CHAPTER IV

Quadronacci numbers

4.1. Introduction.

We discuss generalised Quadronacci numbers – each number is the sum of the previous four, and generalised other such sequences – Fibonacci, Lucas and Tribonacci numbers.

4.2. Fibonacci and Lucas sequences.

Almost all our results are based on extensions of the simple cyclotomic rule

$$\gamma^p - \delta^p = (\gamma - \delta)(\gamma^{p-1} + \gamma^{p-2}\delta + \gamma^{p-3}\delta^2 + \ldots + \delta^{p-1}).$$

We will now see how these ideas recur in defining *Fibonacci* numbers f_p and *Lucas* numbers l_p.

p =	0	1	2	3	4	5	6	7	8	9
f_p =	0	1	1	2	3	5	8	13	21	34
l_p =	2	1	3	4	7	11	18	29	47	76

Each Fibonacci and Lucas number is the sum of the previous two. They can be derived from the following sequence

$$1 \qquad x \qquad x^2 \qquad x^3 \qquad x^4 \qquad x^5 \qquad \ldots$$

where $x^2 = x + 1$, which by the quadratic formula has roots

$$\tau = (1 + \sqrt{5})/2, \qquad \sigma = (1 - \sqrt{5})/2,$$

so that

$$\tau - \sigma = \sqrt{5} \text{ and } \tau + \sigma = 1.$$

Then linear combinations of τ^p and σ^p are also such sums, and

$$\begin{aligned} f_p &= (\tau^p - \sigma^p)/(\tau - \sigma) \\ &= \{((1 + \sqrt{5})/2)^p - ((1 - \sqrt{5})/2)^p\}/\sqrt{5} \\ &= \tau^{p-1} + \tau^{p-2}\sigma + \ldots + \sigma^{p-1} \\ &= \Pi(s = 1, p - 1)[\tau + \sigma(e\uparrow i2\pi s/p)]. \end{aligned}$$

The Lucas number is

$$\begin{aligned} l_p &= (\tau^p + \sigma^p)/(\tau + \sigma) \\ &= \{((1 + \sqrt{5})/2)^p + ((1 - \sqrt{5})/2)^p\}. \end{aligned}$$

For odd p we have seen this is just

$$l_p = \tau^{p-1} - \tau^{p-2}\sigma + \ldots + \sigma^{p-1}$$

and for any p, since $\tau + \sigma = 1$

$$l_p = \Pi(s = 0, p - 1)[\tau - \sigma(e\uparrow i\pi(2s + 1)/p)]. \blacksquare$$

It is also possible to produce sequences in which each number is the sum of a factor times the number before it plus a factor times the number two places before it. By analogy with our treatment of the Fibonacci and Lucas sequences, derived from the sequence

$$1 \qquad x \qquad x^2 \qquad x^3 \qquad x^4 \qquad x^5 \qquad \ldots$$

we now have

$$x^2 = vx + w,$$

where the roots τ_x and σ_x satisfy

$$\tau_x = (v + \sqrt{(v^2 + 4w)})/2, \qquad \sigma_x = (v - \sqrt{(v^2 + 4w)})/2).$$

Once again, linear combinations of τ_x^p and σ_x^p are also such sums, and

$$f_{xp} = (\tau_x^p - \sigma_x^p)/(\tau_x - \sigma_x)$$
$$= \{(v + \sqrt{(v^2 + 4w)})^p - (v - \sqrt{(v^2 + 4w)})^p\}/2^p\sqrt{(v^2 + 4w)}$$
$$= \tau_x^{p-1} + \tau_x^{p-2}\sigma_x + \ldots + \sigma_x^{p-1}$$
$$= \Pi(s = 1, p - 1)[\tau_x + \sigma_x(e\uparrow i2\pi s/p)].$$

The generalised Lucas number is

$$l_{xp} = (\tau_x^p + \sigma_x^p)/(\tau_x + \sigma_x)$$
$$= \{(v + \sqrt{(v^2 + 4w)})^p + (v - \sqrt{(v^2 + 4w)})^p\}/2^p v,$$

which for odd p satisfies

$$l_{xp} = \tau_x^{p-1} - \tau_x^{p-2}\sigma_x + \ldots + \sigma_x^{p-1}$$

and for any p

$$l_{xp} = \Pi(s = 0, p - 1)[\tau_x - \sigma_x(e\uparrow i\pi(2s + 1)/p)]/v.$$

Let the first element of the sequence be K and the second L, then

$$g_{xp} = (\alpha\tau_x^p + \beta\sigma_x^p)/(\tau_x + \sigma_x)$$

satisfies

$$\alpha = (vK/2) + [v(L - vK)/2\sqrt{(v^2 + 4w)}]$$

and

$$\beta = (vK/2) - [v(L - vK)/2\sqrt{(v^2 + 4w)}]. \blacksquare$$

We can generalise these results in a different direction, in the first instance by considering the sequence in which $x^3 = x^2 + x + 1$. This has the solutions [24]

$$\tau = (1/3)[1 + (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3}],$$
$$\sigma = (1/3)[1 + \omega_3(19 + 3\sqrt{33})^{1/3} + \omega_3^2(19 - 3\sqrt{33})^{1/3}],$$
$$\rho = (1/3)[1 + \omega_3^2(19 + 3\sqrt{33})^{1/3} + \omega_3(19 - 3\sqrt{33})^{1/3}],$$

where $\omega_3 = ½[-1 + i\sqrt{3}]$.

