CHAPTER III

Differences and sums of powers

3.1. Introduction.

We analyse the number of solutions (mod 4) of differences and sums of pth and non-matching powers.

We ask: when are the differences of powers of the form 2 (mod 4)?

3.2. Differences and sums of powers.

Standard results in [3] for the case p = 2 are: *Except for positive integers of the form* 4k + 2, *every positive integer can be represented as the difference of two squares.*

Also: *Every odd prime is uniquely the difference of two squares.*

Putting p = 2 in the general binomial expansion, we infer *these primes are in one of the forms:*

$$4k + 1 = (2k + 1)^2 – (2k)^2$$

or

$$4k + 3 = (2k + 2)^2 – (2k + 1)^2. \blacksquare$$

Prime differences of odd prime p *powers are unique for given* p *and of the form*

Prime difference, p odd prime	**Form**
$(4r + 1)^p – (4r)^p$	4s + 1
$(4r + 2)^p – (4r + 1)^p$	4s + 3
$(4r + 3)^p – (4r + 2)^p$	4s + 3
$(4r + 4)^p – (4r + 3)^p$	4s + 1
$(4r + 1 + u)^p – (4r + u)^p$	4s + u(3 – u) + 1

Proof. $(2t + 1)^p – (2t)^p$ is of the form (*terms divisible by* 4) + 2pt + 1.

We define two equivalence classes having remainders of 1 or 3 under division of the above by 4. For t even = 2r, this maps to the equivalence class given by 4s + 1. For t odd = 2r + 1, it is of the form 4s + 3.

$(2t + 2)^p – (2t + 1)^p$ is of the form (*terms divisible by* 4) + 2pt + 3. For t even = 2r, this is of the form 4s + 3, and for t odd = 2r + 1, its partition is 4s + 1.

The expression for the form 4s + u(3 – u) + 1 is adjusted to give the table entries above it for u = 0, 1, 2 and 3. Since the substitution u $\rightarrow$ 4n + u for the prime expression leaves the form in the same equivalence class, the formula extends to arbitrary u $\in$ **N**. ■

Let p *be an odd prime. Differences of prime powers* (*these are non-prime differences if* $x \neq 0$) *are of the form*

Difference, p odd	**Form**
$(4r + 1)^p - (4r - x)^p$	$4s - [(x + 1)(x^2 + 5x - 3)/3]$
$(4r + 2)^p - (4r + 1 - x)^p$	$4s - [(x + 1)(x^2 + 2x - 9)/3]$
$(4r + 3)^p - (4r + 2 - x)^p$	$4s - [(x + 1)(x^2 - x - 9)/3]$
$(4r + 4)^p - (4r + 3 - x)^p$	$4s - [(x + 1)(x^2 - 4x - 3)/3]$
$(4r + 1 + u)^p - (4r + u - x)^p$	$4s + (x + 1)[(3 - u)(u - (x/2)) + (ux/2) - (x/3)(x + ½) + 1]$

Proof. By the first table, chaining together (adding) the adjacent sums

$$[(4r + 1 + u)^p - (4r + u)^p] + [(4r + u)^p - (4r + u - 1)^p]$$

results, by additive epimorphism, in a form equal to the sum of the adjacent forms.

Adding together x adjacent sums gives a number $(4r + 1 + u)^p - (4r + u - x)^p$.

The form expression

$$4s + (x + 1)[(3 - u)(u - (x/2)) + (ux/2) - (x/3)(x + ½) + 1]$$

results from its composition with adjacent sums as

$$4s + (x + 1)u(3 - u) - (1 + 2 + \ldots + x)(3 - 2u) - (1.1 + 2.2 + \ldots + x.x) + (x + 1),$$

where $(1 + 2 + \ldots + x)$ is the arithmetic sum $x(x + 1)/2$ and

$$1^2 + 2^2 + \ldots + x^2 = x[x^2 + (3/2)x + ½]/3 = x(x + 1)(x + ½)/3.$$

Once again, the substitution $u \to 4n + u$ in the expression for the number leaves it in the same form equivalence class. ■

The formulae in the above two tables are unchanged when p is an *odd* number > 1 rather than a prime, though for non-prime p none of the differences of powers is prime.

