CHAPTER I

Exponential factorisation theorems

1.1. Introduction.

In this outline of our conceptual work, a review largely based on elementary methods, which have been common mathematical currency for over two centuries, we describe explorations of the mathematical landscape concerning global field theorems for exponential powers.

In Section 1.2, we introduce an exponential notation.

Section 1.3 restates foundational rules for real exponentiation, providing proofs for the basic *‘binomial exponent’* and *‘geometric exponent’* theorems.

Section 1.4 continues with new cyclotomic variants of the *‘Fermat subtraction’, ‘Fermat addition’* and *‘linear combination’* factorisation theorems, the latter being a formula that is a linear combination of the previous two.

Section 1.5 develops in many variables the linear combination factorisation theorem.

1.2 The a^b = a↑b Ackermann-Knuth notation for exponentiation. [4]

We use sparingly a special symbolism to obtain our results and think a supplementary notation “↑” for a↑b = a^b is suggestive and appropriate. “a↑b” must coexist with the current convention, so we would still use e.g. $\sin^2\theta$. This has the advantages of being more compact than exp{b}, complicated expressions become more visible and simpler to notate, long sequences of exponents of exponents become easy to write and friendly on line spacing, the choice between (a↑b)↑c and a↑(bc) allows flexibility and nuance, and because usually a↑(b↑c) ≠ (a↑b)↑c, the non-associative nature of exponentiation becomes easy to specify.

1.3 Fundamental exponential theorems for real numbers.

Let a, b, … to h, α, β, γ, δ ∈ $\mathbf{R}^+$ be non-negative real numbers and j, k, … to z ∈ **N** be natural numbers. **N** starts from 1, unless 0 is otherwise indicated. Multiplication will take precedence over exponentiation in implicit bracketing. We admit ±[(-a)↑(±p)] and ±[a↑(±b)] as terms, but not otherwise ±[(-a)↑(±b)].

Factorisation of real numbers, or of complex numbers a + ib, is not unique. Natural numbers factorise uniquely, and there is a type of unique factorisation for Heegner numbers, given by p + q√-1, p + q√-2 or ½(p + q√-r), where -r is one of -3, -7, -11, -19, -43, -67 or -163.

Since exponentiation is in general not associative, we need to understand both sides of (a↑b)↑c ≠ a↑(b↑c), which is one of the principal objectives of our paper.

Now (a↑b)↑c = a↑(b x c), (a x b)↑c = (a↑c) x (b↑c), a↑(b + c) = (a↑b) x (a↑c) and the binomial theorem is (a + b)↑q = Σ(r = 0, q)[q!/r!(q – r)!]$a^r b^{q-r}$. We adopt the convention that 0! = 1 *and writing linearly put* a x q = Σ(r = 0, q – 1) a, *where the sum is* 0 *if* q = 0, *and* a↑q = Π(r = 0, q – 1) a, *where the product is* 1 *if* q = 0.

Of course, $a\uparrow(b\uparrow c)$ may be resolved immediately using the formula

$$a\uparrow(b\uparrow(c + d)) = a\uparrow[(b\uparrow c)(b\uparrow d)],$$

or equivalently

$$a\uparrow(b\uparrow c) = a\uparrow\{d[b\uparrow(c - \log_b d)]\}$$
$$= \{a\uparrow[b\uparrow(c - \log_b d)]\}\uparrow d. \blacksquare$$

In comparison with the rule for a^q in terms of *products* above, the binomial theorem for $(1 + m)^q$ gives an inductive formula for p^q, $p > 2$, in terms of *sums*.

$$p^q = q!\{ \Sigma(t = 0, p - 3)[\Sigma(r_{p-2-t} = 0, q - \Sigma(u = 0, t - 1)r_{p-2-u})(1/r_{p-2-t}!)$$
$$[\Sigma(r_0 = 0, q - \Sigma(v = 0, t)r_{p-2-v})(1/r_0![q - \Sigma(w = 0, t + 1)r_{p-2-w}]!)]]\}.$$

Proof. We employ a proof by induction. Using a new variable s, which starts from the first term s = 0, the (s + 1)th term in a binomial expansion of $(1 + p)^q$ is

$$[q!/s!(q - s)!][p\uparrow(q - s)].$$

Hence, putting $s = r_{p-1}$, the sum of all these terms is

$$A_{1+p,\,q} = q!\{ \Sigma(r_{p-1} = 0, q)(1/r_{p-1}!) [1/(q - r_{p-1})!] [p\uparrow(q - r_{p-1})]\},$$

so since the binomial theorem gives

$$2\uparrow(q - r_1) = (1 + 1)\uparrow(q - r_1) = \Sigma(r_0 = 0, q - r_1)\{(q - r_1)!/[r_0!(q - r_1 - r_0)!]\},$$

the binomial expansion, $A_{1+p,\,q}$, for $(1 + p)\uparrow q$, $p = 2$, is equal to our postulated formula

$$3\uparrow q = q![\Sigma(r_1 = 0, q)(1/r_1!)\{ \Sigma(r_0 = 0, q - r_1)[1/r_0!(q - r_1 - r_0)!]\}].$$

Now, assuming the formula for $p\uparrow r$, with $r = q - r_{p-1} \le q$, above

$$A_{1+p,\,q} = q!\{ \Sigma(r_{p-1} = 0, q)(1/r_{p-1}!) [1/(q - r_{p-1})!] \{(q - r_{p-1})! \Sigma(t = 0, p - 3)$$
$$[\Sigma(r_{p-2-t} = 0, q - \Sigma(u = 0, t)r_{p-1-u})(1/r_{p-2-t}!)$$
$$[\Sigma(r_0 = 0, q - \Sigma(v = 0, t + 1)r_{p-1-v})(1/r_0!(q - \Sigma(w = 0, t + 2)r_{p-1-w})!)]]\}\}.$$

Rearranging where t varies from 0 to p – 3, and cancelling terms

$$A_{1+p,\,q} = q!\{ \Sigma(t = 0, p - 3)[\Sigma(r_{p-1} = 0, q)(1/r_{p-1}!)$$
$$[\Sigma(r_{p-2-t} = 0, q - \Sigma(u = 0, t)r_{p-1-u})(1/r_{p-2-t}!)$$
$$[\Sigma(r_0 = 0, q - \Sigma(v = 0, t + 1)r_{p-1-v})(1/r_0!(q - \Sigma(w = 0, t + 2)r_{p-1-w})!)]]\}\},$$

so the second line summation starts from r_{p-2}. This implies

$$A_{1+p,\,q} = q!\{ \Sigma(t = 0, p - 2)[\Sigma(r_{p-1-t} = 0, q - \Sigma(u = 0, t - 1)r_{p-1-u})(1/r_{p-1-t}!)$$
$$[\Sigma(r_0 = 0, q - \Sigma(v = 0, t)r_{p-1-v})(1/r_0!(q - \Sigma(w = 0, t + 1)r_{p-1-w})!)]]\},$$

where t varies from 0, then 1 (e.g. r_{p-2} sum) to p – 2, that is, $A_{1+p,\,q} = (1 + p)\uparrow q$. $\blacksquare$

Using $a\uparrow(\Sigma(j = 0, k) z_j) = \Pi(j = 0, k)(a\uparrow z_j)$, $a\uparrow(p\uparrow q)$ can be expanded out using the above formula to give the *binomial exponent factorisation theorem (BEFT)*, or expanded out fully as $n\uparrow(p\uparrow q)$.