We can consider linear combinations of powers of the above, in which for example

$$\tau + \sigma + \rho = 1,$$

when we get the generalised Lucas formula

$${}_3l_p = (\tau^p + \sigma^p + \rho^p)/(\tau + \sigma + \rho)$$
$$= (1/3)^p\{[1 + (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3}]^p +$$
$$[1 + \omega_3(19 + 3\sqrt{33})^{1/3} + \omega_3^2(19 - 3\sqrt{33})^{1/3}]^p +$$
$$[1 + \omega_3^2(19 + 3\sqrt{33})^{1/3} + \omega_3(19 - 3\sqrt{33})^{1/3}]^p\},$$

with values

p =	0	1	2	3	4	5	6	7	8	9
${}_3l_p$ =	3	1	3	7	11	21	39	71	131	241

We now generalise these numbers to ${}_3g_p$, given by the linear combination

$${}_3g_p = (\alpha\tau^p + \beta\sigma^p + \gamma\rho^p)/(\tau + \mu\sigma + \nu\rho).$$

So, for p = 0, suppose ${}_3g_p$ equals the integer K. Then

$$K(\tau + \mu\sigma + \nu\rho) = (\alpha + \beta + \gamma).$$

For p = 1, we choose ${}_3g_p$ as the integer L, giving

$$(L - \alpha)\tau + (L\mu - \beta)\sigma + (L\nu - K(\tau + \mu\sigma + \nu\rho) + \alpha + \beta)\rho = 0.$$

If we allow μ and ν to be free variables, we have now introduced a constraint by selecting L. For example, we could evaluate γ in terms of α and β:

$$\gamma = [\alpha(L - K\tau) + \beta(L - K\sigma)]/(K\rho - L).$$

For p = 2, we allocate ${}_3g_p$ the integer value M. This additional constraint allows us to express β in terms of α:

$$\alpha[(M - \tau L)\tau(K\rho - L) + (L - K\tau)(M - \rho L)\rho]$$
$$+ \beta[(M - \sigma L)\sigma(K\rho - L) + (L - K\sigma)(M - \rho L)\rho] = 0.$$

We can now fix α by choosing suitable values of μ and ν. To simplify matters, we choose $\mu = \nu = 1$, as in the previous generalised Lucas sequence. Then $\tau + \sigma + \rho = 1$.

Thus the p = 0 case gives

$$\gamma = K - \alpha - \beta$$

and the p = 1 case then gives

$$\beta = [\alpha(\rho - \tau) + L - K\rho]/(\sigma - \rho),$$

leading to

$$\alpha(\tau - \sigma)(\tau - \rho) = M - L(\sigma + \rho) + K\rho\sigma.$$

If we put

$$T = (19 + 3\sqrt{33})^{1/3},$$
$$U = (19 - 3\sqrt{33})^{1/3},$$

then

$$\tau = (1/3)[1 + T + U]$$
$$\sigma = (1/3)[1 + \omega_3 T + \omega_3^2 U]$$
$$\rho = (1/3)[1 + \omega_3^2 T + \omega_3 U],$$

and using UT = 4 gives

$$\alpha[T^2 + U^2 + 4] = 3M - L(2 - T - U) + K(T^2 + U^2 - T - U - 3)/3.$$

We note that

$$[T^2 + U^2 + 4] = [T^6 - 64]/T^2[T^2 - 4]$$
$$= [T^3 - U^3]/[T - U]$$
$$= 198/[T - U]\sqrt{33},$$

so it is possible to get an expression for α with all surds in the numerator:

$$\alpha = (K/3) + \sqrt{33}.[T - U][3M + L(T + U - 2) - K(T + U + 7)/3]/198,$$

giving

$$\beta = (K/3) + \sqrt{33}.[\omega_3 T - \omega_3^2 U]$$
$$[3M + L(\omega_3 T + \omega_3^2 U - 2) - K(\omega_3 T + \omega_3^2 U + 7)/3]/198.$$

The expression for γ is likewise obtained from that for α under the transformation

$$\alpha \to \gamma: T \to \omega_3^2 T, U \to \omega_3 U.$$

Thus it is possible to have integer generalised ${}_3g_p$ sequences satisfying the linear combinations of x^3 = the linear combinations of $(x^2 + x + 1)$, that is, a sequence

p =	0	1	2	3	4	5
${}_3g_p$ =	K	L	M	K + L + M	K + 2L + 2M	2K + 3L + 4M. ■

4.3. Tribonacci sequences.

We now generalise as we did for the previous example involving a quadratic, to consider the cubic equation

$$x^3 = ux^2 + vx + w,$$

which, with $x = y + u/3$, may be written as

$$y^3 + (-u^2/3 - v)y - (2u^3/27) - (vu/3) - w = 0,$$

where we will use the variables

$$q = -u^2/3 - v$$

and

$$r = -(2u^3/27) - (vu/3) - w.$$

The solution of this equation is

$$x = (u/3) + \varepsilon[1/2(-r + \sqrt{}(r^2 + 4q^3/27))]^{1/3}$$
$$+ \mu[1/2(-r - \sqrt{}(r^2 + 4q^3/27))]^{1/3},$$

where the $\{\varepsilon, \mu\}$ pairings are $\{1, 1\}$, $\{\omega_3, \omega_3^2\}$ or $\{\omega_3^2, \omega_3\}$.

