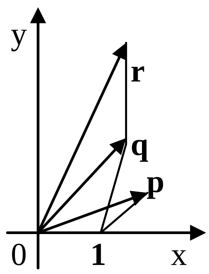
and it can be seen these satisfy (5). This is related to a theorem demonstrated in chapter XV

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{7}$$

In the geometric picture, zero is at the origin of the coordinate system and the sum of two complex numbers is represented by vector addition (this means that the negative of a complex number is the vector going in the opposite direction).



To multiply two complex numbers, we first note that *similar triangles* are defined as to have the same interior angles, but may have different lengths, and even be rotated with respect to one another. We can add a further detail if we wish: reflecting triangles in a mirror image about a line. So in the diagram below the product of **p** and **q** is **r**, **p** subtends an angle η with the x axis, **q** an angle θ with the x axis, **p01** is a similar triangle to **r**0**q** and also **q** is common between two triangles.



So if **p** is represented by p(cos η)**1** + p(sin η)**i**, and **q** by q(cos θ)**1** + q(sin θ)**i**, then from the diagram we have to prove that

$$\mathbf{r} = pq[(\cos\eta)(\cos\theta) - (\sin\eta)(\sin\theta)]\mathbf{1} + pq[(\cos\eta)(\sin\theta) + (\sin\eta)(\cos\theta)]\mathbf{i}, \tag{8}$$

and this can be represented by the formula known to trigonometry

$$\mathbf{r} = pq[\cos(\eta + \theta)]\mathbf{1} + pq[\sin(\eta + \theta)]\mathbf{i}, \tag{9}$$

or equivalently from the algebra of equation (7)

$$\mathbf{r} = (pe^{i\eta})(qe^{i\theta}) = pqe^{i(\eta + \theta)}. \ \square \tag{10}$$

The algebraic approach was introduced by the Irish mathematician William Rowan Hamilton. Again, we treat complex numbers as a set of pairs of numbers, this time with a special sort of algebra. So we write

(a + ib) as (a, b).

Describing the set of complex numbers by ℂ, we have

$$(a, b) + (c, d) \text{ belongs to } \mathbb{C}. \quad (11)$$

$$[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)] \quad (12)$$

$$(a, b) + (c, d) = (c, d) + (a, b) \quad (13)$$

There is a (0, 0) satisfying

$$(a, b) + (0, 0) = (a, b), \quad (14)$$

and there is an additive inverse of (a, b) denoted by (-a, -b), with

$$(a, b) + (-a, -b) = (0, 0). \quad (15)$$

Multiplication belongs to ℂ and satisfies

$$(a, b)(c, d) = (ac – bd, ad + bc), \quad (16)$$

from which it follows that

$$(a, b)[(c, d)(e, f)] = [(a, b)(c, d)](e, f), \quad (17)$$

$$(a, b)(c, d) = (c, d)(a, b), \quad (18)$$

there is a complex number (1, 0) satisfying

$$(1, 0)(a, b) = (a, b), \quad (19)$$

and there exists an inverse $(a, b)^{-1}$ for a and b together ≠ 0, satisfying

$$(a, b)(a, b)^{-1} = (1, 0). \quad (20)$$

This multiplicative inverse is given by

$$(a, -b)/(a^2 + b^2). \; \square \quad (21)$$

1.5. Matrix addition and multiplication.

A square n × n matrix, A, is an array of numbers with n rows and n columns, for example with n = 2

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

where p, q, r and s are numbers. We will say the element at the j^{th} row and k^{th} column starting from 1 is a_{jk}, so in the matrix above

$$a_{11} = p.$$

To add two matrices A and A′, add their corresponding elements. If

$$A' = \begin{bmatrix} p' & q' \\ r' & s' \end{bmatrix},$$

then

$$A + A' = \begin{bmatrix} p + p' & q + q' \\ r + r' & s + s' \end{bmatrix}.$$

To multiply A and A′, for the element in the j^{th} row and k^{th} column, multiply in turn each element from the j^{th} row with the corresponding element from the k^{th} column, adding the results together, so

$$AA' = \begin{bmatrix} pp' + qr' & pq' + qs' \\ rp' + sr' & rq' + ss' \end{bmatrix},$$

where for example the first row of A is $[p \quad q]$ and the first column of A′ is $\begin{bmatrix} p' \\ r' \end{bmatrix}$, so to get the first row and first column element of the matrix, this is (pp′ + qr′).