The *geometric exponent factorisation theorem (GEFT)* for $a\uparrow(b^{cq})$ is

$$a\uparrow(b^{cq}) = a\{ \Pi(r = 0, q - 1) a\uparrow[(b^c - 1)b^{cr}]\}.$$

Proof. Consider the geometric series

$$S_n = \alpha + \alpha b^c + \alpha b^{2c} + \dots \dots + \alpha b^{nc}.$$

The sum is

$$S_n = \alpha[1 - b^{c(n + 1)}]/[1 - b^c].$$

Insert $\alpha = [b^c - 1]$ and $n = q - 1$. Then the sum becomes

$$[b^{cq} - 1] = [b^c - 1]\{(b^c)\uparrow 0 + (b^c)\uparrow 1 + \dots \dots + (b^c)\uparrow(q - 1))\},$$

yielding

$$a\uparrow[b^{cq} - 1] = \Pi(r = 0, q - 1)\ a\uparrow\{[b^c - 1][b^c\uparrow r]\}. \ \blacksquare$$

The geometric series is related to *cyclotomic equations*, studied in the generalised Fermat factorisation theorems of section 2.4.

Incidentally, when the *arithmetic* (a special case of *Bernoulli*) *series* formula

$$a^{q(q+1)} = [\ \Pi(r = 0, q)\ a^{2r}]$$

is combined with the *GEFT* above, it gives the result

$$a^q = (a^{1/2})\{\ \Pi(r = 0, q)\ a\uparrow[(4r - 3/2(q + 1))/(2q + 3)]\},$$

which reduces to

$$q = ½ + \Sigma(r = 0, q)[(4r - 3/2(q + 1))/(2q + 3)],$$

as can be reconfirmed by routine calculation. A permutation of the argument gives

$$q = -3/2 + \Sigma(r = 0, q)[(4r - 3/2(q + 1))/(2q - 1)].$$

Proof. We give a proof motivated by exponentiation. The arithmetic series sum is

$$\Sigma(r = 0, q)\ r = q(q + 1)/2,$$

thus

$$a^{q(q+1)} = [\ \Pi(r = 0, q)\ a^r]^2.$$

Now

$$q(q + 1) = (q + ½)^2 - ¼.$$

It follows that

$$a\uparrow(q + ½)^2 = a^{1/4}[\ \Pi(r = 0, q)\ a^{2r}].$$

Applying the geometric formula gives

$$\begin{aligned} a\uparrow(q + ½)^2 &= a\{\ \Pi(r = 0, 1)\ a\uparrow[(q - ½)((q + ½)\uparrow r)]\} \\ &= a[a\uparrow(q - ½)][a\uparrow[(q - ½)(q + ½)]] \\ &= a[a\uparrow(q - ½)]\uparrow(q + 3/2). \end{aligned}$$

We deduce

$$[a\uparrow(q - ½)]^{(q + 3/2)} = a^{-3/4}[\ \Pi(r = 0, q)a^{2r}],$$

the last product having q + 1 factors, from r = 0 to r = q. This right hand side is

$$[a^{-3(q+1)/4(q+1)}][\ \Pi(r = 0, q)\ a^{2r}] = \Pi(r = 0, q)a^{(2r - 3/4(q+1))}.$$

Thus exponentiating by 1/(q + 3/2)

$$[a^{(q-½)}] = \Pi(r = 0, q)\ [a\uparrow(2r - 3/4(q + 1))]^{2/(2q+3)},$$

giving

$$a^q = a^{1/2}\Pi(r = 0, q)[a\uparrow(4r - 3/2(q + 1))]^{1/(2q+3)}.$$

If instead we interchange (q – ½) and (q + 3/2) and exponentiate by 1/(q – ½)

$$a^q = a^{-3/2}\Pi(r = 0, q)[a\uparrow(4r - 3/2(q + 1))]^{1/(2q-1)}.\ ■$$

1.4 Generalised cyclotomic Fermat factorisation theorems.

The qth *Fermat number* is $F_q = 2\uparrow(2\uparrow q) + 1$. So $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$ and $F_5 = 4294967297$. Our theorems relate to generalised such F_q. As Fermat knew, the extended Mersenne number $F_q - 2 = \Pi(r = 0, q - 1)F_r$. The *second Fermat subtraction factorisation theorem* example contains this result for a, p = 2 and f = 1.

An example of the *first Fermat subtraction factorisation theorem (first FSFT)* is

$$a\uparrow(p^c) = \{[a\uparrow(p^{c-1}) - 1][\ \Sigma(s = 0, p - 1)[(a\uparrow(p^{c-1}))\uparrow s]]\} + 1.$$

The *first FSFT* is

$$a\uparrow[(bp)^c] - f\uparrow[(gp)^h] =$$
$$\{[a\uparrow((bp)^{c-1})]^b - [f\uparrow((gp)^{h-1})]^g\}\{\ \Sigma(s = 0, p - 1)$$
$$[(a\uparrow[(bp)^{c-1}])\uparrow bs][(f\uparrow[(gp)^{h-1}])\uparrow g(p - 1 - s)]\}.$$

Proof. Introducing $\alpha = \gamma^p$ and $\beta = \delta^p$, we explore the *cyclotomic* formula

$$\gamma^p - \delta^p = (\gamma - \delta)(\gamma^{p-1} + \gamma^{p-2}\delta + \gamma^{p-3}\delta^2 + \ldots + \delta^{p-1}),$$

which can be written in the form

$$\alpha - \beta = [\alpha^{1/p} - \beta^{1/p}]\{\ \Sigma(s = 0, p - 1)[\alpha^{s/p}][\beta^{1 - 1/p - s/p}]\}.$$

We obtain our result using

$$\alpha = a\uparrow[(bp)^c] = a\uparrow[(bp)^1(bp)^{c-1}] = (a\uparrow[(bp)^{c-1}])\uparrow bp$$

and similarly

$$\beta = f\uparrow[(gp)^h] = (f\uparrow[(gp)^{h-1}])\uparrow gp.$$ ■

An example of the *second FSFT* is

$$a\uparrow(p^q) - f\uparrow(p^q) = [a - f]\ \Pi(r = 0, q - 1)$$
$$\{\ \Sigma(s = 0, p - 1)[a\uparrow(s(p^{q-1-r}))][f\uparrow((p - 1 - s)(p^{q-1-r}))]\}.$$