Denoting the three solutions for x by τ_x, σ_x and ρ_x respectively, we have

$$\tau_x = (u/3)[1 + T_x + U_x],$$
$$\sigma_x = (u/3)[1 + \omega_3 T_x + \omega_3^2 U_x],$$
$$\rho_x = (u/3)[1 + \omega_3^2 T_x + \omega_3 U_x],$$

where

$$uT_x = [1/2(2u^3 + 9vu + 27w$$
$$+ \sqrt{}(-27u^2v^2 + 486uvw + 108u^3w + 729w^2 - 108v^3))]^{1/3}$$

and

$$uU_x = [1/2(2u^3 + 9vu + 27w$$
$$- \sqrt{}(-27u^2v^2 + 486uvw + 108u^3w + 729w^2 - 108v^3))]^{1/3}.$$

Then $u^2T_xU_x = -3q$, so

$$T_xU_x = 1 + 3(v/u^2).$$

So we investigate the allocation

$${}_3g_{xp} = (\alpha\tau_x^p + \beta\sigma_x^p + \gamma\rho_x^p)/(\tau_x + \sigma_x + \rho_x),$$

where

$$\tau_x + \sigma_x + \rho_x = u.$$

For p = 0, we set ${}_3g_{xp} = K$, so

$$K = (\alpha + \beta + \gamma)/u,$$

that is,

$$\gamma = uK - \alpha - \beta .$$

For p = 1, we put ${}_3g_{xp} = L$, so

$$L = (\alpha\tau_x + \beta\sigma_x + \gamma\rho_x)/u,$$

giving

$$\beta = [\alpha(\rho_x - \tau_x) + uL - uK\rho_x]/(\sigma_x - \rho_x).$$

For p = 2, the allocation is ${}_3g_{xp} = M$, which implies

$$\alpha(\tau_x^2(\sigma_x - \rho_x) + (\rho_x - \tau_x)\sigma_x^2 - \rho_x^2(\sigma_x - \rho_x) - (\rho_x - \tau_x)\rho_x^2] =$$
$$u\{M - L(\sigma_x^p - \rho_x) + [-L + K\rho_x]\sigma_x^2 - L\rho_x^2(\sigma_x - \rho_x^p) - [-L + L\rho_x]\rho_x^2\},$$

or

$$\alpha(\tau_x - \sigma_x)(\tau_x - \rho_x) = u\{M - L(\sigma_x^p - \rho_x) + K\rho_x\sigma_x\}.$$

Expressing the left hand side in terms of T_x and U_x gives

$$\alpha u^2(T_x^2 + T_xU_x + U_x^2)/3,$$

whereas the right hand side is

$$u\{M - Lu(2 - T_x - U_x)/3 + Ku^2(1 - T_x - U_x + T_x^2 - T_xU_x + U_x^2)/9\}.$$

Now

$$T_x^3 - U_x^3 = [T_x - U_x][T_x^2 + T_xU_x + U_x^2],$$

and we will also need

$$T_x^3 + U_x^3 = [T_x + U_x][T_x^2 - T_xU_x + U_x^2].$$

Thus

$$\alpha u[T_x^3 - U_x^3]/3[T_x - U_x] =$$
$$\{M - Lu(2 - T_x - U_x)/3 + Ku^2(1 - T_x - U_x + [T_x^3 + U_x^3]/ [T_x + U_x])/9\},$$

or

$$\alpha = [T_x - U_x]/[T_x^3 - U_x^3]/\{3M/u - L(2 - T_x - U_x)$$
$$+ Ku(1 - T_x - U_x + [T_x^3 + U_x^3]/ [T_x + U_x])/3\}.$$

The expression for β is obtained from that for α under the transformation

$$\alpha \to \beta: T_x \to \omega_3T_x, U_x \to \omega_3^2U_x,$$

and the expression for γ is likewise obtained from that for α under the transformation

$$\alpha \to \gamma: T_x \to \omega_3^2T_x, U_x \to \omega_3U_x.$$ ■

4.2. Quadronacci sequences.

The quartic equation $x^4 = x^3 + x^2 + x + 1$ is also solvable. Putting $x = y + ¼$ gives

$$f(y) = y^4 - (11/8)y^2 - (13/8)y - 339/4^4 = 0,$$

which can be put in the form

$$f(y) = (y^2 - jy + m)(y^2 + jy + n).$$

Then

$$2m = j^2 - 11/8 - 13/(8j),$$
$$2n = j^2 - 11/8 + 13/(8j)$$

and

$$h(j^2) = j^6 - (11/4)j^4 + (115/4^2)j^2 - 13^2/4^3 = 0.$$

Let the resolvent cubic of the above equation be

$$-h(-j^2) = -h(z) = z^3 + az^2 + bz + c$$

with $a = 11/4$, $b = 115/4^2$ and $c = 13^2/4^3$, giving a discriminant for f(y) of

$$D = -16a^4c + 4a^3b^2 + 128a^2c^2 - 144ab^2c + 27b^4 - 256c^3 = 1{,}438{,}756{,}811/4^8.$$

Since the discriminant of the resolvent cubic, D_{res}, is minus the discriminant of f(y), so $D_{res} < 0$, the resolvent cubic has exactly one real root, the roots of j^2 being minus these roots.

The solutions for x are now

$$\tau = ¼ + ½(-j + \sqrt{(11/4 - j^2 - 13/(4j))})$$
$$\sigma = ¼ + ½(-j - \sqrt{(11/4 - j^2 - 13/(4j))})$$
$$\rho = ¼ + ½(j + \sqrt{(11/4 - j^2 + 13/(4j))})$$

and

$$\lambda = ¼ + ½(j - \sqrt{(11/4 - j^2 + 13/(4j))}),$$

where we can choose the positive sign in $\pm\sqrt{j^2}$ from

$$j^2 = (1/3)[11/4 + B + C],$$
$$j^2 = (1/3)[11/4 + \omega_3 B + \omega_3^2 C]$$

or

$$j^2 = (1/3)[11/4 + \omega_3^2 B + \omega_3 C],$$

with real values

$$B = [½(-65 + 3\sqrt{[(3)(563)]})]^{1/3} \approx 3.077469944$$

and

$$C = [½(-65 - 3\sqrt{[(3)(563)]})]^{1/3} \approx -4.549191464,$$

(so $BC = -14$).