Note that, in general (see exercise A at the end of this chapter)

$$AA' \neq A'A,$$

and when this happens we say the matrix multiplication is noncommutative.

1.6. Basic properties of intricate numbers.

A complex number, represented by g = a1 + bi, where i = $\sqrt{-1}$, can also be represented by

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where we may multiply the matrix, say 1, by a to form the matrix

$$a1 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

Here a1 is the *real* part and bi the *imaginary* part of the complex number, with $i^2 = -1$. This representation follows all the rules for a *field*, given in chapter III section 4, which defines the rules for addition and multiplication, including the existence of a multiplicative inverse g^{-1} of a nonzero complex number, satisfying $gg^{-1} = 1$, with

$$g^{-1} = (a1 - bi)/(a^2 + b^2).$$

If we wish to extend this algebra to include all possible 2 × 2 matrices with real elements, then we can introduce two more *basis elements* – the *actual* matrix

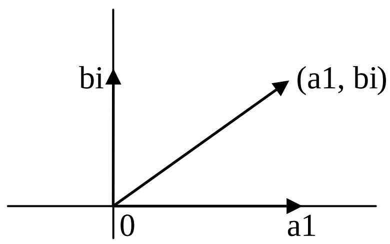
$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the *phantom* matrix

$$\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Just as for complex numbers where we can represent the (a, b) pair of real and imaginary components as vectors in what is called an *Argand diagram*, we can also have a 4-dimensional diagram representing what I call an *intricate number*

$$h = a1 + bi + c\alpha + d\phi.$$



Argand diagram

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ are *linearly independent* if there are no coefficients $a_1, a_2, \ldots a_n$, not all zero, satisfying

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots a_n\mathbf{v}_n = 0.$$

The linearly independent intricate basis elements satisfy

$$1^2 = 1,\ i^2 = -1,\ \alpha^2 = 1,\ \phi^2 = 1,$$
$$1i = i = i1,\ 1\alpha = \alpha = \alpha 1,\ 1\phi = \phi = \phi 1,$$
$$i\alpha = -\phi = -\alpha i,\ i\phi = \alpha = -\phi i \text{ and } \alpha\phi = i = -\phi\alpha. \quad (1)$$

An intricate number can represent uniquely any real 2 × 2 matrix. We will show, and in chapter II extend the ideas below to n × n matrices, that a 2 × 2 real matrix

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

does not have an inverse if its *determinant* det A = ps – rq = 0, in which case it is called a *singular* matrix. For a complex number the basis elements 1 and i have determinant 1. All matrices except the zero matrix for a complex number have multiplicative inverses. We will see in contrast that nonzero intricate numbers may have no multiplicative inverse.

In more detail, the matrix above has the intricate representation

$$h = a1 + bi + c\alpha + d\phi$$
$$= \tfrac{1}{2}(p + s)1 + \tfrac{1}{2}(q - r)i + \tfrac{1}{2}(p - s)\alpha + \tfrac{1}{2}(q + r)\phi. \quad (2)$$

The *intricate conjugate* is (a1 – bi – cα – dϕ). If the multiplicative inverse exists, it is

$$h^{-1} = (a1 - bi - c\alpha - d\phi)/(a^2 + b^2 - c^2 - d^2),$$

so the denominator is non-zero. This denominator is the determinant, because from (2)

$$a^2 + b^2 - c^2 - d^2 = \tfrac{1}{4}\,[(p + s)^2 + (q - r)^2 - (p - s)^2 - (q + r)^2] = ps - rq. \qquad (3)$$