The adjacent differences below for even $p > 0$ *powers are of the form*

Difference, p even	**Form**
$(4r + 1)^p - (4r)^p$	$4s + 1$
$(4r + 2)^p - (4r + 1)^p$	$4s + 3$
$(4r + 3)^p - (4r + 2)^p$	$4s + 1$
$(4r + 4)^p - (4r + 3)^p$	$4s + 3$
$(4r + 1 + u)^p - (4r + u)^p$	$4s + (2u/3)(2u - 5)(u - 2) + 1$

Proof. Differences $(2t + 1)^p - (2t)^p$, where p is even, are of the form

(*terms divisible by* 4) + 2pt + 1 = 4s + 1,

whereas differences $(2t + 2)^p - (2t + 1)^p$ for even p, are of the form

(*terms divisible by* 4) + $2^p - 1$ = 4s + 3.

The expression for the form $4s + (2u/3)(2u - 5)(u - 2) + 1$ is adjusted to give the table entries above it for u = 0, 1, 2 and 3. Once again, since the substitution $u \to 4n + u$ for the difference expression leaves the form in the same equivalence class, the formula extends to arbitrary $u \in$ **N**. ■

Let p > 0 *be even. Differences of even powers are of the form*

Difference, p even	**Form**
$(4r + 1)^p - (4r - x)^p$ $(4r + 2)^p - (4r + 1 - x)^p$ $(4r + 3)^p - (4r + 2 - x)^p$ $(4r + 4)^p - (4r + 3 - x)^p$	$4s - [(x + 1)(x^3 + 7x^2 + 13x - 3)/3]$ $4s - [(x + 1)(x^3 + 3x^2 - x - 9)/3]$ $4s - [(x + 1)(x^3 - x^2 - 3x - 3)/3]$ $4s - [(x + 1)(x^3 - 5x^2 + 7x - 9)/3]$
$(4r + 1 + u)^p - (4r + u - x)^p$	$4s + [(x + 1)/3][(2u - 5)(u - 2)(2u - x)$ $- (4u - 9)ux + (2u - 3)x(2x + 1)$ $- x^2(x + 1) + 3]$

Proof. The proof is similar to that for odd p.

The form expression

$$4s + [(x + 1)/3][(2u - 5)(u - 2)(2u - x) - (4u + 1)ux + (6u + 1)(x(2x + 1)/3) - x^2(x + 1) + 3]$$

results from chaining together (x + 1) forms beginning with

$$4s + (2u/3)(2u - 5)(u - 2) + 1$$

and ending with

$$4s + (2(u - x)/3)(2u - 5 - 2x)(u - 2 - x) + 1.$$

We use the further relation

$$1^3 + 2^3 + \dots + x^3 = x^2(x + 1)^2/4$$

to obtain the final result. ■

We now look at $(4r + 1 + u)^p + (4r + u)^p$, which is of the form

$$4s + (1 + u)^p + u^p.$$

For p even, irrespective of whether u is even or odd, this is of the form

$$4s + (1 + 2v)^p \equiv 4s + 1.$$ ■

Let p > 0 *be even. Sums of* p *powers are of the form*

Sum, p even	**Form**
$(4r + 1)^p + (4r - x)^p$ $(4r + 2)^p + (4r + 1 - x)^p$ $(4r + 3)^p + (4r + 2 - x)^p$ $(4r + 4)^p + (4r + 3 - x)^p$	$4s + 1 + [x(x^3 + 8x^2 + 20x + 10)/3]$ $4s + 1 + [x(x^3 + 4x^2 + 2x - 10)/3]$ $4s + 1 + [x(x^3 - 4x - 6)/3]$ $4s + 1 + [x(x^3 - 4x^2 + 2x - 2)/3]$
$(4r + 1 + u)^p + (4r + u - x)^p$	$4s + 1 - [x/3][(2u - 7)(u - 3)(2u - x - 1)$ $- (4u - 13)(u - 1)(x - 1)$ $+ (2u - 5)(x - 1)(2x - 1)$ $- (x - 1)^2x + 3]$

Proof.

$$(4r + 1 + u)^p + (4r + u - x)^p = [(4r + 1 + u)^p + (4r + u)^p] - [(4r + u)^p - (4r + u - x)^p].$$