To simplify subsequent manipulations, we will also put

$$R = \sqrt{(11/4 - j^2 - 13/(4j))}$$

and

$$S = \sqrt{(11/4 - j^2 + 13/(4j))}.$$

We now choose, analogously to the previous example,

$${}_4g_p = (\alpha\tau^p + \beta\sigma^p + \gamma\rho^p + \delta\lambda^p)/(\tau + \mu\sigma + \nu\rho + \xi\lambda),$$

under the simplified allocation $\mu = \nu = \xi = 1$, and noting that $\tau + \sigma + \rho + \lambda = 1$.

Then the $p = 0$ instance gives, for integer K,

$$K = \alpha + \beta + \gamma + \delta.$$

For $p = 1$, we choose ${}_4g_p$ as the integer L, obtaining

$$L = ½[½(\alpha + \beta + \gamma + \delta) + (-\alpha - \beta + \gamma + \delta)j + (\alpha - \beta)R + (\gamma - \delta)S]$$

or

$$L = ½[½K + (2\gamma + 2\delta - K)j + (2\alpha + \gamma + \delta - K)R + (\gamma - \delta)S].$$

For $p = 2$, we choose an integer ${}_4g_p = M$, so

$$M = ¼[¼(\alpha + \beta + \gamma + \delta) + (-\alpha - \beta + \gamma + \delta)j + (\alpha + \beta + \gamma + \delta)j^2$$
$$+ (\alpha - \beta)(1 - 2j)R + (\gamma - \delta)(1 + 2j)S + (\alpha + \beta)R^2 + (\gamma + \delta)S^2]$$

or

$$M = [L - (K/4)(½ - j)](½ - j) + (\gamma + \delta)j^2$$
$$- ¼(\gamma + \delta - K)R^2 + (\gamma - \delta)jS + ¼(\gamma + \delta)S^2.$$

Then, for $p = 3$, we finally assign the integer ${}_4g_p = N$, which gives

$$N = (1/8)[(1/8)(\alpha + \beta + \gamma + \delta) + (3/4)(-\alpha - \beta + \gamma + \delta)j + (3/2)(\alpha + \beta + \gamma + \delta)j^2$$
$$+ (-\alpha - \beta + \gamma + \delta)j^3 + 3(\alpha - \beta)(¼ - j + j^2)R + 3(\gamma - \delta)(¼ + j + j^2)S$$
$$+ 3(\alpha + \beta)(½ - j)R^2 + 3(\gamma + \delta)(½ + j)S^2 + (\alpha - \beta)R^3 + (\gamma - \delta)S^3],$$

that is

$$8N = [6L - K(1 - 2j)](½ - j)^2 + (2L + K(1 - 2j))R^2$$
$$+ (\gamma + \delta)((3/2) - j)(4j^2 - R^2)$$
$$+ 3(\gamma + \delta)(½ + j)S^2 + (\gamma - \delta)(6j - R^2 + S^2)S.$$

We observe that, if we apply the transformation $S \to -S$, each of the equations for K, L, M and N are invariant under the swap transformations $\gamma \to \delta$ and $\delta \to \gamma$. Also these equations are invariant under the swap transformations $\alpha \to \beta$, $\beta \to \alpha$, if we combine them with the substitution $R \to -R$. The simultaneous transformations $\alpha \to \gamma$, $\gamma \to \alpha$, $\beta \to \delta$, $\delta \to \beta$ in combination with the transformations $j \to -j$, $-j \to j$, $R \to S$, $S \to R$ likewise leave the formulas invariant.

Hence if we obtain a symbolic formula for δ, we can find the formula for γ from it by substituting each occurrence of S by –S in the equation for δ. Similarly, if we obtain a formula for α, the corresponding formula for β is derived by substituting R by –R in the α equation. Moreover, we can obtain the equation for α from the equation for γ by applying for every j, R and S, the transformations $j \to -j$, $-j \to j$, $R \to S$, and $S \to R$.

Since swapping j and -j swaps the values of S and R when these are expanded out in terms of j, the formula described above for α (or by similar methods for β), obtained under a j, -j swap, is still valid after eliminating R^2 and S^2 by substituting expressions that involve j.

The equation for M we now put in the form

$$P\gamma = E + F\delta$$

where

$$P = j^2 + jS - \tfrac{1}{4}R^2 + \tfrac{1}{4}S^2,$$
$$E = M - [L - (K/4)(\tfrac{1}{2} - j)](\tfrac{1}{2} - j) - \tfrac{1}{4}KR^2$$

and

$$F = -j^2 + jS + \tfrac{1}{4}R^2 - \tfrac{1}{4}S^2.$$

The equation for N we allocate as

$$Q\gamma = G + H\delta$$

where

$$Q = ((3/2) - j)(4j^2 - R^2) + 3(\tfrac{1}{2} + j)S^2 + (6j - R^2 + S^2)S,$$
$$G = 8N - [6L - K(1 - 2j)](\tfrac{1}{2} - j)^2 - [2L + K(1 - 2j)]R^2$$

and

$$H = ((-3/2) + j)(4j^2 - R^2) - 3(\tfrac{1}{2} + j)S^2 + (6j - R^2 + S^2)S.$$

Then

$$\delta = [EQ - GP]/[HP - FQ].$$

We establish that

$$P = -F + 2jS$$

and

$$Q = -H + 2(6j - R^2 + S^2)S,$$

so

$$\begin{aligned} HP - FQ &= 2S[Hj - F(6j - R^2 + S^2)] \\ &= 2S\{[-2j^2 + (R^2/2) + (S^2/2)]^2 - R^2S^2\} \\ &= 2S\{8j^4 - 11j^2 + (13/(4j))^2\}. \end{aligned}$$