The *second FSFT* for free parameter m is

$$a\uparrow((bp)^c) - f\uparrow((gp)^h) =$$
$$\{a\uparrow[(b^m)((bp)^{c-m})]\ - f\uparrow[(g^m)((gp)^{h-m})]\}$$
$$\Pi(r = 0, m - 1)\{\ \Sigma(s = 0, p - 1)[a\uparrow(bs((bp)^{c-1-r}))(b\uparrow r))]$$
$$[f\uparrow(g(p - 1 - s)((gp)^{h-1-r})(g\uparrow r))]\}.$$

Proof. Consider the identity

$$\gamma\uparrow(p^m) - \delta\uparrow(p^m) = (\gamma\uparrow(p^{m-1}) - \delta\uparrow(p^{m-1}))$$
$$\{\ \Sigma(s = 0, p - 1)[(\gamma\uparrow(p^{m-1}))\uparrow s][(\delta\uparrow(p^{m-1}))\uparrow(p - 1 - s)]\}.$$

The term $(\gamma\uparrow(p^{m-1}) - \delta\uparrow(p^{m-1}))$ can be replaced for an nth general recursion to give

$$\gamma\uparrow(p^m) - \delta\uparrow(p^m) = (\gamma\uparrow(p^{m-1-n}) - \delta\uparrow(p^{m-1-n}))$$
$$\Pi(r = 0, n)\{\ \Sigma(s = 0, p - 1)$$
$$[(\gamma\uparrow(p^{m-1-r}))\uparrow s][(\delta\uparrow(p^{m-1-r}))\uparrow(p - 1 - s)]\}.$$

Hence allocating the maximum nth replacement to m – 1, our star equation is

(*) $$\gamma\uparrow(p^m) - \delta\uparrow(p^m) = [\gamma - \delta]\ \Pi(r = 0, m - 1)\{\ \Sigma(s = 0, p - 1)$$
$$[(\gamma\uparrow(p^{m-1-r}))\uparrow s][(\delta\uparrow(p^{m-1-r}))\uparrow(p - 1 - s)]\}.$$

Put

$$\alpha = a\uparrow((bp)^c)$$
$$= \gamma\uparrow(p^m),$$

leading to

$$\gamma = \gamma\uparrow(p\uparrow 0) = \gamma\uparrow[(p^m)(p^{-m})] = [\gamma\uparrow p^m]\uparrow(p^{-m})$$
$$= \alpha\uparrow(p^{-m}).$$

Now, as can be confirmed by multiplying both sides by $(b\uparrow{-m})$,

$$p\uparrow{-m} = (bp)^{-m}\, b^m,$$

so

$\gamma = a\uparrow[b^m((bp)^{c-m})]$.

If we likewise allocate

$\beta = f\uparrow(gp)^h$

$= \delta\uparrow(p^m)$,

reducing to

$\delta = \beta\uparrow(p^{-m})$

then we obtain similarly

$\delta = f\uparrow[g^m((gp)^{h-m})]$.

In consequence

$a\uparrow((bp)^c) - f\uparrow((gp)^h) =$

$\{a\uparrow[b^m(bp)^{c-m})] - f\uparrow[g^m(gp)^{h-m}]\}$

$\Pi(r = 0, m-1)\{ \Sigma(s = 0, p-1)$

$[(\alpha\uparrow(p^{-1-r}))\uparrow s][(\beta\uparrow(p^{-1-r}))\uparrow(p-1-s)]\}$.

For the first of the last two terms in [ ], assign

$\alpha = a\uparrow((bp)^c)$

and

$p^{-1-r} = [(bp)^{-1-r}][b^{1+r}]$.

Then this term is

$\{a\uparrow[bs[(bp)^{c-1-r}]b^r]\}$

and the second term is by similar process

$\{f\uparrow[g(p-1-s)[(gp)^{h-1-r}]g^r]\}$. ■

The *first Fermat addition factorisation theorem (first FAFT)* states:

Let g, p $\in$ **N** *be odd numbers and h* $\in$ **N**. *Then*

$a\uparrow((bp)^c) + f\uparrow((gp)^h) =$

$\{(a\uparrow[(bp)^{c-1}])^b + (f\uparrow[(gp)^{h-1}])^g\}\{ \Sigma(s = 0, p-1)$

$[(a\uparrow[(bp)^{c-1}])\uparrow bs][((-f)\uparrow[(gp)^{h-1}])\uparrow g(p-1-s)]\}$.

Proof. Introducing

$\alpha = \gamma^p$, $\beta = \delta^p$,

we revisit the *first FSFT* equation

$\alpha - \beta = [\alpha^{1/p} - \beta^{1/p}]\{ \Sigma(s = 0, p-1)[(\alpha^{s/p})[\beta^{1-1/p-s/p}]\}$.

Using

$\alpha = a\uparrow(bp)^c = a\uparrow[(bp)^1(bp)^{c-1}] = (a\uparrow[(bp)^{c-1}])\uparrow bp$

and similarly

$\beta = f\uparrow(gp)^h = (f\uparrow[(gp)^{h-1}])\uparrow gp$

gives, under the transformation $\beta \to -\beta$, that $\beta^{1/p}$ changes sign as $\delta \to -\delta$, since p is odd, which transforms f $\to$ -f, since gp is odd in $\beta = f\uparrow(gp)^h$. Hence the result. ■

The *second Fermat addition factorisation theorem (second FAFT)* for free parameter m states:

Let g *and* p $\in$ **N** *be odd numbers and h* $\in$ **N**. *Then*

$a\uparrow((bp)^c) + f\uparrow((gp)^h) =$

$\{a\uparrow[(b^m)(bp)^{c-m}] + f\uparrow[(g^m)(gp)^{h-m}]\}$

$\Pi(r = 0, m-1)\{ \Sigma(s = 0, p-1)[a\uparrow(bs[(bp)^{c-1-r}]b^r)]$

$[(-f)\uparrow(g(p-1-s)[(gp)^{h-1-r}]g^r)]\}$.

Proof. The conditions are identical to the *first FAFT*, and the proof follows by close analogy with the *second FSFT* proof. In particular, we note the following result, suitably amended, i.e. we have performed the transformation $\delta \rightarrow -\delta$.