In consequence, the theorem is obtainable from the result for p even in the previous table, under the substitution for that table of $u \to (u - 1)$ and $x \to (x - 1)$. ■

Adjacent sums of odd p > 1 *powers are of the form*

Sum, p odd	**Form**
$(4r + 1)^p + (4r)^p$	$4s + 1$
$(4r + 2)^p + (4r + 1)^p$	$4s + 1$
$(4r + 3)^p + (4r + 2)^p$	$4s + 3$
$(4r + 4)^p + (4r + 3)^p$	$4s + 3$
$(4r + 1 + u)^p + (4r + u)^p$	$4s - (u/3)(2u - 7)(u - 1) + 1$

Proof. For p odd = 2q + 1 > 1 the aforementioned equivalence class for

$$(4r + 1 + u)^p + (4r + u)^p,$$

being $4s + (1 + u)^p + u^p$, if u = 2v + 1 is odd, is of the form

$$4s + (2q + 1)2v + 1 \equiv 4s + 2v + 1.$$

If v is even, it is of the form 4s + 1, and if v is odd it is of the form 4s + 3.

If u is even = 2w with w even, the form is 4s + 1, and if w is odd, the form is 4s + 3.

As before, the expression for the form 4s – (u/3)(2u – 7)(u – 1) + 1 fits the table entries above it for u = 0, 1, 2 and 3. ■

Let p > 1 *be odd. Sums of* p *powers are of the form*

Sum, p odd	**Form**
$(4r + 1)^p + (4r - x)^p$	$4s + 1 + (x/3)[(x^2 + 6x + 2)]$
$(4r + 2)^p + (4r + 1 - x)^p$	$4s + 1 + (x/3)[(x^2 + 3x - 7)]$
$(4r + 3)^p + (4r + 2 - x)^p$	$4s + 3 + (x/3)[(x^2 - 10)]$
$(4r + 4)^p + (4r + 3 - x)^p$	$4s + 3 + (x/3)[(x^2 - 3x - 7)]$
$(4r + 1 + u)^p + (4r + u - x)^p$	$4s - (u/3)(2u - 7)(u - 1) + 1$ $- [x/2][(4 - u)(2u - x - 1)$ $+ (u - 1)(x - 1) - [(x - 1)(2x - 1)/3] + 2]$

Proof. This results from chaining

$$[(4r + 1 + u)^p + (4r + u)^p] - [(4r + u)^p - (4r + u - x)^p]$$

from previous formulae. ■

The tables we have presented are periodic in the variable x, in the sense that x and the variable x + k are in the same equivalence class for some k.

Let k ∈ **N**. *For* p > 0 *even, sums and differences are periodic with* x ≡ x + 2k. *For* p > 1 *odd, sums and differences are periodic with* x ≡ x + 4k.

Proof. The periodicity follows directly from the tables of adjacent sums and differences, the n in x ≡ x + nk for differences being the smallest number which brings the form back to 4s + 4 in (n – 1) successive adjacent additions of entries in the tables. For sums, we are adding together two tables, the periodicity being due to the subtracted differences. ■

We will see next that by putting η = 1 and θ = 0, or η = 0 and θ = 1, allows us to express the terms $(4r + 1 + u)^p$ and $(4r + u - x)^p$ directly.

Let p *be odd and* η *and* θ *complex numbers. Then there exist* s, t *such that*

$$\eta(4r + 1 + u)^p + \theta(4r + u - x)^p =$$
$$\eta[2(t + s) - (1/3)[u^3 - 3u^2 - u - 3]]$$
$$+ \theta[2(t - s) + (1/3)[x^3 - 3(u - 2)x^2 + (3u^2 - 12u + 2)x$$
$$- u^3 + 6u^2 - 8u]].$$

Proof. Add linear combinations of the odd p sum and difference tables for general x. ■

Let p *be even and* η *and* θ *complex numbers. Then there exist* s, t *such that*

$$\eta(4r + 1 + u)^p + \theta(4r + u - x)^p =$$
$$\eta[2(t + s) + (1/3)[2u^3 - 9u^2 + 10u + 3]]$$
$$+ \theta[2(t - s) + (1/3)[x^4 - (4u - 8)x^3 + (6u^2 - 24u + 20)x^2$$
$$- (4u^3 - 24u^2 + 40u - 10)x - (2u^3 - 9u^2 + 10u)]].$$

