2.5. The optimal panel slope.

2.5.1. The optimal slope of a panel, by latitude.

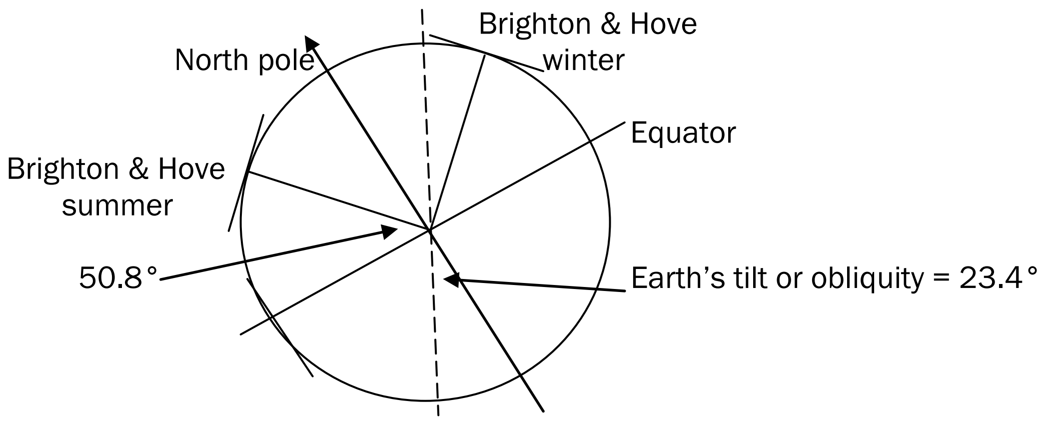
The amount of daily incident solar radiation varies with the time of year and the latitude of the location, where the latitude is measured by the number of degrees north of the Earth's equator. In general, we will denote this angle by θ, where for example Brighton & Hove is at latitude θ = 50.8° N.

We wish to calculate from first principles a good approximation to the slope of a PV array so that the incident solar radiation is at a maximum for a particular latitude averaged over a year.

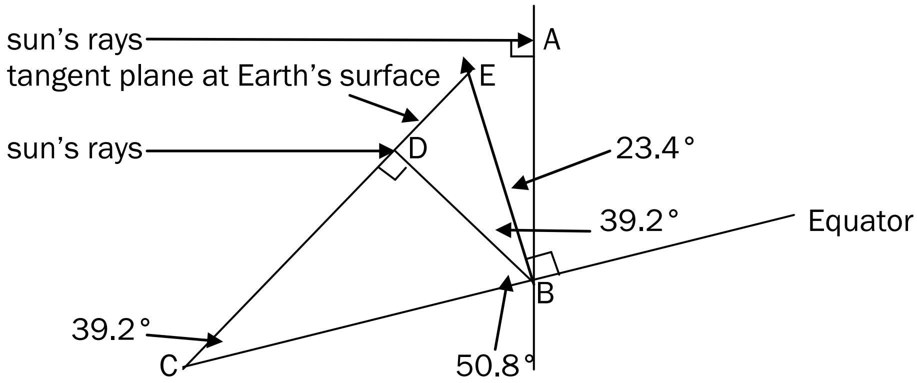
For our calculation we wish to estimate how far from a true value the calculated values at most will be if we adopt a Copernican model of circular motion of the Earth round the sun with uniform velocity and the sun at the centre of this circle.

The distance of the Earth from the sun is very close to 1 astronomical unit, called an AU, which is defined as 149,597,870.7 Km. The nearest approach, called perihelion, is 0.983 AU, and the furthest distance, called aphelion, is 1.017 AU. Their sum, 2 AU, is the longest 'diameter' of the ellipse. Since an ellipse is a squashed circle, we can measure the amount it deviates from a circle by the eccentricity, e, of the ellipse, which for the Earth is 0.0167, so this is very nearly a circle. The focus of the ellipse, which is very near to the centre of the sun, is at a distance given by (perihelion + aphelion) × e = 0.0334 AU from the midpoint of this 'diameter'. The speed of the Earth at perihelion is 30,300 m/s and at aphelion 29,300 m/s. So we see that if we adopt the simplistic Copernican model, we will get results close to the truth, and we can quantify this.