(**) $\gamma\uparrow(p\uparrow m) + \delta\uparrow(p\uparrow m) = [\gamma + \delta]\ \Pi(r = 0, m - 1)\{\ \Sigma(s = 0, p - 1)$
$[(\gamma\uparrow[p\uparrow(m - 1 - r)])\uparrow s][((-\delta)\uparrow[p\uparrow(m - 1 - r)])\uparrow(p - 1 - s)]\}$. ■

We subsequently identify $\varepsilon, \eta, \theta \in \mathbf{C}$ as complex numbers, e = 2.718… and $i = \sqrt{-1}$. We now relax our conditions, to allow complex arithmetic *on*, but not further non- real operations *within*, for example, $a\uparrow b = e\uparrow(i\pi q/p)$ terms. A way of doing this is to consider $a\uparrow b$ expressions as scalars and complex numbers as vectors. This is an idea found in K theory.

We have stated that our factorisation theorems give unique factorisation when applied to natural numbers up to order of factors.

For complex cyclotomics as used in the next lemma, uniqueness of factorisation depends on the class number [30]. That is why, in section 3 on prime number and factorisation theorems, we prefer to use formulae developed for the real case.

The next *FAFT* theorems have *FSFT* analogues. Here is *Lemma 1*.

$$\gamma^p + \delta^p = \Pi(s = 0, p - 1)[\gamma - \delta(e\uparrow i\pi(2s + 1)/p)].$$

Proof. We first note that the leading term in the expansion is γ^p. The trailing term is

$$(-\delta)^p(e\uparrow i\pi[\ \Sigma(s = 0, p - 1)(2s + 1)/p]).$$

Now the following arithmetic series sum has the value

$$\Sigma(s = 0, p - 1)s = p(p - 1)/2,$$

so the Σ summation in [ ] above is

$$[2(p(p - 1)/2) + p]/p = p.$$

Hence, irrespective of whether p is even or odd, since $e\uparrow i\pi(2s + 1) = -1$, the trailing term is δ^p.

Now consider the nth term in the expansion. If $n \neq 0$ or $p - 1$, it consists of a *summation*

$$\Sigma(r = 0, m)\varepsilon_r\gamma^{p - n}\delta^n$$

each ε_r is the product of n factors $(e\uparrow i\pi(2t + 1)/p)$, where the product terms range over all combinations of t from 0 to $p - 1$. If $\Sigma_r\varepsilon_r \neq 0$, it consists of a non-zero vector in the complex plane. Permute the roots under the cyclic transformation $s \rightarrow s + 1$. Then $\varepsilon_r \rightarrow (e\uparrow 2\pi in/p)\varepsilon_r \neq \varepsilon_r$, and $\Sigma_r\varepsilon_r$ remains the same, since it consists of the sum of *all* combinations. The rotation of roots implies the complex sum vector must also be rotated by an ε_r multiplication $\neq 1$, a contradiction unless the sum is zero. ■

The *third Fermat addition factorisation theorem (third FAFT)* is

$$a\uparrow((bp)^c) + f\uparrow((gp)^h) = \Pi(s = 0, p - 1)$$
$$\{(a\uparrow[(bp)^{c-1}])^b - (f\uparrow[(gp)^{h-1}])^g(e\uparrow i\pi(2s + 1)/p)\}.$$

Proof. Put

$$\alpha = \gamma^p,\ \beta = \delta^p.$$

Then using *lemma 1*, we obtain

$$\alpha + \beta = \Pi(s = 0, p - 1)[\alpha^{1/p} - \beta^{1/p}(e\uparrow i\pi(2s + 1)/p)]$$

and with the substitutions

$$\alpha = a{\uparrow}(bp{\uparrow}c) = a{\uparrow}[(bp{\uparrow}1)(bp{\uparrow}(c-1)] = (a{\uparrow}[bp{\uparrow}(c-1)]){\uparrow}bp$$

and

$$\beta = f{\uparrow}(gp{\uparrow}h) = (f{\uparrow}[gp{\uparrow}(h-1)]){\uparrow}gp,$$

this results in the theorem. ■

Lemma 2. Let $p = j(2{\uparrow}k)$ *with* j *odd and* k *non-negative. Then*

$$e{\uparrow}(i\pi(2s+1)/p) = (-1){\uparrow}(1/p) = {}^{p}\sqrt{-1}$$

has a real root (*which is* -1) *only for* $k = 0$.

Proof. Let $k = 0$, so p is an odd number. By applying ${\uparrow}(1/p)$ to the equations below

$$-1 = (-1){\uparrow}p$$

is a solution, for p odd, so ${}^{p}\sqrt{-1}$ has a real root, -1. If $k \neq 0$, then p is even, implying

$$-1 \neq (-1){\uparrow}p \text{ and } -1 \neq 1{\uparrow}p.$$

The norm of the root is 1, so there are no real root of -1 possibilities for p even. ■

The *fourth Fermat addition factorisation theorem (fourth FAFT)* states:
Let $p \in \mathbf{N}$ *be an odd number. Then for free parameter* m

$$a{\uparrow}((bp)^{c}) + f{\uparrow}((gp)^{h}) =$$
$$\{a{\uparrow}[(b^{m})(bp)^{c-m}] + f{\uparrow}[(g^{m})(gp)^{h-m}]\}$$
$$\Pi(r = 0, m-1)\{\ \Pi(s = 0, p-1, \text{omit } (p-1)/2)$$
$$[(a{\uparrow}(b[(bp)^{c-1-r}]b^{r}))$$
$$-(f{\uparrow}(g[(gp)^{h-1-r}]g^{r}))(e{\uparrow}i\pi(2s+1)/p)]\}.$$

Proof. Consider the following factorisation identity, beginning with the 'real root' case, which corresponds to $s = (p-1)/2$, so by *lemma 2*, p is odd:

$$\gamma{\uparrow}(p{\uparrow}m) + \delta{\uparrow}(p{\uparrow}m) = \{\gamma{\uparrow}(p{\uparrow}(m-1)) + \delta{\uparrow}(p{\uparrow}(m-1))]\}$$
$$\Pi(s = 0, p-1, \text{omit } (p-1)/2)$$
$$[\gamma{\uparrow}(p{\uparrow}(m-1)) - [\delta{\uparrow}(p{\uparrow}(m-1))](e{\uparrow}i\pi(2s+1)/p)].$$

The term $\{\gamma{\uparrow}(p{\uparrow}(m-1)) + \delta{\uparrow}(p{\uparrow}(m-1))\}$ can be replaced for an nth general recursion to give

$$\gamma{\uparrow}(p{\uparrow}m) + \delta{\uparrow}(p{\uparrow}m) = \{\gamma{\uparrow}[p{\uparrow}(m-1-n)] + \delta{\uparrow}[p{\uparrow}(m-1-n)]\}$$
$$\Pi(r = 0, n)[\ \Pi(s = 0, p-1, \text{omit } (p-1)/2)$$
$$\{\gamma{\uparrow}[p{\uparrow}(m-1-r)] - (\delta{\uparrow}[p{\uparrow}(m-1-r)])(e{\uparrow}i\pi(2s+1)/p)]\}.$$