Expanding EQ – GP by separating out terms in K, L, M and N, gives

$$\begin{aligned} EQ - GP = {} & K[(Q/4) - (1 - 2j)P][(\tfrac{1}{2} - j)^2 - R^2] \\ & + L[(\tfrac{1}{2} - j)(-Q + 6(\tfrac{1}{2} - j)P) + 2R^2P] \\ & + MQ - 8NP, \end{aligned}$$

or

$$EQ - GP = (K/4)[2(1 + 2j)j^2 + ½(-R^2 + S^2) + j(-R^2 + 5S^2) + 2(1 + 4j)jS + (-R^2 + S^2)S][(½ - j)^2 - R^2] + (L/2)[(-3 + 4j + 4j^2)j^2 + (3/4)(R^2 - S^2) + j(-1 + 3j)(R^2 + 3S^2) + 3(-1 + 4j^2)jS + R^2(S^2 - R^2) + (1 + 2j)R^2S + (-1 + 2j)S^3] + M[((3/2) - j)(4j^2 - R^2) + 3((1/2) + j)S^2 + (6j - R^2 + S^2)S] + 2N[-4j^2 - 4jS + R^2 - S^2],$$

and substituting for R and S in terms of j, reduces the formula to

$$EQ - GP = (K/4)[(39/2) + 11j + 2j^2 + (13/(4j)) + [2j + 8j^2 + 13/(2j)]S][-5/2 - j + 2j^2 + (13/(4j))] + L[(-13/4) + j + 15j^2 + 4j^3 - 4j^4 + 13/(2j) - 169/(16j^2) + [(4j + 4j^3 - 13/(4j)]S] + M[(13/2) + 11j + 6j^2 - 8j^3 + 39/(4j) + [6j + 13/(2j)]S] + N[-8j^2 - (13/j) - 8jS].$$

Thus

$$\delta = [EQ - GP]/[2S\{8j^4 - 11j^2 + (13/(4j))^2\}].$$

We have similarly

$$\gamma = [EH - GF]/[HP - FQ].$$

An analogous argument to that for EQ – GP yields

$$EH - GF = K[(H/4) - (1 - 2j)F][(½ - j)^2 - R^2] + L[(½ - j)(-H + 6(½ - j)F) + 2R^2F] + MH - 8NF,$$

giving, either under the substitution $S \rightarrow -S$ for δ, noting the denominator contains S, or directly

$$EH - GF = (K/4)[-2(1 + 2j)j^2 + ½(R^2 - S^2) + j(R^2 - 5S^2) + 2(1 + 4j)jS + (-R^2 + S^2)S][(½ - j)^2 - R^2] + (L/2)[(3 - 4j - 4j^2)j^2 + (3/4)(-R^2 + S^2) - j(-1 + 3j)(R^2 + 3S^2) + 3(-1 + 4j^2)jS - R^2(S^2 - R^2) + (1 + 2j)R^2S + (-1 + 2j)S^3] + M[((-3/2) + j)(4j^2 - R^2) - 3((1/2) + j)S^2 + (6j - R^2 + S^2)S] + 2N[4j^2 - 4jS - R^2 + S^2],$$

and substituting for R and S in terms of j, enables us to deduce

$$EH - GF = (K/4)[(-39/2) - 11j - 2j^2 - (13/(4j)) + [2j + 8j^2 + 13/(2j)]S][-5/2 - j + 2j^2 + (13/(4j))] + L[(13/4) - j - 15j^2 - 4j^3 + 4j^4 - 13/(2j) + 169/(16j^2) + [(4j + 4j^3 - 13/(4j)]S] + M[(-13/2) - 11j - 6j^2 + 8j^3 - 39/(4j) + [6j + 13/(2j)]S] + N[8j^2 + (13/j) - 8jS].$$

This leads to the result

$$\gamma = [EH - GF]/[2S\{8j^4 - 11j^2 + (13/(4j))^2\}].$$

We obtain α by swap techniques, or otherwise are able to insert values of γ and δ in

$$\alpha = [L - ¼K - (\gamma + \delta - (K/2))j - ½(\gamma - \delta)S]/R - ½(\gamma + \delta - K),$$

which gives

$$\begin{aligned}\alpha = \{&(K/4)[-2(1 - 2j)j^2 + ½(S^2 - R^2) - j(S^2 - 5R^2) \\ &\quad - 2(1 - 4j)jR + (-S^2 + R^2)R][(½ + j)^2 - S^2] \\ &+ (L/2)[(3 + 4j - 4j^2)j^2 \\ &\quad + (3/4)(-S^2 + R^2) + j(-1 - 3j)(S^2 + 3R^2) \\ &\quad - 3(-1 + 4j^2)jR - S^2(R^2 - S^2) \\ &\quad + (1 - 2j)S^2R - (1 + 2j)R^3] \\ &+ M[((-3/2) - j)(4j^2 - S^2) - 3((1/2) - j)R^2 + (-6j - S^2 + R^2)R] \\ &+ 2N[4j^2 + 4jR - S^2 + R^2]\}/[2R\{8j^4 - 11j^2 + (13/(4j))^2\}],\end{aligned}$$