The rotation axis of the Earth is tilted at an angle, called the obliquity, to the vertical of the Earth's orbital plane around the sun. The obliquity is 23° 26′, where 26′ is 26 minutes or 60ths of a degree, but in decimal notation we can represent this number of degrees as 23.4°.



We will compute the angle of the sun's rays at the summer and winter solstices, on 21st June and 21st December respectively. At the longest day in summer, the summer solstice, we define this angle in terms of the obliquity and latitude of the array, the latitude chosen being that at Brighton & Hove, θ = 50.8°.



The Earth's orbital plane is called the ecliptic. In the diagram above the sun's rays meet the vertical to the plane of the ecliptic at right angles. B is at the centre of the Earth, and in this planar representation the line running through CB is in the plane defined by the equator. Brighton & Hove is represented at D, and the tangent plane at D (the flat roof!) is the line ED continued to C.

The obliquity, angle ABE, is 23.4°. The latitude of D is angle DBC = θ = 50.8°.

Since the angles in a triangle sum to 180°

angle BCD + angle DBC + 90° = 180°,

which gives

angle BCD = 90° – θ = 39.2°.

Angle EBC is a right angle, so

angle EBD + angle DBC = 90°,

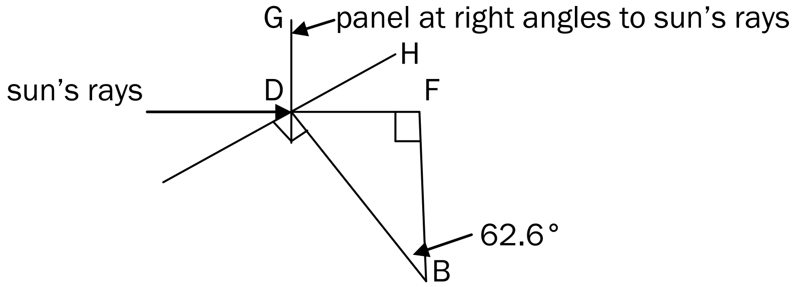
or

angle EBD = 90° – θ = 39.2°.

Thus

angle DBA = 39.2° + 23.4° = 113.4° – θ = 62.6°.

We now look at more detail around the vicinity of D, Brighton & Hove. We want to find angle GDH, which is the best slope of the PV array with the horizontal under the summer solstice assumptions we have made, assuming these are the most relevant. Please excuse the unrepresentative angles in the diagram!



Since DBF is a right angled triangle

angle FBD = 90° – angle FDB,

and since

angle HDB = 90°,

angle HDF = angle HDB – angle FDB

angle HDF = 90° – 90° + angle FBD = angle FBD

and

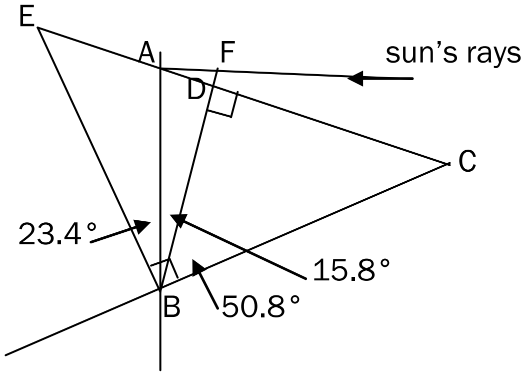
angle GDF = 90° = angle GDH + angle HDF,

we obtain

angle GDH = 90° – angle HDF = 90° – 62.6° = θ – 23.4° = 27.4°,

which is the optimum angle for the panel using this calculation for the sun's rays to be at right angles to the panel at midday of the summer solstice.

We will now look at the situation at the winter solstice, the shortest day of the year.



D is the location of Brighton & Hove and CD is a line tangent to the Earth's surface at D. B again depicts the Earth's centre, and BC represents a line going through the equatorial plane. BE at right angles to BC is the Earth's rotation axis, whereas BA is at right angles to the sun's rays.

We are interested in the triangle FAD, where we will find that angle FAD is the angle at which the sun's rays hit the horizontal, and because the PV panel is assumed to be at right angles to the sun's rays for maximum output, this will give the slope of the PV array with the horizontal.