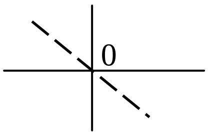
We can verify directly for 2 × 2 matrices A and A′ (A′ could be the zero matrix) that

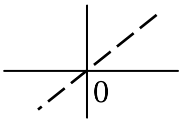
$$(\det A)(\det A') = \det (AA').$$

A matrix is called *nilpotent* if its mth power is zero, which implies that it is singular, proved in the general case by taking an n × n determinant. The singular matrices (ϕ + i) and (α + i) have zero square. □

1.7. Intricate multiplication from the symmetries of a square.

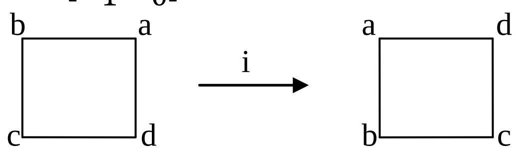
In diagrams we will call a line – – – of the following type on the left a *diagonal*



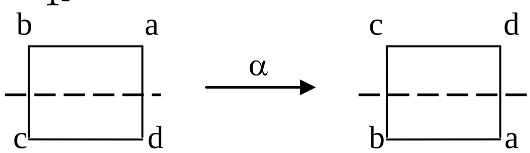


and a line of the type on the right an *antidiagonal*. If 0 is the origin of the coordinate system, the diagonal at an angle of 3π/4 radians anticlockwise from the right horizontal axis, and the antidiagonal at π/4 radians pass through it.

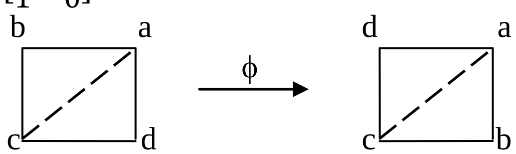
We can represent the group of the symmetries of a square by intricate basis elements. We can represent $i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as a rotation of the square anticlockwise by π/2 radians



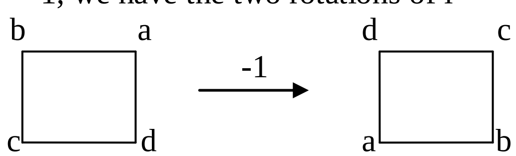
$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ as a reflection about the horizontal axis



and $\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ as a reflection about the antidiagonal



Since $i^2 = -1$, we have the two rotations of i



which is a combined diagonal and antidiagonal, or equivalently a combined horizontal and vertical reflection.

Then, as can be checked, we have the formulas of 1.6.(1). We can represent these formulas by the group multiplication table

$\times$	1	i	α	ϕ
1	1	i	α	ϕ
i	i	-1	$-\phi$	α
α	α	ϕ	1	i
ϕ	ϕ	$-\alpha$	-i	1

and extend the table for multiplication by the further elements -1, -i, $-\alpha$ and $-\phi$.

These are rigid transformations, in that they do not change the shape of the square [Se88]. The group of rigid symmetries of a square is an example of the dihedral group, D_n, the group of all symmetries of a regular polygon with n sides. D_n contains 2n elements generated by a rotation to the next vertex of the polygon and a reflection about an axis of symmetry. □

1.8. Factorisation of intricate numbers.

An intricate number may be represented as $1(p1 + qi + r\alpha + s\phi)$, and 1 may be factorised intricately in an infinite number of distinct ways.