Hence allocating the maximum nth replacement to $m-1$, our equation is

$$\gamma{\uparrow}(p{\uparrow}m) + \delta{\uparrow}(p{\uparrow}m) = [\gamma + \delta]\ \Pi(r = 0, m-1)\{\ \Pi(s = 0, p-1, \text{omit } (p-1)/2)$$
$$\{\gamma{\uparrow}[p{\uparrow}(m-1-r)] - (\delta{\uparrow}[p{\uparrow}(m-1-r)])(e{\uparrow}i\pi(2s+1)/p)\}\}.$$

Our substitutions now follow the steps of the *second FSFT.* Put

$$\alpha = a{\uparrow}(bp{\uparrow}c) = \gamma{\uparrow}(p{\uparrow}m).$$

We recall this leads to

$$\gamma = \alpha{\uparrow}(p{\uparrow}\text{-}m) = \alpha{\uparrow}[(bp{\uparrow}\text{-}m)(b{\uparrow}m)] = a{\uparrow}[(b{\uparrow}m)(bp{\uparrow}(c-m))].$$

If we likewise allocate

$$\beta = f{\uparrow}(gp{\uparrow}h) = \delta{\uparrow}(p{\uparrow}m),$$

then we obtain similarly

$$\delta = f{\uparrow}[(g{\uparrow}m)(gp{\uparrow}(h-m))].$$

Consequently

$a\uparrow(bp\uparrow c) + f\uparrow(gp\uparrow h) =$

$\{a\uparrow[(b\uparrow m)(bp\uparrow(c - m))] + f\uparrow[(g\uparrow m)(gp\uparrow(h - m))]\}$

$\Pi(r = 0, m - 1)\{ \Pi(s = 0, p - 1, \text{omit } (p - 1)/2)$

$[(\alpha\uparrow[p\uparrow(-1 - r)]) - (\beta\uparrow[p\uparrow(-1 - r)])(e\uparrow i\pi(2s + 1)/p)]\}.$

For the first term of the last expression in [ ], assign

$\alpha = a\uparrow(bp\uparrow c)$

and

$p\uparrow(-1 - r) = [bp\uparrow(-1 - r)](b\uparrow(1 + r)).$

Then this term is

$\{a\uparrow(b[bp\uparrow(c - 1 - r)](b\uparrow r))\}$

and in comparable manner the second term is

$\{(f\uparrow(g[gp\uparrow(h - 1 - r)](g\uparrow r)))(e\uparrow i\pi(2s + 1)/p)\}.$ ■

Under the *FAFT* constraints, say p odd, each of the FAFT and FSFT equations is equivalent by "if and only if" equivalence to the identity $(\gamma, \delta) = (\gamma, \delta)$, so under FAFT constraints their equaliser (intersection) is also equivalent to this identity. Thus we can also form linear combinations of *FAFT* and *FSFT* equations. These theorems we call *linear combination factorisation theorems – LCFT*. Here is a typical example.

Let g *and* p $\in$ **N** *be odd, h* $\in$ **N**. *Put*

$A = a\uparrow[(b^{m})(bp)^{c - m}]$

$B = f\uparrow[(g^{m})(gp)^{h - m}]$

$C_r = \Sigma(s = 0, (p - 1)/2)\{a\uparrow(2bs[(bp)^{c - 1 - r}]b^{r})\}$

$\{f\uparrow(g(p - 1 - 2s)[(gp)^{h - 1 - r}]g^{r})\}$

$D_r = \Sigma(s = 0, (p - 3)/2)\{a\uparrow(b(2s + 1)[(bp)^{c - 1 - r}]b^{r})\}$

$\{f\uparrow(g(p - 2 - 2s)[(gp)^{h - 1 - r}]g^{r})\}.$

Then

$\eta[a\uparrow((bp)^{c})] + \theta[f\uparrow((gp)^{h})] = ½[(\eta + \theta)(A + B)\ \Pi(r = 0, m - 1)(C_r - D_r) +$

$(\eta - \theta)(A - B)\ \Pi(r = 0, m - 1)(C_r + D_r)].$ ■

Our formula for difference of powers may be generalised by putting q = jk and

$\gamma^{q} - \delta^{q} = [(\gamma\uparrow k) - (\delta\uparrow k)]\{ \Sigma(s = 0, j - 1)[\gamma\uparrow ks][\delta\uparrow k(j - 1 - s)]\}.$

Accordingly, with $j = 2^{n}$, ω_{2j} a primitive 2jth root of unity and k odd

$\gamma^{q} + \delta^{q} = [(\gamma\uparrow k) + (\omega_{2j}\delta)\uparrow k]\{\Sigma(s = 0, j - 1)[\gamma\uparrow ks][(-\omega_{2j}\delta)\uparrow k(j - 1 - s)]\},$

or with the same $j = 2^{n}$ and k odd

$\gamma^{q} + \delta^{q} = [(\gamma\uparrow 2^{n}) + (\delta\uparrow 2^{n})]$

$\{\Sigma(s = 0, k - 1)[\gamma\uparrow 2^{n}s][(-(\delta\uparrow 2^{n}))\uparrow(k - 1 - s)]\}.$

This can be compared with the binomial identity

$\gamma^{q} + \delta^{q} = [(\gamma\uparrow k) + (\delta\uparrow k)]^{j} -$

$\{\Sigma(s = 1, j - 1)[(j - 1)!/s!(j - 1 - s)!][\gamma\uparrow ks][\delta\uparrow k(j - 1 - s)]\}.$ ■

Thus for q even as above an option is

$Z = \eta(\gamma^{q}) + \theta(\delta^{q}) = ½[(\eta + \theta)(\gamma\uparrow[2^{n}] + \delta\uparrow[2^{n}])U + (\eta - \theta)(\gamma\uparrow[2^{n}] - \delta\uparrow[2^{n}])V],$

with

$U = \Sigma(s = 0, k - 1)[\gamma\uparrow(2^n s)]\{[-(\delta\uparrow 2^n)]\uparrow(k - 1 - s)\}$

and

$V = \Sigma(s = 0, k - 1)[\gamma\uparrow(2^n s)]\{\delta\uparrow[2^n(k - 1 - s)]\}$. ■

The general *LCFT* formula for any p is then

$Z = ½[(1 - (-1)^p)($*formula for odd* p$) + (1 + (-1)^p)($*formula for even* p$)]$. ■

1.5 Extension of the LCFT to many variables.

We now give a modified example of the *LCFT* (note that here firstly, p is odd)

$Z = \eta(\gamma\uparrow p) + \theta(\delta\uparrow p) = ½[(\eta + \theta)(\gamma + \delta)X + (\eta - \theta)(\gamma - \delta)Y]$,

generated from essentially γ replaced by the sum of variables $\Sigma(i = 0, n - 1)\gamma_i$, and δ replaced by the sum $\Sigma(i = 0, n - 1)\delta_i$. This is suitably general, because if, for example, there are less variables δ_i than γ_i, say j of them, we can set δ_i to zero for $j \leq i < n - 1$. We will first give an example restricted to the case n = 3.