Angle EBA is the Earth's obliquity, 23.4°. Angle DBC is the magnitude of the latitude of D, θ = 50.8°. Since

angle EBA + angle ABD + angle FBC = angle EBC = θ = 90°,

we have

23.4° + angle ABD + θ = 90°,

or

angle ABD = 66.6° – θ = 15.8°.

Now

angle AFD = angle AFB,

and triangle FAD contains a right angle, so

angle FAB + angle AFD + angle FBA = 180°,

giving

90° + angle AFD + 66.6° – θ = 180°

angle AFD = 90° – 66.6° + θ,

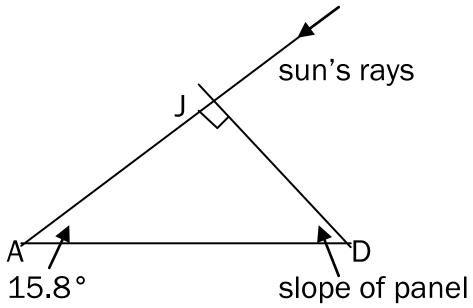
and since angle ADF is a right angle

angle FAD + 90° + (90° – 66.6° + θ) = 180°,

which means

angle FAD = 66.6° – θ = 15.8°.

Thus at the winter solstice, if the panel is at an angle to maximise its output, the slope of the panel, angle JAD in the diagram below, satisfies in the right angled triangle JAD



$$90° + (66.6° - \theta) + \text{angle JDA} = 180°,$$
$$\text{angle JDA} = 23.4° + \theta = 74.2°.$$

We now have the angle for maximum power output due to the sun's rays at the summer and winter solstices, and under the same assumptions, we wish to calculate the maximum power output over the whole year.

We have already mentioned that the effect of cloud is present more in winter than in summer. This will affect the calculation. Also, of course, winter days are shorter than summer ones. Rather than do an explicit calculation to factor in these effects directly, we will take data from *Figure* 3.

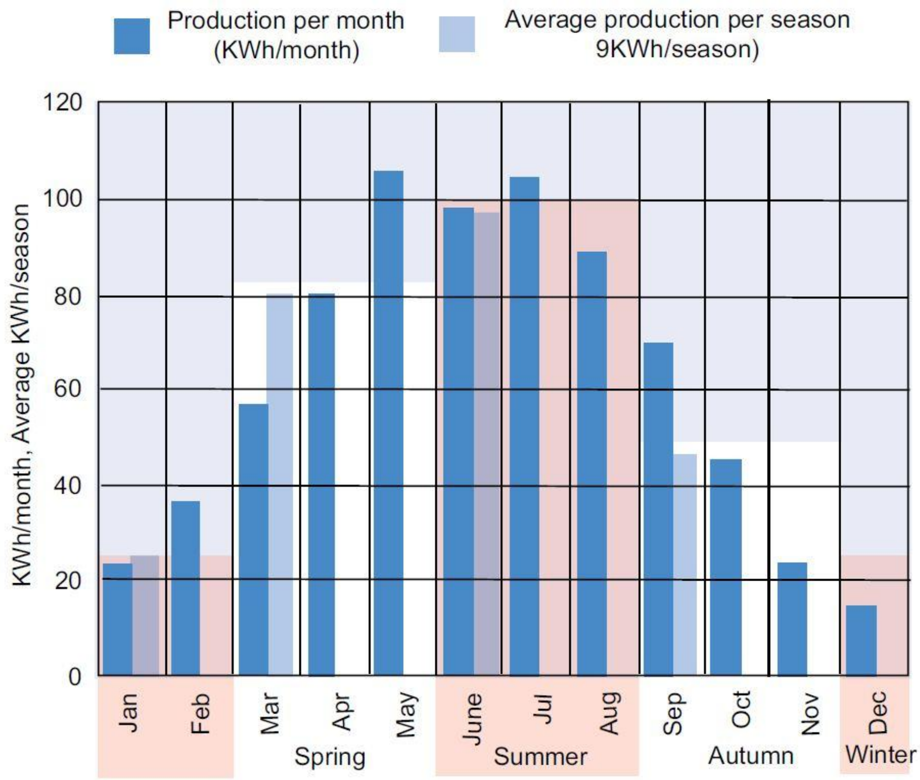
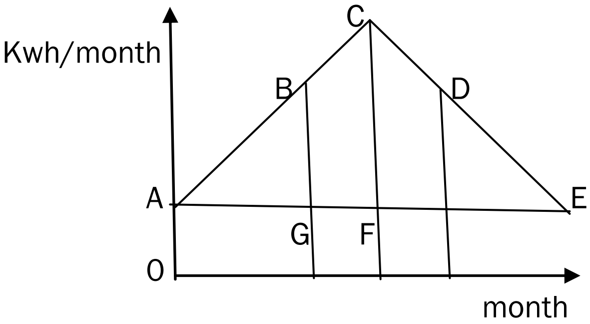