Proof. Add linear combinations of even p sum and difference tables for general x. ■

If we consider $(4r + v)^p + (4r + v)^{p-1}$ and $(4r + v)^p - (4r + v)^{p-1}$ for both p even and p odd and (p > 1 and r > 0), we obtain the following table.

p	**Formula**	**v**	**Form**
odd	$(4r + v)^p + (4r + v)^{p-1}$	0	4s
		1	4s + 2
		2	4s
		3	4s
		v	4s + v(v – 3)(v – 2)
	$(4r + v)^p - (4r + v)^{p-1}$	0	4s
		1	4s
		2	4s
		3	4s + 2
		v	4s + (v/3)(v – 2)(v – 1)
even	$(4r + v)^p + (4r + v)^{p-1}$	0	4s
		1	4s + 2
		2	4s + 2
		3	4s
		v	4s – v(v – 3)
	$(4r + v)^p - (4r + v)^{p-1}$	0	4s
		1	4s
		2	4s + 2
		3	4s + 2
		v	4s – (v/3)(2v – 7)(v – 1)

■

Suppose, as an example, we wanted to obtain, mod 4, the value of

$$(4r + v)^p + (4r + v)^{p-y}$$

for p odd and y even. The mod 4 form consists of one term for p odd:

$$[(4r + v)^p + (4r + v)^{p-1}],$$

with (y/2) terms considered for (p – 1) even (the example uses y = 4):

$-[(4r+v)^{p-1} - (4r+v)^{p-2}] - [(4r+v)^{p-3} - (4r+v)^{p-4}]$,□

and (y – 2)/2 terms, considered for (p – 2) odd:

$-[(4r+v)^{p-2} - (4r+v)^{p-3}]$.

By this and similar techniques we obtain

p	Formula	y	Form
odd	$(4r+v)^p + (4r+v)^{p-y}$	odd	$4s + [v(v-3)/2][y(v-1) + (v-3)]$
		even	$4s + [2v/3](2v-5)(v-2) + [yv/6](v-5)(v-1)$
	$(4r+v)^p - (4r+v)^{p-y}$	odd	$4s - [v(v-1)/6][y(v-5) - 3(v-3)]$
		even	$4s + [yv/2](v-3)(v-1)$
even	$(4r+v)^p + (4r+v)^{p-y}$	odd	$4s + [v(v-3)/2][-y(v-1) - (v-3)]$
		even	$4s - [2v/3](v^2 - 3v - 1) + [yv/6](v-5)(v-1)$
	$(4r+v)^p - (4r+v)^{p-y}$	odd	$4s - [v(v-1)/6][y(v-5) + 3(v-3)]$
		even	$4s - [yv/2](v-3)(v-1)$

To check, we may have to use

$(v/3)(v-5)(v-1) = v(v-3)(v-1) \pmod 4$,

since there may be more than one way to obtain these formulae. ■

Of special note is that, by subtracting terms like $(4r+v)^p - (4r+v)^{p-1}$, for p even we can now obtain terms elliptic in u and x from the previously derived sums and differences of powers for p odd.

Formula, p even, r > 0	Form
$(4r+1+u)^p - (4r+u-x)^p$	$4s - (2x^3/3) - 3x^2 - (4x/3) + 1$ $+ 2u(-u + 2 + x^2 + 3x - ux)$
$(4r+1+u)^p + (4r+u-x)^p$	$4s + (2x^3/3) + 3x^2 + (4x/3) + 1$ $+ 2u(-x + u - 3)$

Using a typical formula such as

$$(4r+1+u)^{p-y} - (4r+u-x)^p = (4r+1+u)^p - (4r+u-x)^p - [(4r+1+u)^p - (4r+1+u)^{p-y}]$$

and putting v = (1 + u) or v = (u – x), the sum and difference tables of section 5 for p odd, and the above table for p even, then give formulae (mod 4) for arbitrary sums and differences of powers. Thus for both p even and p odd we obtain the following elliptic curves in u and x, linear in y, reduced mod 4.