All integers have integer valued intricate factorisations in an infinite number of ways. This follows because

$$(a^2 + b^2 - c^2 - d^2) = (a1 + bi + c\alpha + d\phi)(a1 - bi - c\alpha - d\phi)$$

and any integer may be represented for integers a, b, c, d by $(a^2 + b^2 - c^2 - d^2)$, since

$$(a^2 - c^2) = (a + c)(a - c)$$

so if $(a - c) = 1$, $(a + c)$ can have any odd value, thus looking at $(b^2 - d^2)$, this can have any odd value, and varying over $(a + c)$ any even integer can be found, and likewise, if $(a - c) = 2$, the product with $(a + c)$ forms an arbitrary multiple of 4, so keeping $(b^2 - d^2)$ odd, any odd integer can be found. □

1.9. Intricate products under non-commutation.

Say we wished to find the value of

$$(a + bi + c\alpha + d\phi)(p + qi + r\alpha + s\phi) = (t + ui + v\alpha + w\phi)(a + bi + c\alpha + d\phi). \qquad (1)$$

We can multiply both right hand sides by the intricate conjugate $(a - bi - c\alpha - d\phi)$, or put directly intricate parts equal to give as an intermediate stage

<u>real part</u>

$$a(p - t) - b(q - u) + c(r - v) + d(s - w) = 0 \qquad (2)$$

<u>imaginary part</u>

$$a(q - u) + b(p - t) + c(s + w) - d(r + v) = 0 \qquad (3)$$

<u>actual part</u>

$$a(r - v) + b(s + w) + c(p - t) - d(q + u) = 0 \qquad (4)$$

<u>phantom part</u>

$$a(s - w) - b(r + v) + c(q + u) + d(p - t) = 0, \qquad (5)$$

so on substituting for $(q - u)$, $(r - v)$ and $(s - w)$ from (3), (4) and (5) in equation (2)

$$(a^2 + b^2 - c^2 - d^2)(p - t) + (bc - bc)(s + w)$$
$$+ (bd - bd)(r + v) + (cd - cd)(q + u) = 0 \qquad (6)$$

giving for $G = (a^2 + b^2 - c^2 - d^2)$ with $G \neq 0$

$$t = p, \qquad (7)$$

so that on substituting $t = p$ in (2) to (5) with the result

$$b(q - u) - d(s - w) - c(r - v) = 0,$$
$$a(q - u) + c(s + w) - d(r + v) = 0,$$
$$d(q + u) - b(s + w) - a(r - v) = 0,$$
$$c(q + u) + a(s - w) - b(r + v) = 0,$$

we find (this needs pen and paper to work out; for equation (8) which follows, multiply the first equation by b, the second by a, the third by d and the fourth by c)

$$Gu = q(a^2 + b^2 + c^2 + d^2) + 2[r(-bc - ad) + s(ac - bd)], \quad (8)$$
$$Gv = r(a^2 - b^2 - c^2 + d^2) + 2[q(bc - ad) + s(ab - cd)] \quad (9)$$

and

$$Gw = s(a^2 - b^2 + c^2 - d^2) + 2[q(ac + bd) - r(ab + cd)]. \square \quad (10)$$

1.10. The J representation.

An intricate number $p1 + qi + r\alpha + s\phi = p1 + JK$ satisfies

$$(qi + r\alpha + s\phi)^2 = (\pm qi \pm r\alpha \pm s\phi)^2 = -q^2 + r^2 + s^2. \square$$

See chapter XV for a definition and the properties of cos, sin, cosh and sinh.

When $J^2 = 0$ we obtain for J the parameterisation

$$e^\rho[\pm i \pm \cos\sigma\,\alpha \pm \sin\sigma\,\phi],$$

when $J^2 = -1$

$$\pm\cosh\rho\, i \pm \sinh\rho\cos\sigma\,\alpha \pm \sinh\rho\sin\sigma\,\phi,$$

and when $J^2 = 1$

$$\pm\sinh\rho\, i \pm \cosh\rho\cos\sigma\,\alpha \pm \cosh\rho\sin\sigma\,\phi. \square$$

If $J_1{}^2 = \pm 1$ and $J_2{}^2 = \pm 1$, where $J_1 = bi + c\alpha + d\phi \neq J_2 = qi + r\alpha + s\phi$, then it is possible to write

$$J_1J_2 = a + J_3f,$$

where $J_3{}^2 = 0$ or ± 1 and J_3 is intricate. Then $J_2J_1 = (J_1J_2)^*$, the intricate conjugate, and

$$(J_1J_2)(J_2J_1) = \pm 1 = (a + J_3f)(a - J_3f)$$
$$= a^2 \pm f^2, \text{ or } a^2 \text{ if } J_3{}^2 = 0. \square$$

1.11. Composite intricate basis elements.

Composite basis elements are obtained from other basis elements using operators like + and × and satisfy the same formal properties as the original basis elements.