and correspondingly, by substituting for R and S in terms of j, we compute

$$\begin{aligned}\alpha = \{&(K/4)[(-39/2) + 11j - 2j^2 + (13/(4j)) \\ &\quad + [-2j + 8j^2 - 13/(2j)]R][-5/2 + j + 2j^2 - (13/(4j))] \\ &+ L[(13/4) + j - 15j^2 + 4j^3 + 4j^4 + 13/(2j) + 169/(16j^2) \\ &\quad + [(-4j - 4j^3 + 13/(4j)]R] \\ &+ M[(-13/2) + 11j - 6j^2 - 8j^3 + 39/(4j) - [6j + 13/(2j)]R] \\ &+ N[8j^2 - (13/j) + 8jR]\}/[2R\{8j^4 - 11j^2 + (13/(4j))^2\}].\end{aligned}$$

Thus we can find β from

$$\beta = K - \alpha - \gamma - \delta,$$

or by alternative methods using swap techniques, either of which give us

$$\begin{aligned}\beta = \{&(K/4)[2(1 - 2j)j^2 + ½(-S^2 + R^2) - j(-S^2 + 5R^2) \\ &\quad - 2(1 - 4j)jR + (-S^2 + R^2)R][(½ + j)^2 - S^2] \\ &+ (L/2)[(-3 - 4j + 4j^2)j^2 \\ &\quad + (3/4)(S^2 - R^2) + j(1 + 3j)(S^2 + 3R^2) \\ &\quad - 3(-1 + 4j^2)jR + S^2(R^2 - S^2) \\ &\quad + (1 - 2j)S^2R - (1 + 2j)R^3] \\ &+ M[((3/2) + j)(4j^2 - S^2) + 3((1/2) - j)R^2 + (-6j - S^2 + R^2)R] \\ &+ 2N[-4j^2 + 4jR + S^2 - R^2]\}/[2R\{8j^4 - 11j^2 + (13/(4j))^2\}],\end{aligned}$$

and substituting for R and S in terms of j, produces the formula

$$\begin{aligned}\beta = \{&(K/4)[(39/2) - 11j + 2j^2 - (13/(4j)) \\ &\quad + [-2j + 8j^2 - 13/(2j)]R][-5/2 - j + 2j^2 - (13/(4j))] \\ &+ L[(-13/4) - j + 15j^2 - 4j^3 - 4j^4 - 13/(2j) - 169/(16j^2) \\ &\quad - [(4j + 4j^3 - 13/(4j)]R] \\ &+ M[(13/2) - 11j + 6j^2 + 8j^3 - 39/(4j) - [6j + 13/(2j)]R] \\ &+ N[-8j^2 + (13/j) + 8jR]\}/[2R\{8j^4 - 11j^2 + (13/(4j))^2\}].\end{aligned}$$

So it is possible to compute integer generalised ${}_4g_p$ sequences satisfying the linear combinations of x^4 = the linear combinations of $(x^3 + x^2 + x + 1)$, that is, a sequence

p =	0	1	2	3	4	5	6	7
${}_4l_p$ =	4	1	3	7	15	26	51	99
${}_4g_p$ =	K	L	M	N	K + L + M + N	K + 2L + 2M + 2N	2K + 3L + 4M + 4N	4K + 6L + 7M + 8N

etc. The sequence may be extended for negative p, and the formulae are also valid for K, L, M and N complex numbers. ■

As an extension of *Lemma 1* in section 2.2, we mention the result

Let h_1, h_2 *and* h_3 *be the three imaginary unit quaternions. Then*

$${}_4l_p = (\tau^p + \sigma^p + \rho^p + \lambda^p)$$
$$= \Pi(s = 0, p - 1)[\tau - \sigma(e\uparrow h_1\pi(2s + 1)/p)$$
$$- \rho(e\uparrow h_2\pi(2s + 1)/p) - \lambda(e\uparrow h_3\pi(2s + 1)/p)]. \blacksquare$$

We can also consider linear combinations of roots of $x^4 = tx^3 + ux^2 + vx + w$. Putting $x = y + t/4$ gives

$$f(y) = y^4 - [(3t^2/8) + u]y^2 - [(t^3/8) + (ut/2) + v]y$$
$$- [(3t^4/4^4) + (t^2u/16) + (tv/4) + w] = 0,$$

which can be put in the form

$$f(y) = (y^2 - jy + m)(y^2 + jy + n).$$

Then

$$2m = j^2 - [(3t^2/8) + u] - [(t^3/8) + (tu/2) + v]/j,$$
$$2n = j^2 - [(3t^2/8) + u] + [((t^3/8) + (tu/2) + v]/j$$

and

$$h(j^2) = j^6 - [(3t^2/4) + 2u]\, j^4 + [(3t^4/4^2) + t^2u + tv + 4w + u^2]j^2$$
$$- \ [(t^3/8) + ut/2 + v]^2 = 0.$$

The solutions for x are now

$$\tau_x \ = t/4 + ½(-j + \surd(j^2 - 4n))$$
$$\sigma_x = t/4 + ½(-j - \surd(j^2 - 4n))$$
$$\rho_x = t/4 + ½(j + \surd(j^2 - 4m))$$

and

$$\lambda_x = t/4 + ½(j - \surd(j^2 - 4m)).$$

If we write

$$j^2 = k^2 + [t^2/4 + 2u/3],$$

then

$$k^6 + [(-u^2/3) + tv + 4w]k^2 + [t^2w + (2u^3/27) - (tuv/3) + (8uw/3) - v^2] = 0,$$

which using new variables q and r, we put in the form

$$k^6 + qk^2 + r = 0,$$

so

$$j^2 = [t^2/4 + 2u/3] + \varepsilon[1/2(-r + \surd(r^2 + 4q^3/27))]^{1/3} + \theta[1/2(-r - \surd(r^2 + 4q^3/27))]^{1/3},$$

where the $\{\varepsilon, \theta\}$ pairings are again $\{1, 1\}$, $\{\omega_3, \omega_3{}^2\}$ or $\{\omega_3{}^2, \omega_3\}$.