p	Formula	y	Form
odd	$(4r+1+u)^p +$ $(4r+u-x)^{p-y}$	odd	$4s - (u/3)(2u-7)(u-1) + 1 -$ $(x/2)[(4-u)(2u-x-1) + (u-1)(x-1) -$ $[(x-1)(2x-1)/3] + 2] +$ $[(u-x)(u-x-1)/6][y(u-x-5) - 3(u-x-3)]$

		even	$4s - (u/3)(2u - 7)(u - 1) + 1 -$ $(x/2)[(4 - u)(2u - x - 1) + (u - 1)(x - 1) -$ $[(x - 1)(2x - 1)/3] + 2] -$ $[y(u - x)/2](u - x - 3)(u - x - 1)]$
	$(4r + 1 + u)^{p-y} +$ $(4r + u - x)^{p}$	odd	$4s - (u/3)(2u - 7)(u - 1) + 1 -$ $(x/2)[(4 - u)(2u - x - 1) + (u - 1)(x - 1) -$ $[(x - 1)(2x - 1)/3] + 2] +$ $[(u + 1)u/6][y(u - 4) - 3(u - 2)]$
		even	$4s - (u/3)(2u - 7)(u - 1) + 1 -$ $(x/2)[(4 - u)(2u - x - 1) + (u - 1)(x - 1) -$ $[(x - 1)(2x - 1)/3] + 2] - [y(u + 1)/2](u - 2)u$
	$(4r + 1 + u)^{p} -$ $(4r + u - x)^{p-y}$	odd	$4s + (x + 1)[(3 - u)(u - (x/2)) + (ux/2) -$ $(x/3)(x + ½) + 1] -$ $[(u - x)(u - x - 1)/6][y(u - x - 5) - 3(u - x - 3)]$
		even	$4s + (x + 1)[(3 - u)(u - (x/2)) + (ux/2) -$ $(x/3)(x + ½) + 1] +$ $[y(u - x)/2](u - x - 3)(u - x - 1)]$
	$(4r + 1 + u)^{p-y} -$ $(4r + u - x)^{p}$	odd	$4s + (x + 1)[(3 - u)(u - (x/2)) + (ux/2) -$ $(x/3)(x + ½) + 1] +$ $[(u + 1)u/6][y(u - 4) - 3(u - 2)]$
		even	$4s + (x + 1)[(3 - u)(u - (x/2)) + (ux/2) -$ $(x/3)(x + ½) + 1] - [y(u + 1)/2](u - 2)u$
even $r > 0$	$(4r + 1 + u)^{p} +$ $(4r + u - x)^{p-y}$	odd	$4s + (2x^3/3) + 3x^2 + (4x/3) + 1 +$ $2u(-u + 2 + x^2 + 3x - ux) +$ $[(u - x)(u - x - 1)/6][y(u - x - 5) - 3(u - x - 3)]$
		even	$4s + (2x^3/3) + 3x^2 + (4x/3) + 1 +$ $2u(-u + 2 + x^2 + 3x - ux) +$ $[y(u - x)/2](u - x - 3)(u - x - 1)]$
	$(4r + 1 + u)^{p-y} +$ $(4r + u - x)^{p}$	odd	$4s + (2x^3/3) + 3x^2 + (4x/3) + 1 +$ $2u(-u + 2 + x^2 + 3x - ux) +$ $[(u + 1)u/6][y(u - 4) + 3(u - 2)]$
		even	$4s + (2x^3/3) + 3x^2 + (4x/3) + 1 +$ $2u(-u + 2 + x^2 + 3x - ux) + [y(u + 1)/2](u - 2)u$
	$(4r + 1 + u)^{p} -$ $(4r + u - x)^{p-y}$	odd	$4s - (2x^3/3) - 3x^2 - (4x/3) + 1 +$ $2u(-u + 2 + x^2 + 3x - ux) +$ $[(u - x)(u - x - 1)/6][y(u - x - 5) + 3(u - x - 3)]$
		even	$4s - (2x^3/3) - 3x^2 - (4x/3) + 1 +$ $2u(-u + 2 + x^2 + 3x - ux) - [y(u + 1)/2](u - 2)u$
	$(4r + 1 + u)^{p-y} -$ $(4r + u - x)^{p}$	odd	$4s - (2x^3/3) - 3x^2 - (4x/3) + 1 +$ $2u(-u + 2 + x^2 + 3x - ux) +$ $[(u + 1)u/6][y(u - 4) + 3(u - 2)]$
		even	$4s - (2x^3/3) - 3x^2 - (4x/3) + 1 +$ $2u(-u + 2 + x^2 + 3x - ux) + [y(u + 1)/2](u - 2)u$

The corresponding formulae for e.g.

$$\eta(4r + 1 + u)^{p} + \theta(4r + u - x)^{p-y}$$

can be obtained from the above sums and differences.