The *multinomial theorem* is

$(\gamma_0 + \gamma_1 + \dots + \gamma_{n-1})^p = \Sigma p!/(p_0!p_1! \dots p_{n-1}!)[(\gamma_0\uparrow p_0) \dots (\gamma_{n-1}\uparrow p_{n-1})]$,

where the sum is extended over all non-negative p_i with $\Sigma p_i = p$.

For example, if n = 3 = p, then

$(\gamma_0 + \gamma_1 + \gamma_3)^3 = (\gamma_0^2 + 3\gamma_1^2 + 3\gamma_2^2)\gamma_0 + (\gamma_1^2 + 3\gamma_2^2 + 3\gamma_0^2)\gamma_1 + (\gamma_2^2 + 3\gamma_0^2 + 3\gamma_1^2)\gamma_2$
$+ 6\gamma_0\gamma_1\gamma_2$.

Let p *be odd. Then the expansion of*

$(\gamma_0 + \gamma_1 + \dots + \gamma_{n-1})^p$

can be written as a sum

$\Sigma_{y,\text{combinations for } y}\, J_{r \dots x}(\gamma_r \dots \gamma_x)$

where the number of terms in the sequence r, … x *is* y, *with* y *odd* $\leq$ n *and each coefficient* $J_{r \dots x}$ *is invariant under any transformation* $\gamma_s \rightarrow -\gamma_s$, $r \geq s \geq x$.

Proof. Each term in the expansion is a scalar, α, times a product $\Pi_t\gamma_t^q$, with $\Sigma q = p$, p odd. This may be represented by $\alpha\Pi_j\gamma_j^u\Pi_k\gamma_k^v$, with u even and v odd. Hence it may be represented by even parity terms invariant under $\gamma_s \rightarrow -\gamma_s$ given by $\alpha\Pi_j\gamma_j^u\Pi_k\gamma_k^{v-1}$, each multiplied by odd parity terms $(\gamma_r \dots \gamma_x) = \Pi_k\gamma_k$. Collecting together all the even parity terms, we put

$J_{r \dots x} = \Sigma_i\alpha_i\Pi_j\gamma_j^u\Pi_k\gamma_k^{v-1}$.

If there exists any term $J_{r \dots x}\Pi_k\gamma_k$, where k ranges over an even number of values, then $J_{r \dots x} = \Sigma_i\alpha_i\Pi_k\gamma_k^w$, where Σw is even for each i, since $J_{r \dots x}$ is of even parity, which contradicts for each i that p = Σw + the range of k values, is odd. Hence y = the range of k values, is odd. ■

If $p \geq n$, *with* p, n *odd, then the number of* $J_{r \dots x}$ *terms is* 2^{n-1}. *If* p < n, *the number of terms is*

$\Sigma(k = 0, (p - 1)/2)[n!/(2k + 1)!(n - 2k - 1)!]$.

Proof. If $p \geq n$, then the number of terms is the number of combinations of odd $\gamma_r \ldots\gamma_x$

$= n + n(n-1)(n-2)/2.3 + \ldots + n!/(2k+1)!(n-2k-1)! + \ldots + 1$

$= ½(1+1)^n = 2^{n-1}$.

If $p < n$, n odd, it is the same series truncated after the $½(p+1)$th term. ■

Consider for $n = 3$, p odd,

$(\gamma_0 + \gamma_1 + \gamma_2)^p - (\delta_0 + \delta_1 + \delta_2)^p = [\gamma_0 + \gamma_1 + \gamma_2 - \delta_0 - \delta_1 - \delta_2]$

$[\Sigma(s = 0, p-1)[(\gamma_0 + \gamma_1 + \gamma_2)\uparrow s][(\delta_0 + \delta_1 + \delta_2)\uparrow(p-1-s)]]$,

which by the above theorem can be equated to

$J_0\gamma_0 + J_1\gamma_1 + J_2\gamma_2 + J_{012}\gamma_0\gamma_1\gamma_2 + K_0\delta_0 + K_1\delta_1 + K_2\delta_2 + K_{012}\delta_0\delta_1\delta_2$,

where if p were $= 1$ then we would have $J_{012} = K_{012} = 0$.

We wish to add together the following combination

$a_0[(\gamma_0 + \gamma_1 + \gamma_2)^p - (\delta_0 + \delta_1 + \delta_2)^p] + a_1[(-\gamma_0 + \gamma_1 + \gamma_2)^p - (\delta_0 + \delta_1 + \delta_2)^p]$

$a_2[(\gamma_0 - \gamma_1 + \gamma_2)^p - (\delta_0 + \delta_1 + \delta_2)^p] + b_0[(\gamma_0 + \gamma_1 + \gamma_2)^p - (-\delta_0 - \delta_1 - \delta_2)^p]$

$b_1[(\gamma_0 + \gamma_1 + \gamma_2)^p - (\delta_0 - \delta_1 - \delta_2)^p] + b_2[(\gamma_0 + \gamma_1 + \gamma_2)^p - (-\delta_0 + \delta_1 - \delta_2)^p]$.

This linear combination may be equated to

$\eta_0J_0\gamma_0 + \eta_1J_1\gamma_1 + \eta_2J_2\gamma_2 + \eta_{012}J_{012}\gamma_0\gamma_1\gamma_2 +$

$\theta_0K_0\delta_0 + \theta_1K_1\delta_1 + \theta_2K_2\delta_2 + \theta_{012}K_{012}\delta_0\delta_1\delta_2$,

where, for example,

$J_0\gamma_0 = (-J_0)(-\gamma_0)$

$J_{012}\gamma_0\gamma_1\gamma_2 = (-J_{012})(-\gamma_0)(-\gamma_1)(-\gamma_2)$,

giving

$\eta_0 = a_0 - a_1 + a_2 + b_0 + b_1 + b_2$

$\eta_1 = a_0 + a_1 - a_2 + b_0 + b_1 + b_2$

$\eta_2 = a_0 + a_1 + a_2 + b_0 + b_1 + b_2$

$-\theta_0 = a_0 + a_1 + a_2 - b_0 + b_1 - b_2$

$-\theta_1 = a_0 + a_1 + a_2 - b_0 - b_1 + b_2$

$-\theta_2 = a_0 + a_1 + a_2 - b_0 - b_1 - b_2$.