Figure 3. [MCS]. *The chart above illustrates the seasonal variation in the energy production of solar panels. I believe that this is based on typical data on how solar panel systems are expected to perform*.

From the diagram, we will make the gross approximation that solar energy production is at a maximum of 100 Kwh/month at the summer solstice, and decreases uniformly as a straight line to 20 Kwh/month at the winter solstice.



At which value is the area under AB equal to the area under BC? This should give an approximation to the average month for which the slope of the panel should be maximised. To do this we see that the area of triangle ACF is proportional to $(AC)^2$, and the area of triangle ABG, which has the same angles as triangle ACF, is proportional to $(AB)^2$. We have

½(area ACF + rectangle below it) = (area ABG + rectangle below it).

The month after the winter solstice, m, for which this occurs then satisfies

$$\tfrac{1}{2}(\tfrac{1}{2} \times 6 \times 80 + 6 \times 20) = (\tfrac{1}{2} \times m \times m \times 80/6) + m \times 20,$$

giving

$$m^2 + 3m - 27 = 0$$

or by the standard solution of a quadratic equation, and since this solution is positive

$$m = \tfrac{1}{2}[-3 + \sqrt{(9 + 4 \times 27)}] = 3.91 \text{ months.}$$

Thus m corresponds to 18th April.

The ratio of m to 6 months is now r = 0.652. We will use this as the factor by which we incline the new slope angle, which is at $\beta_w = 23.4° + \theta = 74.2°$ calculated for maximisation at the winter solstice and $\beta_s = \theta - 23.4° = 27.4°$ calculated for the summer solstice. So we will assume for average maximisation, using the equation of a straight line, it should be at

$$\beta_w + 0.652(\beta_s - \beta_w) = [74.2 - (0.652 \times 46.8)]° = 43.7°.$$

In review, Doly García pointed out that the assumption that this corresponds to optimum output summed over a year may be unjustified. Recalculations have shown that this objection is correct. Comparisons between the current and the modified approach are given in section 2.7.2 on maximum output.

We have not yet finished. The above calculation only concerns the angle suitable for direct solar radiation. It does not include the effects of scattering of light and light from clouds. We must include these factors in our calculation.

We saw in section 2.2 that these two effects correspond on average to 25% of the radiation from the sun reaching Earth's surface, whereas the effect of direct sunlight was 22.5%. Thus the first effect is 1.1 times that of the second. We will assume that light from indirect sources is on average the same from all points of the sky. For indirect radiation, the optimal panel configuration is horizontal, if we assume that a panel at an angle has its source of radiation restricted by the flat surface, so that it does not come from all angles. Thus at an angle of 43.7° a sector of 43.7/180 = 0.24 of the ambient indirect radiation is blocked out, leaving 76% remaining.

However, we also need to take into account that we are assuming the power output varies as the sine of the incident radiation to the plane of the PV panel. If this angle is γ the normalised power component varies from sin(γ_1) = sin(43.7°) through sin(90°) = 1 to sin(180°) = 0, and the total summation is the integral

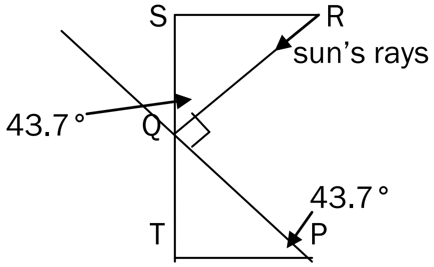
$$\int_{\gamma_1}^{\pi} \sin\gamma \, d\gamma = {}_{\gamma_1}^{\pi}[\cos\gamma] = 1 + \cos\gamma_1,$$

where

cos(43.7°) = 0.723,

and we are assuming this integral is invariant over the day. For a horizontal panel the same factor is 2. Thus the power is reduced to (100×1.723)/2 = 86.15% from indirect radiation.