To simplify subsequent manipulations, we will also put

$$R = \surd(j^2 - 4n)$$

and

$$S = \surd(j^2 - 4m).$$

We now choose

$${}_4g_{xp} = (\alpha\tau_x{}^p + \beta\sigma_x{}^p + \gamma\rho_x{}^p + \delta\lambda_x{}^p)/(\tau_x + \mu\sigma_x + \nu\rho_x + \xi\lambda_x),$$

under the simplified allocation $\mu = \nu = \xi = 1$, and noting that $\tau_x + \sigma_x + \rho_x + \lambda_x = t$.

Then the p = 0 instance gives, for ${}_4g_{xp} = K$,

$$Kt = \alpha + \beta + \gamma + \delta.$$

For p = 1, we choose ${}_4g_{xp}$ as the value L, obtaining

$$Lt = ½t[½(\alpha + \beta + \gamma + \delta) + (-\alpha - \beta + \gamma + \delta)(j/t) + (\alpha - \beta)(R/t) + (\gamma - \delta)(S/t)]$$

or

$$Lt = ½t[½Kt + (2\gamma + 2\delta - Kt)(j/t) + (2\alpha + \gamma + \delta - Kt)(R/t) + (\gamma - \delta)(S/t)].$$

For p = 2, we choose a value ${}_4g_{xp}$ = M, so

$$Mt = ¼t^2[¼(\alpha + \beta + \gamma + \delta) + (-\alpha - \beta + \gamma + \delta)(j/t) + (\alpha + \beta + \gamma + \delta)(j/t)^2 + (\alpha - \beta)(1 - 2(j/t))(R/t) + (\gamma - \delta)(1 + 2(j/t))(S/t) + (\alpha + \beta)(R/t)^2 + (\gamma + \delta)(S/t)^2]$$

or

$$Mt = t^2\{[L - (Kt/4)(½ - (j/t)](½ - (j/t)) + (\gamma + \delta)(j/t)^2 - ¼(\gamma + \delta - Kt)R^2/t^2 + (\gamma - \delta)(jS/t^2) + ¼(\gamma + \delta)(S^2/t^2)\}.$$

Then, for p = 3, we finally assign the value ${}_4g_{xp}$ = N, which gives

$$Nt = (1/8)t^3[(1/8)(\alpha + \beta + \gamma + \delta) + (3/4)(-\alpha - \beta + \gamma + \delta)(j/t) + (3/2)(\alpha + \beta + \gamma + \delta)(j/t)^2 + (-\alpha - \beta + \gamma + \delta)(j/t)^3 + 3(\alpha - \beta)(¼ - (j/t) + (j/t)^2)(R/t) + 3(\gamma - \delta)(¼ + (j/t) + (j/t)^2)(S/t) + 3(\alpha + \beta)(½ - (j/t))(R/t)^2 + 3(\gamma + \delta)(½ + (j/t))(S/t)^2 + (\alpha - \beta)(R/t)^3 + (\gamma - \delta)(S/t)^3],$$

that is

$$8Nt = t^3\{[6L - Kt(1 - 2(j/t)](½ - (j/t))^2 + (2L + Kt(1 - 2(j/t)))(R/t)^2 + (\gamma + \delta)((3/2) - (j/t))(4(j/t)^2 - (R/t)^2) + 3(\gamma + \delta)(½ + (j/t))(S/t)^2 + (\gamma - \delta)(6(j/t) - (R/t)^2 + (S/t)^2)(S/t)\}.$$

Our comments on swapping α, β, γ and δ symbols made in the previous example where we had t = u = v = w = 1 still apply, including the statement that swapping j and -j swaps the values of S and R when these are expanded out in terms of j, since this swaps m and n. Thus the formula described for α, say, obtained under a j, -j swap, is still valid after eliminating R^2 and S^2 by substituting expressions that involve j.

The equation for M we now put in the form

$$P\gamma = E + F\delta$$

where

$$P = t^2[(j/t)^2 + jS/t^2 - ¼(R/t)^2 + ¼(S/t)^2],$$
$$E = Mt - t^2[L - (Kt/4)(½ - (j/t))](½ - (j/t)) - ¼Kt(R/t)^2]$$

and

$$F = t^2[-(j/t)^2 + jS/t^2 + ¼(R/t)^2 - ¼(S/t)^2].$$

The equation for N we allocate as

$$Q\gamma = G + H\delta$$

where

$$Q = t^3\{((3/2) - (j/t))(4(j/t)^2 - (R/t)^2) + 3(½ + (j/t))(S/t)^2 + (6(j/t) - (R/t)^2 + (S/t)^2)(S/t)\},$$
$$G = 8Nt - t^3\{[6L - Kt(1 - 2(j/t))](½ - (j/t))^2 - [2L + Kt(1 - 2(j/t))](R/t)^2\}$$

and

$$H = t^3\{((-3/2) + (j/t))(4(j/t)^2 - (R/t)^2) - 3(½ + (j/t))(S/t)^2 + (6(j/t) - (R/t)^2 + (S/t)^2)(S/t)\}.$$