We also have the supplementary equations

$\eta_{012} = a_0 - a_1 - a_2 + b_0 + b_1 + b_2$

$-\theta_{012} = a_0 + a_1 + a_2 - b_0 + b_1 + b_2$.

A little linear algebra then gives

$a_1 = (\eta_2 - \eta_0)/2$,

$a_2 = (\eta_2 - \eta_1)/2$

and

$a_0 = (\eta_0 + \eta_1 - \eta_2 - \theta_2)/2$.

Likewise

$b_1 = (\theta_2 - \theta_0)/2$,

$b_2 = (\theta_2 - \theta_1)/2$

and

$b_0 = (\theta_0 + \theta_1 - \theta_2 + \eta_2)/2$.

These results, easily extended to many variables, imply

$$\eta_{012} = \eta_0 + \eta_1 - \eta_2$$

and

$$\theta_{012} = \theta_0 + \theta_1 - \theta_2.$$

We can express $J_0\gamma_0$, $J_1\gamma_1$, $J_2\gamma_2$ and $J_{012}\gamma_0\gamma_1\gamma_2$ in terms of $(\gamma_0 + \gamma_1 + \gamma_2)^p$.

$$J_0\gamma_0 = \tfrac{1}{4}[(\gamma_0 + \gamma_1 + \gamma_2)^p - (-\gamma_0 + \gamma_1 + \gamma_2)^p + (\gamma_0 - \gamma_1 + \gamma_2)^p + (\gamma_0 + \gamma_1 - \gamma_2)^p].$$
$$J_1\gamma_1 = \tfrac{1}{4}[(\gamma_0 + \gamma_1 + \gamma_2)^p + (-\gamma_0 + \gamma_1 + \gamma_2)^p - (\gamma_0 - \gamma_1 + \gamma_2)^p + (\gamma_0 + \gamma_1 - \gamma_2)^p].$$
$$J_2\gamma_2 = \tfrac{1}{4}[(\gamma_0 + \gamma_1 + \gamma_2)^p + (-\gamma_0 + \gamma_1 + \gamma_2)^p + (\gamma_0 - \gamma_1 + \gamma_2)^p - (\gamma_0 + \gamma_1 - \gamma_2)^p].$$
$$J_{012}\gamma_0\gamma_1\gamma_2 = \tfrac{1}{4}[(\gamma_0 + \gamma_1 + \gamma_2)^p - (-\gamma_0 + \gamma_1 + \gamma_2)^p - (\gamma_0 - \gamma_1 + \gamma_2)^p - (\gamma_0 + \gamma_1 - \gamma_2)^p].$$

We now introduce the symbol $X_{uvw,xyz}$, in which u is 0 if γ_0 is positive, and u is 1 if γ_0 is negative, and similarly with v for γ_1 and w for γ_2. We adopt a similar type of convention for x, y, z with respect to δ_0, δ_1 and δ_2. Thus we have the expression for p odd

$$(\gamma_0 + \gamma_1 + \gamma_2)^p - (\delta_0 + \delta_1 + \delta_2)^p = [\gamma_0 + \gamma_1 + \gamma_2 - \delta_0 - \delta_1 - \delta_2]X_{000,000},$$

with

$$X_{000,000} = [\ \Sigma(s = 0, p - 1)[(\gamma_0 + \gamma_1 + \gamma_2)\uparrow s][(\delta_0 + \delta_1 + \delta_2)\uparrow(p - 1 - s)]].$$

Hence we have the following *theorem.*

$$\eta_0J_0\gamma_0 + \eta_1J_1\gamma_1 + \eta_2J_2\gamma_2 + (\eta_0 + \eta_1 - \eta_2)J_{012}\gamma_0\gamma_1\gamma_2 +$$
$$\theta_0K_0\delta_0 + \theta_1K_1\delta_1 + \theta_2K_2\delta_2 + (\theta_0 + \theta_1 - \theta_2)K_{012}\delta_0\delta_1\delta_2$$
$$= \tfrac{1}{2}\ [(\eta_0 + \eta_1 - \eta_2 - \theta_2)(\gamma_0 + \gamma_1 + \gamma_2 - \delta_0 - \delta_1 - \delta_2)X_{000,000}$$
$$+ (\eta_2 - \eta_0)(-\gamma_0 + \gamma_1 + \gamma_2 - \delta_0 - \delta_1 - \delta_2)X_{100,000}$$
$$+ (\eta_2 - \eta_1)(\gamma_0 - \gamma_1 + \gamma_2 - \delta_0 - \delta_1 - \delta_2)X_{010,000}$$
$$+ (\theta_0 + \theta_1 - \theta_2 + \eta_2)(\gamma_0 + \gamma_1 + \gamma_2 + \delta_0 + \delta_1 + \delta_2)X_{000,111}$$
$$+ (\theta_2 - \theta_0)(\gamma_0 + \gamma_1 + \gamma_2 - \delta_0 + \delta_1 + \delta_2)X_{000,011}$$
$$+ (\theta_2 - \theta_1)(\gamma_0 + \gamma_1 + \gamma_2 + \delta_0 - \delta_1 + \delta_2)X_{000,101}].\ \blacksquare$$

Consider the general case for p odd

$$(\Sigma(i = 0, n - 1)\gamma_i)^p - (\Sigma(i = 0, n - 1)\delta_i)^p = [\Sigma(i = 0, n - 1)(\gamma_i - \delta_i)]$$
$$[\ \Sigma(s = 0, p - 1)[(\Sigma(i = 0, n - 1)\gamma_i)\uparrow s][(\Sigma(i = 0, n - 1)\delta_i)\uparrow(p - 1 - s)]],$$

which by the above theorem can be equated to

$$\Sigma_{y,\text{combinations for } y}\ J_{r\ \dots\ x}(\gamma_r\ \dots\ \gamma_x) - \Sigma_{y,\text{combinations for } y}\ K_{r\ \dots\ x}(\delta_r\ \dots\ \delta_x);$$

we have introduced $K_{r\ \dots\ x}$ for the coefficient of $(\delta_r \dots \delta_x)$ similar to $J_{r\ \dots\ x}$ for $(\gamma_r\ \dots\ \gamma_x)$.

If $p < n$, then for $y > p$, $J_{r\ \dots\ x} = K_{r\ \dots\ x} = 0$.

We wish now to add together linear combinations of 2n independent equations, each of a type similar to the above, expressed in the n variables γ_i and the n variables δ_i.