To determine composite basis elements obtained from addition, first set

$$J_1 = ui + v\alpha + w\phi, \qquad J_1{}^2 = -1 = -u^2 + v^2 + w^2, \quad (1)$$
$$J_2 = xi + y\alpha + z\phi, \qquad J_2{}^2 = -1 = -x^2 + y^2 + z^2. \quad (2)$$

Then for $1 \geq L, M \geq 0$, if the denominator is positive, putting

$$J = (J_1L + J_2M)/\sqrt{[(uL + xM)^2 - (vL + yM)^2 - (wL + zM)^2]},$$

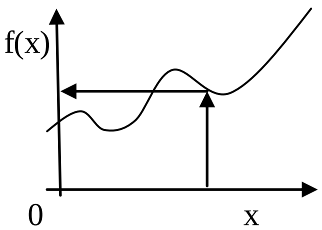
since $i\alpha = -\alpha i$, etc., J satisfies the same type of constraint,

$$J^2 = [J_1{}^2L^2 + (J_1J_2 + J_2J_1)LM + J_2{}^2M^2]/[(uL + xM)^2 - (vL + yM)^2 - (wL + zM)^2],$$

so on using (1) and (2)

$$J^2 = -1.$$

We use next the following notations. A mapping, also called a function, can be represented by a set of *ordered pairs*, shown below in the *graph*



We often muddy the waters when using the word function, as either the ordered pair (x, f(x)) or just f(x). We can write this function as f(x), or in the notation

$$f: x \to f(x),$$

or even as

$$x \overset{f}{\to} f(x).$$

The set of x might be denoted by [0, 1], which is the set of all numbers between 0 and 1, with 0 and 1 included. A mapping is continuous if it is not discontinuous, so there are no gaps greater than a finite constant in adjacent values of f(x) as x is squeezed closer between two values.

If $u > 0$, $x < 0$, a continuous mapping

$$[L, M]: [1, 0] \overset{t}{\to} [0, 1] \qquad (3)$$

carries (uL + xM) impossibly through zero, which also happens in the complex case by setting $v = w = y = z = 0$ in equations (1) and (2), but if u and x are of the same sign, then the positive constraint on the denominator is unnecessary, because $-u^2 + v^2 + w^2 = -1$, $-x^2 + y^2 + z^2 = -1$, so $(uL + xM)^2 > (vL + yM)^2 + (wL + zM)^2$. □

Next, multiplicatively, let $\mathcal{J}^2 = -1$, $\mathcal{A}^2 = 1$ and $\mathcal{F}^2 = 1$, where we put

$$\mathcal{J} = qi + r\alpha + s\phi,$$
$$\mathcal{A} = bi + c\alpha + d\phi,$$
$$\mathcal{F} = ei + f\alpha + g\phi,$$

and we allocate

$$\mathcal{A}\mathcal{F} = \mathcal{J}. \qquad (4)$$

Since $\mathcal{J}$ does not have a real part, it follows from the relations

$-be + cf + dg = 0$, (real part)
$cg – df = q$, (i part)
$bg – de = r$ (α part)

and

$-bf + ce = s$ (ϕ part)

that

$$\mathcal{A}\mathcal{F} = -\mathcal{F}\mathcal{A} = \mathcal{J}. \qquad (5)$$

Multiplying (4) on the left by $\mathcal{A}$

$$\mathcal{F} = \mathcal{A}\mathcal{J},$$

and multiplying on the right by $\mathcal{F}$

$$\mathcal{A} = \mathcal{J}\mathcal{F}.$$

Correspondingly, multiplying (5) on the right by $\mathcal{A}$ and the left by $\mathcal{F}$ gives