For the corresponding situation for direct sunlight, we need to know how much the power is reduced by having a flat panel. Since the angle 43.7° is optimum at month m, the direct radiation is at right angles to the PV plane at that month. In the diagram



PQ is the panel. SQ is the vertical component of the sun's rays which are travelling along RQ. If the panel were horizontal, it would occupy part of PT and the total radiation on the panel would be reduced from a factor of 1 to cos(γ_1) = cos(43.7°).

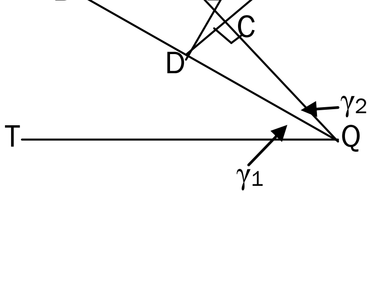
At month m we will compare the radiation on a horizontal panel with a panel at angle γ_1 = 43.7° to the horizontal. We will compare the indirect and direct radiation and their sum.

For a horizontal panel the indirect radiation we have seen is boosted by a factor 1.11 with respect to direct radiation, which has a vertical component cos(γ_1). Their sum is therefore

1.11 + cos(γ_1) = 1.11 + 0.72 = 1.83.

For the panel at an angle γ_1, the indirect radiation we have just seen is 1.11×(1 + cos(γ_1))/2, the direct radiation factor is 1, and their sum is 1.56 + [0.555 × cos(γ_1)] = 1.96. It can be seen that the increased power at Brighton & Hove on adopting a panel at 43.7° over a horizontal panel is not large. There is a 1% increase.

To find the maximum power output at month m, consider a variation where the angle of the array is incremented from γ_1 = 43.7° by an additional amount γ_2. We will find the value of γ_2 is negative.



The adjusted panel is now at QA, and the original panel was at QB. Light rays hit QA along the old trajectory RE, and we want to find the vertical component to QA of this along SC. To do this we perform a similar calculation to the ones we have done previously to find that angle EDC is γ_2. Relative to a length or power value of ED = 1, the value of CD is $\cos(\gamma_2)$. The slope of the adjusted panel is $\gamma_1 + \gamma_2 = 43.7° + \gamma_2$.

To find the maximum power output we give a quick course in differential calculus using trigonometric functions like cos and sin. I am assuming the reader has done calculus before.

The derivative, or slope, of a function $f(x) = x^n$ is

$$\frac{df(x)}{dx} = nx^{n-1}.$$

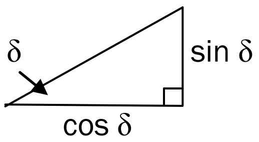
The exponential function is defined as a function whose derivative is given by the function itself. Its value is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots,$$

where the number n! (called n factorial) = n × (n − 1) × (n − 2) × ... × 1.

The Pythagoras theorem for a hypotenuse of length 1 can be stated as

$$(\cos\delta)^2 + (\sin\delta)^2 = 1,$$



which we write by an abuse of notation as

$$\cos^2\delta + \sin^2\delta = 1.$$

If we represent the imaginary number

$$i = \sqrt{(-1)},$$

so

$$i^2 = -1,$$

then we see

$$\cos\delta = \frac{e^{-i\delta} + e^{i\delta}}{2},$$

$$\sin\delta = i\frac{e^{-i\delta} - e^{\delta}}{2}$$

satisfies the Pythagoras theorem for a hypotenuse of length 1. Expanded out these are

$$\cos\delta = 1 - \frac{\delta^2}{2} + \cdots$$

$$\sin\delta = \delta - \frac{\delta^3}{3!} + \cdots$$

It then follows from the definition of the exponential function that

$$e^{i\delta} = \cos\delta + i\sin\delta.$$

Thus

$$e^{i(A+B)} = e^{iA}e^{iB}$$

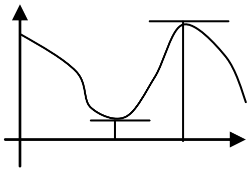
$$= (\cos A + i\sin A)(\cos B + i\sin B) = (\cos(A+B) + i\sin(A+B))$$

so that on comparing real and imaginary parts

$$\cos(A + B) = \cos A\cos B - \sin A\sin B,$$

$$\sin(A + B) = \cos A\sin B + \sin A\cos B.$$

A local maximum or minimum of a function occurs when the slope or derivative of a function is zero.