Define the symbol $\xi(i,j)$ by

$$\xi(i,j) = 1 \text{ for } i = j$$
$$= 0 \text{ for } i \neq j.$$

Then our general linear combination is

$$\Sigma(s = 0, n - 1)[a_s(\Sigma(r = 0, n - 1)(\gamma_r - 2\xi(s,r - 1)\gamma_{r-1}))^p - (\Sigma(r = 0, n - 1)\delta_r)^p]$$
$$+ \Sigma(s = 0, n - 1)[b_s(\Sigma(r = 0, n - 1)\gamma_r)^p - (\Sigma(r = 0, n - 1)(-\delta_r + 2\xi(s,r - 1)\delta_r))^p].$$

With y odd, our linear combination equals

$$\Sigma_{y,\text{combinations for } y}\ \eta_{r\ \ldots\ x} J_{r\ \ldots\ x}(\gamma_r \ldots \gamma_x) + \Sigma_{y,\text{combinations for } y}\ \theta_{r\ \ldots\ x} K_{r\ \ldots\ x}(\delta_r \ldots \delta_x),$$

where

$$J_{r\ \ldots\ x}(\gamma_r \ \ldots \gamma_x) = -J_{r\ \ldots\ x}[(-\gamma_r) \ldots (-\gamma_x)],$$
$$K_{r\ \ldots\ x}(\delta_r \ \ldots \delta_x) = -K_{r\ \ldots\ x}[(-\delta_r) \ldots (-\delta_x)].$$

This gives

$$\eta_r = \Sigma(s = 0, n - 1)[a_s - 2\xi(r + 1,s)a_{r + 1} + b_s]$$

and

$$-\theta_r = \Sigma(s = 0, n - 1)[a_s - b_s + 2\xi(r + 1,s)b_{r + 1}].$$

Introducing, say, $\eta(012)$ for η_{012}, we also have the supplementary equations

$$\eta(r_0 \ldots r_{y - 1}) = \Sigma(s = 0, n - 1)[a_s - 2\Sigma(t = 0, y - 1)\xi(r_{t + 1},s)a_{t + 1} + b_s]$$
$$-\theta(r_0 \ldots r_{y - 1}) = \Sigma(s = 0, n - 1)[a_s - b_s + 2\Sigma(t = 0, y - 1)\xi(r_{t + 1},s)b_{t + 1}],$$

where we consider the set of variables $(r_0 \ldots r_{y - 1})$ to be any suitable combination for y.

Applying linear algebra, where s goes from 0 to n – 2, we obtain

$$a_{s + 1} = (\eta_{n - 1} - \eta_s)/2,$$
$$a_0 = ((-n + 2)\eta_{n - 1} + \Sigma(s = 0, n - 2)\eta_s - \theta_{n - 1})/2.$$

Likewise

$$b_{s + 1} = (\theta_{n - 1} - \theta_s)/2,$$
$$b_0 = ((-n + 2)\theta_{n - 1} + \Sigma(s = 0, n - 2)\theta_s + \eta_{n - 1})/2.$$

These results imply

$$\eta(r_0 \ldots r_{y - 1}) = ½[(-n + 4)\eta_{n - 1} + (-n + 2)\theta_{n - 1}] - \Sigma(s = 0, n - 1)[\Sigma(t = 0, y - 1)\xi(r_{t + 1},s)(\eta_{n - 1} - \eta_t)].$$
$$-\theta(r_0 \ldots r_{y - 1}) = ½[(n - 4)\theta_{n - 1} + (-n + 2)\eta_{n - 1}] + \Sigma(s = 0, n - 1)[\Sigma(t = 0, y - 1)\xi(r_{t + 1},s)(\theta_{n - 1} - \theta_t)].$$

$J_{r\ \ldots\ x}(\gamma_r \ \ldots \gamma_x)$ can then be computed explicitly.

We now introduce the symbol $X(v_0v_1 \ldots v_{n - 1}, x_0x_1 \ldots x_{n - 1})$, in which v_i is 0 if γ_i is positive, and v_i is 1 if γ_i is negative, and likewise for x_i with respect to δ_i. Thus we have the expression

$$(\Sigma(s = 0, n - 1)\gamma_s)^p - (\Sigma(s = 0, n - 1)\delta_s)^p = [\Sigma(s = 0, n - 1)\gamma_s - \Sigma(s = 0, n - 1)\delta_s]X(0 \ldots 0,0 \ldots 0)$$

with

$$X(0 \ldots 0,0 \ldots 0) = [\Sigma(r = 0, p - 1)[(\Sigma(s = 0, n - 1)\gamma_s)\uparrow r][(\Sigma(s = 0, n - 1)\delta_s)\uparrow(p - 1 - r)]].$$

Writing, say, $J(012)$ for J_{012} and $\gamma(0)$ for γ_0, we have the following *theorem.*

$$\Sigma_{y,\text{combinations for } y}\ \eta(r_0 \ldots r_{y - 1})J(r_0 \ldots r_{y - 1})(\gamma(r_0) \ldots \gamma(r_{y - 1}))$$
$$+ \Sigma_{y,\text{combinations for } y}\ \theta(r_0 \ldots r_{y - 1})K(r_0 \ldots r_{y - 1})(\delta(r_0) \ldots \delta(r_{y - 1}))$$
$$= \Sigma_{y,\text{combinations for } y}\ ½[(-n + 4)\eta_{n - 1} + (-n + 2)\theta_{n - 1} - 2\Sigma(s = 0, n - 1)$$
$$[\Sigma(t = 0, y - 1)\xi(r_{t + 1},s)(\eta_{n - 1} - \eta_t)]]$$
$$[\Sigma(s = 0, n - 1)[(1 - 2v_s)\gamma_s - \delta_s]X(v_0v_1 \ldots v_{n - 1}, 0 \ldots 0)]$$
$$- \Sigma_{y,\text{combinations for } y}\ ½[(n - 4)\theta_{n - 1} + (-n + 2)\eta_{n - 1} + 2\Sigma(s = 0, n - 1)$$
$$[\Sigma(t = 0, y - 1)\xi(r_{t + 1},s)(\theta_{n - 1} - \theta_t)]]$$
$$[\Sigma(s = 0, n - 1)[\gamma_s + (1 - 2x_s)\delta_s]X(0 \ldots 0, x_0x_1 \ldots x_{n - 1})]. \blacksquare$$

If q is even = $(2^n)k$, with k odd, then a formula for even q is obtained from the above formula under the transformation $p \to k$, $\gamma_s \to \gamma_s \uparrow 2^n$, $\delta_s \to \delta_s \uparrow 2^n$.

Alternatively, for q even, a complex cyclotomic formula can be utilised, where we set $q = jk = (2^n)k$, with k odd and ω_{2j} a primitive 2jth root of unity, obtainable from the above under the transformation $p \to j$, $\gamma_s \to \gamma_s \uparrow k$, $\delta_s \to (-\omega_{2j}\delta_s)\uparrow k$.

We recall, given u = p or q, the general *LCFT* formula for Z is then

$Z = ½\{[1 – (-1)^u]$(*formula for* u *odd*) + $[1 + (-1)^u]$(*formula for* u *even*)$\}$. ■