The above formulae have a finite number of integer solutions mod 4, arising from the periodicity in u, x and y – a maximum of $2^8 = 256$ possibilities, given that p has two sets of solutions, for even and odd, and the first exponent is or is not less than the second. Explicit calculations reduce this – less than the number of solutions derived in elliptic curve theory from a direct application of Siegel's theorem [26], [27]. The theorem of Mazur puts limits on the torsion subgroup, giving crudely less than 240 possibilities [20], [21]. ■

Rather than use reduction mod 4, we can proceed as follows.

$$(q + v)^p - (q + v)^{p-1} = qs + v^{p-1}(v - 1),$$

thus by 'chaining'

$$(q + v)^p - (q + v)^{p-y-1} = qs + v^{p-y-1}(v^{y+1} - 1). ■$$

3.3. Differences of powers of the form 2 (mod 4).

René Schoof [25] poses the problem – *for exponents* ≥ 2, *when are differences of powers of the form* 2 (mod 4)? We ask, *does* 2 *only* = $3^3 - 5^2$ *in powers*?

We will write such differences in the form (A + B) where

$$A = m^p - n^p$$

and

$$B = n^p - n^{p-y},$$

or of the form (C + A), where

$$C = m^{p-y} - m^p.$$

Since B and C are always even, so is A.

We first investigate the cases for A – where A is even.

A factorises as

$$(m - n)(m^{p-1} + m^{p-2}n + \dots + n^{p-1}).$$

Because A is not odd, so $n \neq m - 1$, the factorisation exists.

If m and n are even, then $A \equiv 0 \pmod 4$.

Otherwise, m and n are odd.
If p is even, then

A = (m – n)(an even number of terms, each of which is odd)
$\equiv 0 \pmod 4$.

If p is odd, if both m and n are of the form 4r + 1, or both are of the form 4r + 3, then m – n = 4r, so

$$A \equiv 0 \pmod 4.$$

If p is odd, if m is of the form 4r + 1 and n is of the form 4r + 3, or vice-versa, then since we are dealing with

A = (m – n)(an odd number of terms, each of which is odd),

then

$$A \equiv 2 \pmod 4.$$

This exhausts all the cases for the structure of $A = m^p - n^p$, for A even.

We investigate

$$B = n^p – n^{p-y}.$$

Write

$$n = 4r + j.$$

Then

$$B = (4r + j)^{p-y}((4r + j)^y – 1).$$

If $n = 4r + 1$, then

$$B \equiv (4r + 1)(4r + 1 – 1) \equiv 4r \equiv 0 \pmod 4,$$

so for $n = 4r + 1$ there is only one case, irrespective of whether y is even or odd.

If $n = 4r + 3$, and y is even, then

$$(4r + 3)^y \equiv 1 \pmod 4, \text{ so } B \equiv 4r \equiv 0 \pmod 4.$$

If $n = 4r + 3$, and y is odd, then

$$B = (4r + 3)^{p-y}(4r + 2) \equiv 4r + 2 \equiv 2 \pmod 4.$$

If n is even, if $n = 4r$ then $B \equiv 0 \pmod 4$. If $n = 4r + 2$, if $p – y = 1$ then

$$B \equiv (4r + 2)(4r + 3) \equiv 2 \pmod 4,$$

and if $p – y = 1$, y is even and p is odd for $n = 4r + 2$, then

$$B \equiv (4r + 2)(4r + 1) \equiv 2 \pmod 4,$$

otherwise if $p – y \neq 1$, then

$$B \equiv 0 \pmod 4.$$

We note that if n is negative, then if the case n positive is of the form $4r + 1$, then -n is of the form $4r + 3$, likewise $n = 4r + 3$ implies $-n = 4r + 1$. This is a salient feature of our calculations, since if $p – y$ is even, we may need to include the case $(-n)^{p-y} = n^{p-y}$.

This exhausts all the cases covering B.