$$\mathcal{F} = -\mathcal{J}\mathcal{A},$$
$$\mathcal{A} = -\mathcal{F}\mathcal{J},$$

and we have established an equivalence of algebras for

$\mathcal{J} \leftrightarrow i$,

$\mathcal{A} \leftrightarrow \alpha$

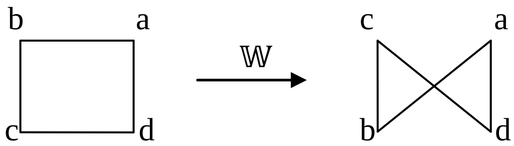
and

$\mathcal{F} \leftrightarrow \phi$. □

1.12. Nonrigid transformations of intricate type.

We can change a square nonrigidly, and these transformations correspond to number systems.

The twisted transformation, $\mathbb{W}$,



reduces the area of the unit square in orthogonal space, where an n-dimensional Pythagoras's theorem holds, to half its area as a twisted object, but we can also embed the square in 3 dimensions, rotating a line by π radians on moving its centre along a line at right angles, as a result modifying its area because of the twist I speculate to at least $2^{-3/2}\pi$ units.

The intricate twisted $\mathbb{W}$ operator may be adjoined to an intricate number

$$h' = a'1 + b'i + c'\alpha + d'\phi$$

to form

$$h'\mathbb{W} = \mathbb{W}h'.$$

This number intersects any intricate number h nowhere except at 0, so except at 0 it forms a separate equivalence class, or partition. The twisted $\mathbb{W}$ operator satisfies

$$\mathbb{W}^2 = 1,$$

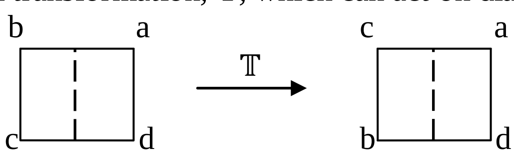
and we can construct numbers of the type

$$h + h'\mathbb{W},$$

where h and h′ are intricate and this number is uniquely expressed, even at 0 when h = h′ = 0.

We will see in chapter II that numbers of this extended intricate type may be represented hyperintrically. □

The torn transformation, $\mathbb{T}$, which can act on diagonals, or vertical and horizontal axes



is shown above with a vertical axis, in which the entire left hand side is reflected about a horizontal axis, the right hand side remaining fixed. For this transformation we need to say which side, left or right, the vertical axis belongs to. The torn transformation retains the area of the torn square. It is nonrigid because the relationship of points in the square changes relative to the square. □

1.13. Exercises.

(A) Check

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & ac \\ bd & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & cb \\ ad & 0 \end{bmatrix}.$$

(B) Find a way to memorise the intricate basis element equations (1) of section 1.6.

(C) Define triticate numbers by the following basis elements (ω is a third root of unity = $e^{2\pi i/3}$ = $\cos(2\pi/3) + i\sin(2\pi/3)$, where $\omega^3 = 1$).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix}.$$

Show this basis is linearly independent.

(D) Are there any other representations like the above using cube roots of unity that you can construct?

(E) Develop the theory of triticate numbers by analogy or otherwise with the treatment of intricate numbers.

(F) Are there other representations, say for penticate numbers for prime p = 5, other prime numbers, and composite numbers (a product of primes)?

(G) Look up the class number on the internet (the book by Conway and Guy, *The book of numbers* [CG00], is also a good reference). What sort of divisors of these numbers (we might call them polyticate numbers) are there?

(H) Verify for 2×2 matrices A and A′ that

$$(\det A)(\det A') = \det(AA').$$

Show $(1 + \phi)^2 = 2(1 + \phi)$.

Show that a 2×2 matrix is nilpotent (meaning a power of the matrix is zero) implies that it is singular, but the reverse implication does not hold: $(1 + \phi)$ is singular but not nilpotent.