From the definition of cos and sin in terms of the series above

$$\frac{d \sin \delta}{d\delta} = \cos \delta,$$

$$\frac{d \cos \delta}{d\delta} = -\sin \delta.$$

After this brief diversion, we now have the mathematical machinery to find the slope of the PV panel for the maximum panel power output.

For the adjusted panel at an angle $\gamma_1 + \gamma_2$, at month m the indirect radiation we have just seen is

$$1.11 \times (1 + \cos(\gamma_1 + \gamma_2))/2,$$

with $\gamma_1 = 43.7°$, whereas the previous discussion has shown that for direct radiation the value is reduced to the corresponding level

$$\cos(\gamma_2).$$

The sum of these two values is therefore

$$0.555 + (0.555 \times \cos(\gamma_1 + \gamma_2)) + \cos(\gamma_2),$$

and using the expression for $\cos(\gamma_1 + \gamma_2) = \cos(43.7° + \gamma_2)$,

$$\cos(43.7° + \gamma_2) = \cos(43.7°) \cos(\gamma_2) - \sin(43.7°) \sin(\gamma_2),$$

or

$$\cos(43.7° + \gamma_2) = 0.723 \cos(\gamma_2) - 0.691 \sin(\gamma_2),$$

the sum is

$$0.556 + 1.401 \cos(\gamma_2) - 0.384 \sin(\gamma_2).$$

To find the maximum or minimum of this, we have seen we take the derivative with respect to γ_2 and equate it to zero. Thus

$$-1.401 \sin(\gamma_2) - 0.384 \cos(\gamma_2) = 0,$$

or

$$\tan(\gamma_2) = -0.384/1.401 = -0.274,$$

$$\gamma_2 = -15.3°,$$

but this is a maximum, since if we take its derivative again, it is negative, which means the slope is decreasing as we pass through the maximum. Thus the slope of the panel should be

$$\gamma_1 + \gamma_2 = 28.4°.$$

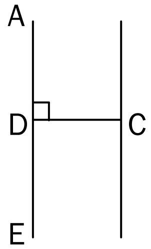
The number we generated for a horizontal panel was 1.83. Using a panel at 28.4° generates a number 0.556 + 0.488 + 0.965 = 2.01, thus the boost in this case is additionally 10% from the horizontal panel case. Note that the computations are for mid day. Outside of midday the effective slope is less.

panel from horizontal	**0°**	**28.4°**	**35°**	**43.7°**
computed boost	0%	10%	9%	1%

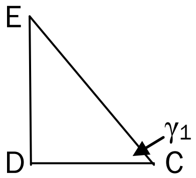
2.5.2. Non south facing panels.

At what angle should a PV panel be if not completely south facing?

Let us consider a panel at angle γ_1 to the horizontal.



horizontal cross-section

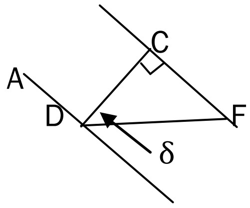


vertical cross-section

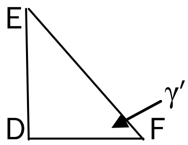
The height of the panel is ED, so

$$\tan \gamma_1 = ED/DC.$$

If the panel is now turned by an angle δ,



horizontal cross-section



vertical cross-section about DF

then in the diagram

$$DF \cos \delta = DC.$$

If the angle to the horizontal taken through the vertical slice containing DF is γ', then

$$\tan \gamma' = ED/DF = DC \tan \gamma_1 \cos \delta/DC = \tan \gamma_1 \cos \delta.$$

Now if the sun's rays are parallel to DF, if we are considering a non south facing panel to be sloped at angle γ_1, we need at Brighton & Hove to specify that the adjusted value of the slope of the panel along DC is

$$\gamma_1 = \tan^{-1}(\tan \gamma'/\cos \delta)$$

to maximise output. Thus the panel should be sloped more.