We now work out explicitly configurations for C, which is negative for $y > 0$. The argument parallels that for B exactly. We write

$$m = 4r + k,$$

so that

$$C = (4r + k)^{p-y}(1 – (4r + k)^y).$$

Allocating $m = 4r + 1$ gives

$$C \equiv (4r + 1)(1 – 4r – 1) \equiv -4r \equiv 0 \pmod 4,$$

whereas if $m = 4r + 3$, first considering y even,

$$(4r + 3)^y \equiv 1 \pmod 4,$$

so in this case

$$C \equiv (4r + 1)(1 – 4r – 1) \text{ or } (4r + 3)(1 – 4r – 1) \equiv 0 \pmod 4,$$

and for the y odd case

$$C \equiv (4r + 1)(1 – 4r – 3) \text{ or } (4r + 3)(1 – 4r – 3) \equiv 2 \pmod 4.$$

The case for m even is likewise similar, which gives $C \equiv 0 \pmod 4$ if $p – y \neq 1$.

The comment for B on $(-n)^{p-y}$ solutions also holds for C with $(-m)^{p-y}$ solutions. This is apposite, since a reversal of sign for m in both A and C results in a cancellation of both $(-m)^p$ terms in (C + A).

If (A + B) or (C + A) ≡ 2 (mod 4), then for (A + B), A ≡ 0 (mod 4) and B ≡ 2 (mod 4), or vice-versa, and similarly for (C + A).

For (A + B), if A ≡ 0 (mod 4), then m and n are even, so n = 4r + 2, y is odd and p is even or y is even and p is odd, and p – y = 1, which breaks the stipulation of the problem that p – y ≠ 1, so there are no such cases.

If A ≡ 2 (mod 4), then p is odd, m is of the form 4r + 1, n is like 4r + 3 or vice-versa, and B ≡ 0 (mod 4), so if n = 4r + 1 this satisfies, but if n = 4r + 3, then y is even.

Thus, if we consider A with B ≡ 0 (mod 4), then the lowest such positive numbers A are for p odd > 1, with m = 4r + 1 and n = 4r + 3 or vice-versa,

$3^3 – 1^3 = 26$, $5^3 – 3^3 = 98$, $7^3 – 5^3 = 218$, $3^5 – 1^5 = 242$, $7^3 – 1^3 = 342$, etc.

For the same A ≡ 2 (mod 4) with |m| > |n|, where |m| is the positive magnitude $\sqrt{(m^2)}$, we generate additional (A + B) entries for n = 4r + 1 with freedom for y (which can also be negative), i.e. the additional terms for low powers

$7^3 – (-3)^4 = 262$, $7^3 – 5^2 = 318$, $7^3 – (-3)^2 = 334$

and if |m| ≤ |n| we obtain, for low values of m and n, the (A + B) terms

$3^3 – 5^2 = 2$, $3^3 – (-3)^2 = 18$, $3^5 – 5^3 = 118$, $3^5 – 5^2 = 218$, $3^5 – (-3)^2 = 234$, etc.

For n = 4r + 3 and |m| > |n|, we have the extra lowest term $5^5 – 3^3 = 3098$.

We now realise that all terms with the (A + B) structure: positive m > positive n and y ≥ 0 are ≥ 26, the lower powers and values of m and n generating the lowest (A + B).

If A ≡ 0 (mod 4) for (A + C), then m = 4r + 2. In the case C ≡ 2 (mod 4), p – y = 1 again – which does not satisfy the problem criteria.

Accordingly, A ≡ 2 (mod 4) and C ≡ 0 (mod 4). For |m| > |n| and b = 4r + 1, y is even, so we can consider a low value of (C + A) corresponding to a relatively high value of m, e.g.

$15^3 – 5^5 = 250$.

In this and the next case, if |m| < |n| and y > 0, all (C + A) terms are negative.

For (C + A), if n = 4r + 3, so m = 4r + 1, p – y is free to range above 1. If |m| > |n| then we have low terms like

$(-7)^2 – 3^3 = 22$, $5^4 – 3^5 = 382$, $9^3 – 3^5 = 486$, $(-7)^4 – 3^5 = 2158$, etc.

Some of these terms can be quite small, for example

$13^3 – 3^7 = 10$. ■