Then

$$\delta = [EQ - GP]/[HP - FQ].$$

We establish that

$$P = -F + 2(jS/t^2)$$

and

$$Q = -H + 2[6(j/t) - (R/t)^2 + (S/t)^2](S/t),$$

so

$$\begin{aligned} HP - FQ &= 2(S/t)[H(j/t) - F(6(j/t) - (R/t)^2 + (S/t)^2)] \\ &= 2(S/t)\{[-2(j/t)^2 + ((R/t)^2/2) + ((S/t)^2/2)]^2 - (R/t)^2(S/t)^2\} \\ &= 2(S/t)\{[(8j^2 - 3t^2 - 8u)j^2/t^4] + 4[((t^3/8) + (ut/2) + v)^2/(j^2t^4)]\}. \end{aligned}$$

Expanding EQ – GP by separating out terms in K, L, M and N, gives

$$\begin{aligned} EQ - GP = {} & (Kt/4)[(Qt^2) - (1 - 2(j/t))Pt^3][(½ - (j/t))^2 - (R/t)^2] \\ & + L[(½ - (j/t))(-Qt^2 + 6(½ - (j/t))Pt^3) + 2(R/t)^2Pt^3] \\ & + MtQ - 8NtP, \end{aligned}$$

or

$$\begin{aligned} EQ - GP = {} & (Kt^6/4)[2(1 + 2(j/t))(j/t)^2 + ½(-(R/t)^2 + (S/t)^2) \\ & \quad + (j/t)(-(R/t)^2 + 5(S/t)^2) \\ & \quad + 2(1 + 4(j/t))(jS/t^2) + (-(R/t)^2 + (S/t)^2)(S/t)] \\ & \quad [(½ - (j/t))^2 - (R/t)^2] \\ & + (Lt^5/2)[(-3 + 4(j/t) + 4(j/t)^2)(j/t)^2 \\ & \quad + (3/4)((R/t)^2 - (S/t)^2) + (j/t)(-1 + 3(j/t))((R/t)^2 + 3(S/t)^2) \\ & \quad + 3(-1 + 4(j/t)^2)(jS/t^2) + (R/t)^2((S/t)^2 - (R/t)^2) \\ & \quad + (1 + 2(j/t))(R/t)^2(S/t) + (-1 + 2(j/t))(S/t)^3] \\ & + Mt^4[((3/2) - (j/t))(4(j/t)^2 - (R/t)^2) \\ & \quad + 3((1/2) + (j/t))(S/t)^2 + (6(j/t) - (R/t)^2 + (S/t)^2)(S/t)] \\ & + 2Nt^3[-4(j/t)^2 - 4(jS/t^2) + (R/t)^2 - (S/t)^2], \end{aligned}$$

and substituting for R and S in terms of j, reduces the formula to

$$\begin{aligned} EQ - GP = {} & (Kt^6/4)[2(1 + 2(j/t))(j/t)^2 + 2(n - m)/t^2 \\ & \quad + 4(j/t)[(j^2 + n - 5m)/t^2] \\ & \quad + 2(1 + 4(j/t))(jS/t^2) + 4((n - m)/t^2)(S/t)] \\ & \quad [(1/4 - (j/t) + 4(n/t^2)] \\ & + (Lt^5/2)[(-3 + 4(j/t) + 4(j/t)^2)(j/t)^2 \\ & \quad + (3(m - n)/t^2) \\ & \quad + 4(j/t)(-1 + 3(j/t))((j/t)^2 - (n +3m)/(t^2)) \\ & \quad + 3(-1 + 4(j/t)^2)(jS/t^2) + 4((j^2 - 4n)/t^2)(n - m)/t^2) \\ & \quad + (1 + 2(j/t))((j^2 - 4n)/t^2)(S/t) \\ & \quad + (-1 + 2(j/t))((j^2 - 4m)/t^2)(S/t)] \\ & + Mt^4[((3/2) - (j/t))[4(j/t)^2 - ((j^2 - 4n)/t^2)] \\ & \quad + 3((1/2) + (j/t))((j^2 - 4m)/t^2) \\ & \quad + (6(j/t) + 4((n - m)/t^2)(S/t)] \\ & + 8Nt^3[-(j/t)^2 - (jS/t^2) - (n - m)/t^2)]. \end{aligned}$$

Thus

$$\delta = [EQ - GP]/[2(S/t)\{[(8j^2 - 3t^2 - 8u)j^2/t^4] + 4[((t^3/8) + (ut/2) + v)^2/(j^2t^4)]\}].$$

We have similarly

$\gamma = [EH - GF]/[HP - FQ]$,

given by the expression for δ under the substitution $S \rightarrow -S$. Note the denominator contains S.

We also obtain α by swap techniques. The equation for α is now obtained from the equation for γ by applying the transformations $j \rightarrow -j$, $-j \rightarrow j$, $R \rightarrow S$, and $S \rightarrow R$ for every j, R and S.

Thus, by swapping, we can then find β by substituting R by -R in the α equation.

So it is possible to compute a formula ${}_4g_{xp} = (\alpha\tau_x^p + \beta\sigma_x^p + \gamma\rho_x^p + \delta\lambda_x^p)/t$, where the linear combinations of pth powers of roots satisfies $x^4 = (tx^3 + ux^2 + vx + w)$, that is, for a sequence

$p =$	0	1	2	3	4	5
${}_4g_{xp} =$	K	L	M	N	$wK + vL + uM + tN$	$twK + (tv + w)L + (tu + v)M + (t^2 + u)N$, etc.

■